

REMARKS ON FRINK'S METRIZATION TECHNIQUE AND APPLICATIONS

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Abstract. In this paper, we give a simple counterexample to show again the limits of Frink's construction [17, page 134] and then use Frink's metrization technique to answer two conjectures posed by Berinde and Choban [5], and to calculate corresponding metrics induced by some b -metrics known in the literature. We also use that technique to prove a metrization theorem for 2-generalized metric spaces, and to deduce the Banach contraction principle in b -metric spaces and 2-generalized metric spaces from that in metric spaces.

Key Words and Phrases: Metrization, quasi-metric, b -metric, 2-generalized metric, fixed point.

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1. INTRODUCTION AND PRELIMINARIES

The metrization problem is concerned with conditions under which a topological space X is metrizable [11], where for a function $d : X \times X \rightarrow [0, \infty)$ satisfying some axioms and generating a topology \mathcal{T} on X , and for a metric $D : X \times X \rightarrow [0, \infty)$, the topological space (X, \mathcal{T}) is called *metrizable* by the metric d if \mathcal{T} and the metric topology induced by d coincide. Recall that a space X is a *metric space* if there exists a *metric* $D : X \times X \rightarrow [0, +\infty)$ that satisfies the following conditions for all $x, y, z \in X$.

- I. $D(x, y) = 0$ if and only if $x = y$.
- II. The symmetry: $D(x, y) = D(y, x)$.
- III. The triangle inequality: $D(x, z) \leq D(x, y) + D(y, z)$.

Some generalizations of the triangle inequality (III) were introduced such as

- IV. The generalized triangle inequality: If $D(x, y) < \varepsilon$ and $D(y, z) < \varepsilon$ then $D(x, z) < 2\varepsilon$.

V. The uniform regular property: For every $\varepsilon > 0$ there exists $\phi(\varepsilon) > 0$ such that if $D(x, y) < \phi(\varepsilon)$ and $D(y, z) < \phi(\varepsilon)$ then $D(x, z) \leq \varepsilon$.

In 1993 Czerwik [12] introduced the notion of a *b-metric* with a coefficient 2. This notion was generalized later with a coefficient $K \geq 1$ [13]. In 2010 Khamsi and Hussain [22] reintroduced the notion of a *b-metric* under the name *metric-type*. Another notion of metric-type, called *s-relaxed_p metric* was introduced in [15, Definition 4.2], see also [20]. A *b-metric* is called *quasi-metric* in [30]. Quasi-metric spaces play an important role in the study of Gromov hyperbolic metric spaces [35, Final remarks], and in the study of optimal transport paths [39]. For convenience the names *b-metric* and *b-metric space* will be used in what follows. It is clear that condition (V) reduces to (IV) if $\phi(\varepsilon) = \frac{\varepsilon}{2}$, and every *b-metric space* (X, D, K) is a space with the distance function D satisfying (I), (II) and (V) with $\phi(\varepsilon) = \frac{\varepsilon}{2K}$.

Recall that a *distance space* is a pair (X, D) consisting of a set X and a function $D : X \times X \rightarrow [0, \infty)$ satisfying $D(x, y) + D(y, x) = 0$ if and only if $x = y$ [5, Definition 2.1]. Note that the convergence in a distance space (X, D) is defined by the usual way, that is, $\lim_{n \rightarrow \infty} x_n = x$ if $\lim_{n \rightarrow \infty} D(x_n, x) = 0 = \lim_{n \rightarrow \infty} D(x, x_n)$. Similarly, a sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n, m \rightarrow \infty} D(x_n, x_m) = 0$. The convergence in (X, D) generates a topology \mathcal{T} , called the *sequential topology* on (X, D) , in the sense of Franklin [16, page 108]: a subset U is called *open* in (X, \mathcal{T}) if for each $x \in U$ and $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) there exists n_0 such that $x_n \in U$ for all $n \geq n_0$. For each $x \in X$ and each $r > 0$ the set

$$B(x, r, D) = \{y \in X : D(x, y) < r\}$$

is called a *ball with center x and radius r* . There is another topology $\mathcal{T}(D)$ on (X, D) : a subset U of X is called *open* if for each $x \in U$ there exists $r_x > 0$ such that $B(x, r_x, D) \subset U$. The topology $\mathcal{T}(D)$ is called the *topology induced by the distance D* , see also [5, Definition 2.1]. As in the proof of [3, Proposition 3.3.(1)], $\mathcal{T}(D)$ is exactly the topology \mathcal{T} provided D is symmetric.

