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FIXED POINT THEORY IN TERMS OF A METRIC AND OF AN ORDER RELATION

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Abstract. We consider a nonempty set X endowed with a metric d an order relation $\preceq$ and an operator $f : X \to X$, which satisfies two main assumptions:
(1) f is generalized monotone with respect to $\preceq$;
(2) f is a (generalized) contraction with respect to d on a certain subset Y of $X \times X$.
In the above terms, we will present conditions under which:
(i) f has a unique fixed point in X;
(ii) f is a Picard operator;
(iii) the fixed point problem for f is well-posed;
(iv) f has the Ostrowski property;
(v) f has the shadowing property;
(vi) f satisfies to some Gronwall type inequalities.
Then, we will apply these results to study some problems related to integral and differential equations. Several open questions are discussed.
Key Words and Phrases: Metric space, ordered set, ordered metric space, contraction, generalized contraction, increasing operator, decreasing operator, progressive operator, regressive operator, generalized monotone operator, fixed point, (weakly) Picard operator, stability, Gronwall lemma, open problem.
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