

MEANS AND CONVERGENCE OF SEMIGROUP ORBITS

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Abstract. In this paper we prove the following general theorem. Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of E . Assume that C has nonempty interior and is locally uniformly rotund. Let \mathcal{F} be a commutative nonexpansive semigroup acting on C . If \mathcal{F} has no fixed point in the interior of C , then there exists a unique point \tilde{x} on the boundary of C such that each orbit of \mathcal{F} converges in norm to \tilde{x} . We also establish analogous results for semigroups and mappings which are asymptotically nonexpansive in the intermediate sense.

Key Words and Phrases: Asymptotically nonexpansive in the intermediate sense, fixed point, iterates, locally uniformly rotund set, nonexpansive mapping, semigroup of mappings, uniform convexity.

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