In 1917 Chittenden [10] showed that a space with a distance function satisfying (I), (II) and (V), that was also called a *CF-metric space* [5, Definition 3.2], is metrizable. Consequently, every *b-metric space* is metrizable [24, page 114]. Chittenden's proof was somewhat long and complicated and, although the existence of a distance function satisfying (III) is proved, it is not defined directly in terms of the original distance function satisfying (V). In 1937 Frink [17, page 133] presented a simple and direct proof of the fact that a topological space with a distance function satisfying (I), (II) and (IV), and also (V), is metrizable without relying on Chittenden's theorem. Frink's metrization technique is also called the *chain approach*.

Frink's metrization technique impacted many results. In 1998 Aïmar *et al.* [1] improved Frink's metrization technique to give a direct proof of a theorem of Macías and Segovia in [30] on the metrization of a *b-metric space* (X, D, K) . In 2006 Schroeder showed some limits of Frink's construction, by providing a counterexample of a *b-metric space* (X, D, K) , for which the function d defined by Frink's metrization technique (see (1.1) below) is not a metric [35, Example 2]. In 2009 Paluszynski and

Stempak [31] also improved Frink's metrization technique to produce a metric d from a given b -metric space (X, D) .

In 2000 Branciari [9] introduced a notion of a ν -generalized metric space. This notion was studied by many authors, see [23], [25] and the references given there. Some authors constructed functions that are 2-generalized metrics but are not metrics [9, 3. Example], [14, Examples 1 & 2], [26, Example 1], and stated many fixed point theorems in ν -generalized metric spaces. However, the metrization of ν -generalized metric spaces was rarely studied. Recently a sufficient condition for ν -generalized metric spaces to be metrizable was proved [27, Corollary 2.6].

Many authors transferred results from metric spaces to b -metric spaces and other generalized metric spaces [2], [7], [32]. However, it is necessary to work carefully in generalized metric fixed point theory, since various fixed point theorems in generalized metric spaces, except for b -metric spaces and ν -generalized metric spaces, can be deduced from the corresponding fixed point theorems in metric spaces [2, 4. Conclusions], [21]. In 2013 Berinde and Choban [5] presented a similar situation in the case of b -metric spaces. They asserted that working in b -metric spaces (X, D, K) makes sense since the associate metric d given by (1.1) is not always a metric [5, page 28]. Berinde and Choban introduced the notion of an F -distance space and proposed some conjectures. Note that there were some typos in [5, Conjecture 6.2] that make a misunderstanding in the conjecture. By a private communication with the corresponding author of that paper the conjecture is restated as follows.

Question 1.1 ([5], Conjecture 6.1). *Let (X, ρ) be an F -distance space and $T : X \rightarrow X$ be a map such that $\rho(Tx, Ty) \leq \lambda\rho(x, y)$ for some $\lambda \in [0, 1)$ and all $x, y \in X$.*

- (1) *Is the sequence $\{x_n\}$ Cauchy, where $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ and some $x_1 \in X$?*
- (2) *Does there exist a unique fixed point of T if the space (X, ρ) is complete?*

Question 1.2 ([5], Conjecture 6.2). *Let (X, ρ) be a symmetric distance space, $(X, \mathcal{T}(\rho))$ be Hausdorff compact, and $T : X \rightarrow X$ be a map such that $\rho(Tx, Ty) \leq \lambda\rho(x, y)$ for some $\lambda \in [0, 1)$ and all $x, y \in X$. Does there exist a unique fixed point of T ?*

In this paper, we are interested in studying Frink's metrization technique. In Section 2 we construct a simple counterexample to show again the limits of Frink's construction [17, page 134]. In Section 3 we show that the Banach contraction principle in b -metric spaces can be deduced from the Banach contraction principle in metric spaces and then calculate corresponding metrics induced by some b -metrics known in the literature. In Section 4 we give answers to Question 1.1 and Question 1.2. In Section 5 we prove a metrization condition for 2-generalized metric spaces and show that the Banach contraction principle in 2-generalized metric spaces can be deduced from the Banach contraction principle in metric spaces.

