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WEAK AND STRONG CONVERGENCE RESULTS FOR SUM OF TWO MONOTONE OPERATORS IN REFLEXIVE BANACH SPACES

FERDINARD U. OGBUISI*, LATEEF O. JOLAOSO** AND YEKINI SHEHU***

*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa;
Department of Mathematics, University of Nigeria, Nsukka, Nigeria
E-mail: ferdinard.ogbuisi@unn.edu.ng, fudochukwu@yahoo.com

**School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa
E-mail: 216074984@stu.ukzn.ac.za, jollatanu@yahoo.co.uk

***Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, People's Republic of China
E-mail: yekini.shehu@unn.edu.ng

Abstract. The purpose of this paper is to study an inclusion problem which involves the sum of two monotone operators in a real reflexive Banach space. Using the technique of Bregman distance, we study the operator $Res_T^f \circ A^f$ which is the composition of the resolvent of a maximal monotone operator T and the antiresolvent of a Bregman inverse strongly monotone operator A and prove that $0 \in Tx + Ax$ if and only if x is a fixed point of the composite operator $Res_T^f \circ A^f$. Consequently, weak and strong convergence results are given for the inclusion problem under study in a real reflexive Banach space. We apply our results to convex optimization and mixed variational inequalities in a real reflexive Banach space. Our results are new, interesting and extend many related results on inclusion problems from both Hilbert spaces and uniformly smooth and uniformly convex Banach spaces to more general reflexive Banach spaces.
Key Words and Phrases: Maximal monotone operators, antiresolvent operators, Bregman distance, inclusion problem, reflexive Banach spaces.
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