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ORBITAL FIXED POINT CONDITIONS IN GEODESIC SPACES

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Abstract. Many metric fixed point results can be formulated in an abstract 'convexity structure' setting. This discussion contains a review of some of these, as well as a discussion of other results which seem to require a bit more structure on the space. A metric space (X, d) is said to be Γ-uniquely geodesic if Γ is a family of geodesic segments in X and for each $x, y \in X$ there is a unique geodesic $[x, y] \in \Gamma$ with endpoints x and y. Let X be Γ-uniquely geodesic and let $\mathfrak{C}(X)$ denote the family of all bounded closed convex (relative to Γ) subsets of X. Assume that the family $\mathfrak{C}(X)$ is compact in the sense that every descending chain of nonempty subsets of $\mathfrak{C}(X)$ has a nonempty intersection. This is a brief discussion of what additional conditions on a mapping $T : K \to K$ for $K \in \mathfrak{C}(X)$ always assure that has at least one fixed point. In particular it is shown that if the balls in X are Γ-convex and if the closure of a Γ-convex set in X is again Γ-convex then a mapping $T : K \to K$ always has a fixed point if it is nonexpansive with respect to orbits in the sense of Amini-Harandi, et al., and if for each $x \in K$ with $x \neq T(x)$,

$$\inf_{m\in\mathbb{N}} \left\{ \limsup_{n\to\infty} d\left(T^m(x), T^n(x)\right) \right\} < diam\left(O(x)\right).$$

Mappings of the above type include those which are pointwise contractions in the sense that for each $x \in K$ there exists $\alpha(x) \in (0, 1)$ such that

$$d\left(T(x), T(y)\right) \leq \alpha(x)\, d(x, y) \text{ for all } y \in K.$$

The results discussed here extend known results if K is a weakly compact (convex) subset of a Banach space. A number of open questions are raised in connection with characterizations of normal structure in certain geodesic spaces.

Key Words and Phrases: Normal structure, compact convexity structures, nonexpansive mappings, fixed points, dimihishing orbital diameters, pointwise contractions, mappings nonexpansive with respect to orbits, strictly contractive mappings, geodesic spaces.
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References

[1] M.A. Alghamdi, W.A. Kirk, N. Shahzad, *Remarks on convex combinations in geodesic spaces,* J. Nonlinear Convex Anal., **15**(2014), no. 1, 49-59.

[2] A. Amini-Harandi, M. Fakhar, H.R. Hajishari, *Weak fixed point property for nonexpansive mappings with respect to orbits in Banach spaces,* J. Fixed Point Theory Appl., **18**(2016), no. 3, 601-607.

[3] D. Ariza, C. Li, G. López-Acedo, *The Schauder fixed point theorem in geodesic spaces,* J. Math. Anal. Appl., **417**(2014), no. 1, 345-360.

[4] L.P. Belluce, W.A. Kirk, *Fixed-point theorems for certain classes of nonexpansive mappings,* Proc. Amer. Math. Soc., **20**(1969), 141-146.

[5] J. Bogin, *A generalization of a fixed point theorem of Goebel, Kirk and Shimi,* Canad. Math. Bull., **19**(1976), no. 1, 7-12.

[6] M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature,* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319, Springer-Verlag, Berlin, 1999.

[7] M.S. Brodskiĭ, D.P. Milman, *On the center of a convex set,* (Russian), Doklady Akad. Nauk SSSR (N.S.), **59**(1948), 837-840.

[8] F. Browder, *Nonexpansive nonlinear operators in a Banach space,* Proc. Nat. Acad. Sci. U.S.A., **54**(1965), 1041-1044.

[9] T. Büber, W.A. Kirk, *A constructive proof of a fixed point theorem of Soardi,* Math. Japon., **41**(1995), no. 2, 233-237.

[10] A.A. Gillespie, B.B. Williams, *Fixed point theorem for nonexpansive mappings on Banach spaces with uniformly normal structure,* Applicable Anal., **9**(1979), no. 2, 121-124.

[11] D. Göhde, *Zum Prinzip der kontraktiven Abbildung,* (German), Math. Nachr., **30**(1965), 251-258.

[12] N. Hussain, M. A. Khamsi, *On asymptotic pointwise contractions in metric spaces,* Nonlinear Anal., **71**(2009), no. 10, 4423-4429.

[13] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory,* Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2001.

[14] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances,* Amer. Math. Monthly, **72**(1965), 1004-1006.

[15] W.A. Kirk, *Nonexpansive mappings and the weak closure of sequences of iterates,* Duke Math. J., **36**(1969), 639-645.

[16] W.A. Kirk, *On nonlinear mappings of strongly semicontractive type,* J. Math. Anal. Appl., **27**(1969), 409-412.

[17] W.A. Kirk, *Fixed point theorems for nonexpansive mappings,* 1970, Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968), 162-168, Amer. Math. Soc., Providence, R.I.

[18] W.A. Kirk, *An abstract fixed point theorem for nonexpansive mappings,* Proc. Amer. Math. Soc., **82**(1981), no. 4, 640-642.

[19] W.A. Kirk, *Nonexpansive mappings and normal structure in Banach spaces,* Proceedings of Research Workshop on Banach Space Theory (Iowa City, Iowa, 1981), 113-127, Univ. Iowa, Iowa City, IA, 1982.

[20] W.A. Kirk, N. Shahzad, *Normal structure and orbital fixed point conditions,* J. Math. Anal. Appl., **463**(2018), no. 2, 461-476.

[21] W.A. Kirk, X.K. Xu, *Asymptotic pointwise contractions,* Nonlinear Anal., **69**(2008), no. 12, 4706-4712.

[22] T.C. Lim, *Characterizations of normal structure,* Proc. Amer. Math. Soc., **43**(1974), 313-319.

[23] E. Llorens-Fuster, *Orbitally nonexpansive mappings,* Bull. Aust. Math. Soc., **93**(2016), no. 3, 497-503.

[24] A. Nicolae, *Generalized asymptotic pointwise contractions and nonexpansive mappings involving orbits,* Fixed Point Theory Appl., **2010**(2010), Art. ID 458265, 19 pp.

[25] G. Oldani, D. Roux, *Convexity structures and Kannan maps,* Nonlinear Functional Analysis and its Applications (Maratea, 1985), 327-333, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 173, Reidel, Dordrecht, 1986.

[26] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature,* IRMA Lectures in Mathematics and Theoretical Physics, 6, European Mathematical Society (EMS), Zürich, 2005.

[27] J.-P. Penot, *Fixed point theorems without convexity,* Analyse Non Convexe (Proc. Colloq., Pau, 1977). Bull. Soc. Math. France Mém. No. **60**(1979), 129-152.

[28] S. Prus, *Geometrical Background of Metric Fixed Point Theory,* Handbook of Metric Fixed Point Theory, 93–132, Kluwer Acad. Publ., Dordrecht, 2001.

[29] S. Reich, I. Shafrir, *Nonexpansive iterations in hyperbolic spaces,* Nonlinear Anal., **15**(1990), no. 6, 537-558.

[30] P.M. Soardi, *Existence of fixed points of nonexpansive mappings in certain Banach lattices,* Proc. Amer. Math. Soc., **73**(1979), no. 1, 25-29.

[31] C.S. Wong, *Close-to-normal structure and its applications,* J. Funct. Anal., **16**(1974), 353-358.

[32] C.S. Wong, *On Kannan maps,* Proc. Amer. Math. Soc., **47**(1975), 105-111.

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