Now we recall notions and properties which are useful in what follows.

Definition 1.3 ([13]). Let X be a nonempty set, $K \geq 1$ and $D : X \times X \rightarrow [0, +\infty)$ be a function such that for all $x, y, z \in X$,

- (1) $D(x, y) = 0$ if and only if $x = y$.

- (2) $D(x, y) = D(y, x)$.
 (3) $D(x, z) \leq K [D(x, y) + D(y, z)]$.

Then D is called a b -metric on X and (X, D, K) is called a b -metric space.

Theorem 1.4 ([17], pages 134-135). *Let (X, D) be a space satisfying (I), (II) and (IV). For any $x, y \in X$, define*

$$d(x, y) = \inf \left\{ \sum_{i=1}^n D(x_i, x_{i+1}) : x_1 = x, x_2, \dots, x_{n+1} = y \in X, n \in \mathbb{N} \right\}. \quad (1.1)$$

Then

- (1) For all $x, x_1, \dots, x_n, y \in X$,

$$D(x, y) \leq 2D(x, x_1) + 4D(x_1, x_2) + \dots + 4D(x_{n-1}, x_n) + 2D(x_n, y). \quad (1.2)$$

 (2) d is a metric on X .
 (3) For all $x, y \in X$,

$$\frac{D(x, y)}{4} \leq d(x, y) \leq D(x, y). \quad (1.3)$$

In particular,

- (1) $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, d) .
 (2) A sequence $\{x_n\}$ is Cauchy in (X, D) if and only if it is Cauchy in (X, d) .
 (3) The distance space (X, D) is metrizable by the metric d .

Theorem 1.5 ([17], page 135). *Let (X, δ) be a space satisfying (I), (II) and (V). For all $\varepsilon \geq 0$, put $\psi(\varepsilon) = \min\{\phi(\varepsilon), \frac{\varepsilon}{2}\}$, and put*

$$r_1 = 1, \dots, r_{n+1} = \psi(r_n), \dots$$

and for all $x, y \in X$, define

$$D(x, y) = \begin{cases} 1 & \text{if } D(x, y) \geq r_1 \\ \frac{1}{2^n} & \text{if } r_n > D(x, y) \geq r_{n+1}. \end{cases}$$

Then

- (1) The distance space (X, D) satisfies (I), (II) and (IV).
 (2) $\lim_{n \rightarrow \infty} x_n = x$ in (X, δ) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) . In particular, the distance space (X, δ) is metrizable by the metric d defined as in (1.1).

Remark 1.6. The conclusions of Theorem 1.4 are still true if any strict inequality in (IV) is replaced by the corresponding inequality.

Theorem 1.7 ([1], Theorem I). *Let (X, D, K) be a b -metric space. Then there exists $0 < \beta \leq 1$, depending only on K , such that*

$$d(x, y) = \inf \left\{ \sum_{i=1}^n D^\beta(x_i, x_{i+1}) : x_1 = x, x_2, \dots, x_{n+1} = y \in X, n \in \mathbb{N} \right\} \quad (1.4)$$

is a metric on X satisfying $\frac{1}{2}D^\beta \leq d \leq D^\beta$. In particular, if D is a metric then $d = D$.

