Abstract. Many metric fixed point results can be formulated in an abstract 'convexity structure' setting. This discussion contains a review of some of these, as well as a discussion of other results which seem to require a bit more structure on the space. A metric space \((X,d)\) is said to be \(\Gamma\)-uniquely geodesic if \(\Gamma\) is a family of geodesic segments in \(X\) and for each \(x,y \in X\) there is a unique geodesic \([x,y] \in \Gamma\) with endpoints \(x\) and \(y\). Let \(X\) be \(\Gamma\)-uniquely geodesic and let \(C(X)\) denote the family of all bounded closed convex (relative to \(\Gamma\)) subsets of \(X\). Assume that the family \(C(X)\) is compact in the sense that every descending chain of nonempty subsets of \(C(X)\) has a nonempty intersection. This is a brief discussion of what additional conditions on a mapping \(T : K \to K\) for \(K \in C(X)\) always assure that has at least one fixed point. In particular it is shown that if the balls in \(X\) are \(\Gamma\)-convex and if the closure of a \(\Gamma\)-convex set in \(X\) is again \(\Gamma\)-convex then a mapping \(T : K \to K\) always has a fixed point if it is nonexpansive with respect to orbits in the sense of Amini-Harandi, et al., and if for each \(x \in K\) with \(x \neq T(x)\),

\[
\inf_{m \in \mathbb{N}} \left\{ \limsup_{n \to \infty} d(T^m(x), T^n(x)) \right\} < \text{diam}(O(x)).
\]

Mappings of the above type include those which are pointwise contractions in the sense that for each \(x \in K\) there exists \(\alpha(x) \in (0,1)\) such that

\[
d(T(x), T(y)) \leq \alpha(x) d(x, y)
\]

for all \(y \in K\).

The results discussed here extend known results if \(K\) is a weakly compact (convex) subset of a Banach space. A number of open questions are raised in connection with characterizations of normal structure in certain geodesic spaces.

Key Words and Phrases: Normal structure, compact convexity structures, nonexpansive mappings, fixed points, diminishing orbital diameters, pointwise contractions, mappings nonexpansive with respect to orbits, strictly contractive mappings, geodesic spaces.

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