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VISCOSITY APPROXIMATION METHODS FOR GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND MONOTONE INCLUSIONS IN HADAMARD SPACES

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Abstract. The main purpose of this paper is to introduce and study some viscosity-type proximal point algorithms for approximating a common solution of monotone inclusion problem and fixed point problem. We obtained strong convergence of the proposed algorithms to a common solution of minimization problem and fixed point problem for a generalized asymptotically nonexpansive mapping which is also a unique solution of some variational inequality problems in Hadamard spaces. Our results extend and complement some recent results in this direction.
Key Words and Phrases: Generalize asymptotically nonexpansive mappings, monotone inclusions, fixed point, strong convergence, viscosity approximation, CAT(0) space.
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