Theorem 1.8 ([31], Proposition on page 4308). *Let (X, D, K) be a b -metric space, $0 < p \leq 1$ satisfying $(2K)^p = 2$, and for all $x, y \in X$,*

$$d(x, y) = \inf \left\{ \sum_{i=1}^n D^p(x_i, x_{i+1}) : x_1 = x, x_2, \dots, x_n, x_{n+1} = y \in X, n \in \mathbb{N} \right\}. \quad (1.5)$$

Then d is a metric on X satisfying $\frac{1}{4}D^p \leq d \leq D^p$. In particular, if D is a metric then $d = D$.

Definition 1.9 ([9], Definition 2.1). Let X be a nonempty set, $\nu \in \mathbb{N}$, $\nu \geq 1$ and $\rho : X \times X \rightarrow [0, +\infty)$ be a function such that for any $x, y \in X$ and for any family x_1, \dots, x_ν of pairwise distinct elements in $X \setminus \{x, y\}$,

- (1) $\rho(x, y) = 0$ if and only if $x = y$.
- (2) $\rho(x, y) = \rho(y, x)$.
- (3) $\rho(x, y) \leq \rho(x, x_1) + \rho(x_1, x_2) + \dots + \rho(x_\nu, y)$.

Then ρ is called a ν -generalized metric on X and (X, ρ) is called a ν -generalized metric space. A sequence $\{x_n\}$ is called convergent to x in (X, ρ) if $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$. A sequence $\{x_n\}$ is called Cauchy if $\lim_{n, m \rightarrow \infty} \rho(x_n, x_m) = 0$. A generalized metric space (X, ρ) is called complete if each Cauchy sequence is a convergent sequence.

Theorem 1.10 ([18], Theorem 3.3). *Let (X, D, K) be a complete b -metric space and $T : X \rightarrow X$ be a map such that $D(Tx, Ty) \leq \lambda D(x, y)$ for all $x, y \in X$ and some $\lambda \in [0, \frac{1}{K})$. Then T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.*

Definition 1.11 ([5], Definition 3.3). Let X be a nonempty set and $\rho : X \times X \rightarrow [0, +\infty)$ be a function such that for all $x, y, z \in X$,

- (1) $\rho(x, y) = 0$ if and only if $x = y$.
- (2) For every $\varepsilon > 0$, there exists $\phi(\varepsilon) > 0$ such that if $\rho(x, y) \leq \phi(\varepsilon)$ and $\rho(y, z) \leq \phi(\varepsilon)$ then $\rho(x, z) \leq \varepsilon$ and $\rho(z, x) \leq \varepsilon$.

Then ρ is called an F -distance on X and (X, ρ) is called an F -distance space.

2. REMARKS ON FRINK'S METRIZATION TECHNIQUE

In this section, we construct a simple counterexample to show again the limits of Frink's construction [17, page 134]. In 2006 Schroeder constructed a counterexample showing that for given b -metric space (X, D, K) the distance function d defined by (1.1) is not a metric [35, Example 2]. The following example, that is simpler than [35, Example 2], also shows that Theorem 1.4.(1) and Theorem 1.4.(2) do not hold if a space satisfying (I), (II) and (IV) is replaced by a b -metric space.

Example 2.1. Let $X = \mathbb{R}$, and $D(x, y) = |x - y|^2$ for all $x, y \in X$. Then for all $x, y \in X$,

$$|x - z|^2 \leq 2(|x - y|^2 + |y - z|^2).$$

So (X, D, K) is a b -metric space with $K = 2$. However, we find that for n large enough,

$$\begin{aligned} 2D\left(0, \frac{1}{n}\right) + 4D\left(\frac{1}{n}, \frac{2}{n}\right) + \dots + 4D\left(\frac{n-2}{n}, \frac{n-1}{n}\right) + 2D\left(\frac{n-1}{n}, 1\right) &\leq \frac{4}{n} \\ &< 1 \\ &= D(0, 1). \end{aligned}$$

Then Theorem 1.4.(1) does not hold. We also find that for all n ,

$$d(0, 1) \leq D\left(0, \frac{1}{n}\right) + D\left(\frac{1}{n}, \frac{2}{n}\right) + \dots + D\left(\frac{n-2}{n}, \frac{n-1}{n}\right) + D\left(\frac{n-1}{n}, 1\right) \leq \frac{4}{n}.$$

Letting $n \rightarrow \infty$ yields $d(0, 1) = 0$. Then d is not a metric. So Theorem 1.4.(2) does not hold.

For the case D being a b -metric, Frink's metrization technique was revised in [1] and [31], see Theorem 1.7 and Theorem 1.8. Note that Frink reproved Chittenden's theorem in [10] by using the technique in the proof of Theorem 1.4, see [17, pages 134-135]. Then he used Chittenden's theorem to obtain the metrization of a space under conditions of Alexandroff and Urysohn, Niemytski and Wilson, and some others. We next present detailed proofs for these results, which will be useful in next sections. Notice that the condition corresponding to (C) in Corollary 2.4 originally given by Alexandroff and Urysohn implied that all sets of \mathcal{G}_n are open. Frink [17, page 136] called a collection of sets G_{n_1}, \dots, G_{n_k} a *chain* joining a and b provided $a \in G_{n_1}$, $b \in G_{n_k}$ and two successive sets of the chain have a common point. Then he defined

$$\begin{aligned} d(a, b) = \inf \left\{ \sum_{r=1}^k \frac{1}{2^{n_r}} : a \in G_{n_1}, b \in G_{n_k}, G_{n_r} \in \mathcal{G}_{n_r} \text{ for all } r = 1, \dots, k \right. \\ \left. \text{and } G_{n_r} \cap G_{n_{r+1}} \neq \emptyset \text{ for all } r = 1, \dots, k-1 \right\}. \end{aligned}$$

Frink asserted that d is a metric on X . This technique was used later to show that a space with a distance function satisfying Niemytski and Wilson's conditions is metrizable [17, page 137], and to prove some other results [17, Theorems 1, 2, 3 & 4]. However, the following example shows that the above d is not a metric. This implies that the Frink's argument in [17, page 136] is not suitable.

Example 2.2. Let $X = \mathbb{R}$ with the usual metric d and $\mathcal{G}_n = \{B(x, \frac{2}{n}, d) : x \in X\}$ for all n . Then \mathcal{G}_n 's satisfy all assumptions of Corollary 2.4. However, for $a = 0$ and $b = 1$, define $G_{n_r} = B(\frac{r-1}{n}, \frac{2}{n})$ for all $r = 1, \dots, n+1$, then $0 \in G_{n_1}$, $1 \in G_{n_{n+1}}$ and $G_{n_r} \cap G_{n_{r+1}} \neq \emptyset$ for all $r = 1, \dots, n$. Since $G_{n_r} \in \mathcal{G}_n$ for all $r = 1, \dots, n+1$, it follows that $d(0, 1) \leq \sum_{r=1}^{n+1} \frac{1}{2^{n_r}} = \frac{n+1}{2^n}$. Letting $n \rightarrow \infty$ yields $d(0, 1) = 0$. This implies that d is not a metric on X .

Corollary 2.3 (Chittenden's theorem). *Let (X, ρ) be a space satisfying (I), (II) and (V). Then (X, ρ) is metrizable.*

Proof. For any $\varepsilon > 0$, define $\psi(\varepsilon) = \min\{\phi(\varepsilon), \frac{\varepsilon}{2}\}$. Therefore, for all $x, y, z \in X$, if $\rho(x, y) < \psi(\varepsilon)$ and $\rho(y, z) < \psi(\varepsilon)$ then $\rho(x, z) < \varepsilon$. For each $n \in \mathbb{N}$, define $r_1 = 1, \dots, r_{n+1} = \psi(r_n), \dots$. Then $\lim_{n \rightarrow \infty} r_n = 0$. Define

$$D(x, y) = D(y, x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } \rho(x, y) \geq r_1 \\ \frac{1}{2^n} & \text{if } r_n > \rho(x, y) \geq r_{n+1}. \end{cases}$$

We claim that D satisfies (IV). On the contrary, suppose that there exist $\varepsilon > 0$ and $x, y, z \in X$ satisfying $D(x, y) < \varepsilon$, $D(y, z) < \varepsilon$ and $D(x, z) \geq 2\varepsilon$. Since $D(x, z) \leq 1$, it follows that $2\varepsilon \leq 1$, and so $\varepsilon \leq \frac{1}{2}$. Then there exists $n_0 \geq 1$ satisfying $\frac{1}{2^{n_0+1}} < \varepsilon \leq \frac{1}{2^{n_0}}$. This implies $D(x, y) < \frac{1}{2^{n_0}}$ and $D(y, z) < \frac{1}{2^{n_0}}$. Therefore $\rho(x, y) < r_{n_0+1} = \psi(r_{n_0})$ and $\rho(y, z) < r_{n_0+1} = \psi(r_{n_0})$. Then $\rho(x, z) < r_{n_0}$. This gives $D(x, z) \leq \frac{1}{2^{n_0}} < 2\varepsilon$, a contradiction. So D satisfies (IV). It is clear that D also satisfies (I) and (II). By Theorem 1.4, (X, D) is metrizable by the metric d defined as in (1.1).

We next prove that $\lim_{n \rightarrow \infty} x_n = x$ in (X, ρ) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) . Indeed, if $\lim_{n \rightarrow \infty} x_n = x$ in (X, ρ) then $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$. For each $\varepsilon > 0$, there exists n_0 such that $\frac{1}{2^{n_0}} < \varepsilon$. There also exists n_1 such that $\rho(x_n, x) < r_{n_0}$ for all $n \geq n_1$. Since $\rho(x_n, x) < r_{n_0}$, we have $D(x_n, x) \leq \frac{1}{2^{n_0}}$, and so $D(x_n, x) < \varepsilon$ for all $n \geq n_1$. This implies that $\lim_{n \rightarrow \infty} D(x_n, x) = 0$, and thus $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) .

Next, let $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) . Note that for each $\varepsilon > 0$ there exists n_0 such that $r_{n_0} < \varepsilon$. Since $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) , there exists n_1 such that $D(x_n, x) < \frac{1}{2^{n_0}}$ for all $n \geq n_1$. Therefore $\rho(x_n, x) \leq r_{n_0} < \varepsilon$ for all $n \geq n_1$. This implies that $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$, and so $\lim_{n \rightarrow \infty} x_n = x$ in (X, ρ) .

By the above, $\lim_{n \rightarrow \infty} x_n = x$ in (X, ρ) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) . Since (X, D) is metrizable by the metric d , we get that (X, ρ) is metrizable by the metric d . \square

Corollary 2.4 (Alexandroff and Urysohn). *Let X be a space and \mathcal{G}_n 's be families of subsets of X satisfying the following.*

- (A) *If $G_n, G'_n \in \mathcal{G}_n$ and $G_n \cap G'_n \neq \emptyset$ for some $n > 1$ then there exists $G_{n-1} \in \mathcal{G}_{n-1}$ such that $G_n \subset G_{n-1}$ and $G'_n \subset G_{n-1}$.*
- (B) *If $a \neq b$ then there exists n such that $\{a, b\} \not\subset G_n$ for all $G_n \in \mathcal{G}_n$.*
- (C) *If $S_n(x) = \bigcup \{G_n \in \mathcal{G}_n : x \in G_n\}$ then $\{S_n(x) : n \in \mathbb{N}\}$ forms a complete system of neighborhoods of the point x .*

Then X is metrizable.

Proof. Define a function $D : X \times X \rightarrow [0, \infty)$ as follows

$$D(a, b) = D(b, a) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \text{ and } \{a, b\} \not\subset G_n \text{ for all } n \\ \frac{1}{2^n} & \text{if } a \neq b \text{ and } n = \max\{k : \{a, b\} \subset G_k\}. \end{cases}$$

Then D satisfies (I), (II) and (IV). By Theorem 1.4, (X, D) is metrizable by the metric d defined by (1.1).

Next, we shall prove that $\lim_{n \rightarrow \infty} x_n = x$ in the topological space X if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) . Indeed, if $\lim_{n \rightarrow \infty} x_n = x$ in the topological space X then for each $\varepsilon > 0$ there exists n_0 such that $\frac{1}{2^{n_0}} < \varepsilon$. Since $\lim_{n \rightarrow \infty} x_n = x$ in the given topological space X , there exists n_1 such that $x_n \in S_{n_0}(x)$ for all $n \geq n_1$. So there exists $G_{n_0} \in \mathcal{G}_{n_0}$ such that $\{x_n, x\} \subset G_{n_0}$. Therefore $D(x_n, x) \leq \frac{1}{2^{n_0}} < \varepsilon$ for all $n \geq n_1$. This implies that $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) .

If $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) then $\lim_{n \rightarrow \infty} D(x_n, x) = 0$. For each $\varepsilon > 0$ there exists n_0 such that $\frac{1}{2^{n_0}} < \varepsilon$. There also exists n_1 such that $D(x_n, x) < \frac{1}{2^{n_0}}$ for all $n \geq n_1$. Since $D(x_n, x) < \frac{1}{2^{n_0}}$, it follows that $\{x_n, x\} \subset G_{n_0}$ for some $G_{n_0} \in \mathcal{G}_{n_0}$ and all $n \geq n_1$. Then $x_n \in S_{n_0}(x)$ for all $n \geq n_1$. Therefore $\lim_{n \rightarrow \infty} x_n = x$ in the topological space X .

By the above, $\lim_{n \rightarrow \infty} x_n = x$ in the topological space X if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) . Since (X, D) is metrizable, the topological space X is metrizable. \square

Now, we recall Niemytski and Wilson's conditions. Note that (VIa), (VIb) and (VIc) are equivalent [17, page 137], and they are all denoted by (VI).

VIa. The local axiom of the triangle: Given a point a and a number $\varepsilon > 0$, there exists a number $\phi(a, \varepsilon) > 0$ such that if $D(a, b) < \phi(a, \varepsilon)$ and $D(c, b) < \phi(a, \varepsilon)$ then $D(a, c) < \varepsilon$.

VIb. Coherent: If $\lim_{n \rightarrow \infty} D(a, a_n) = 0$ and $\lim_{n \rightarrow \infty} D(a_n, b_n) = 0$ then $\lim_{n \rightarrow \infty} D(a, b_n) = 0$.

VIc. Wilson's condition IV: For each point a and each $k > 0$, there is $r > 0$ such that if b is a point for which $D(a, b) \geq k$ and c is any point then $D(a, c) + D(b, c) \geq r$.

Corollary 2.5 (Niemytski and Wilson). *Let (X, ρ) be a space satisfying (I), (II) and (VI). Then (X, ρ) is metrizable.*

Proof. We may assume that (X, ρ) satisfy (VIa). For any $\varepsilon > 0$ define

$$\phi'(a, \varepsilon) = \min \left\{ \phi(a, \varepsilon), \frac{\varepsilon}{2} \right\} \text{ and } \psi(a, \varepsilon) = \phi'(a, \phi'(a, \varepsilon)).$$

For any $x \in X$ and all $n \in \mathbb{N}$, define $r_1(x) = 1$ and $r_{n+1}(x) = \psi(x, r_n(x))$. Then $\lim_{n \rightarrow \infty} r_n(x) = 0$. Define $V_n(x) = B(x, r_n(x), \rho)$ and $\mathcal{G}_n = \{V_n(x) : x \in X\}$. Then all assumptions of Corollary 2.4 are satisfied, and so (X, ρ) is metrizable by the metric d induced from the distance D as in the proof of Corollary 2.4. \square

3. APPLICATIONS TO b -METRIC SPACES

In this section, we show that the Banach contraction principle in b -metric spaces can be deduced from the Banach contraction principle in metric spaces. We also use the formula (1.5) to calculate corresponding metrics induced by some b -metrics known in the literature.

We find that every b -metric space (X, D, K) is metrizable with the metric d defined by (1.5). Note that, on transferring fixed point theorems in metric spaces to b -metric spaces, the contraction constants were assumed to be in $[0, \frac{1}{K}) \subset [0, 1)$, see for instance first fixed point theorems in b -metric spaces [6, Theorems 3.1-3.3] and recent results [29, Definition 2.1, Theorems 2.2 & 2.4]. By using the corresponding

metric of given b -metric defined by (1.5), we next show that the contraction constant $\lambda \in [0, \frac{1}{K}]$ in Theorem 1.10 can be relaxed to $\lambda \in [0, 1)$.

Theorem 3.1. *Let (X, D, K) be a complete b -metric space and $T : X \rightarrow X$ be a map such that $D(Tx, Ty) \leq \lambda D(x, y)$ for all $x, y \in X$ and some $\lambda \in [0, 1)$. Then T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.*

Proof. Let $p = \log_{2K} 2$. Then $0 < p \leq 1$ and $(2K)^p = 2$. So d defined by (1.5) is a metric on X . Moreover, $\frac{1}{4}D^p \leq d \leq D^p$. Since (X, D, K) is complete, (X, d) is a complete metric space. For all $x_1 = x, x_2, \dots, x_{n+1} = y \in X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} d(Tx, Ty) &= \inf \left\{ \sum_{i=1}^n D^p(y_i, y_{i+1}) : y_1 = Tx, y_2, \dots, y_{n+1} = Ty \in X, n \in \mathbb{N} \right\} \\ &\leq \sum_{i=1}^n D^p(Tx_i, Tx_{i+1}) \\ &\leq \lambda^p \sum_{i=1}^n D^p(x_i, x_{i+1}). \end{aligned}$$

This implies that

$$\begin{aligned} d(Tx, Ty) &\leq \lambda^p \inf \left\{ \sum_{i=1}^n D^p(x_i, x_{i+1}) : x_1 = x, x_2, \dots, x_{n+1} = y \in X, n \in \mathbb{N} \right\} \\ &= \lambda^p d(x, y). \end{aligned}$$

Since $\lambda^p \in [0, 1)$, T is a contraction map on a complete metric space (X, d) . By the Banach contraction map principle on metric spaces, T has a unique fixed point x^* and $\lim_{n \rightarrow \infty} T^n x = x^*$ in (X, d) . Note that $\frac{1}{4}D^p \leq d \leq D^p$. Then $\lim_{n \rightarrow \infty} T^n x = x^*$ in (X, D, K) . \square

Next, by using the formula (1.5), we calculate the corresponding metric d induced by certain b -metric D known in the literature. Two following b -metric spaces were usually used as “interesting ones” to prove the difference between the setting of b -metric and the setting of metric [24, page 113]. By using Theorem 1.7 and Theorem 1.8, we can get corresponding metrics induced by these b -metrics as follows.

Example 3.2. Let $0 < p \leq 1$, and

$$\ell^p = \left\{ \{x_n\} : x_n \in \mathbb{R}, n \in \mathbb{N}, \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

and $D(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}$ for all $x = \{x_n\}, y = \{y_n\} \in \ell^p$. Then D is a b -metric with the coefficient $K = 2^{\frac{1}{p}}$ [8, Example 1.3]. Define $q = \frac{p}{p+1}$. Then

$(2.2^{\frac{1}{p}})^q = 2$. By Theorem 1.8, (ℓ^p, D, K) is metrizable by the metric d defined by

$$d(x, y) = \inf \left\{ \sum_{i=1}^n D^q(x_i, x_{i+1}) : x_1 = x, x_2, \dots, x_{n+1} = y \in \ell^p, n \in \mathbb{N} \right\}.$$

We find that $D^q(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p+1}}$ for all $x, y \in \ell^p$. Then D^q is a metric on ℓ^p . By Theorem 1.7, $d = D^q$. Then the corresponding metric d is defined by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p+1}}$ for all $x, y \in \ell^p$.

Example 3.3. Let $0 < p \leq 1$, and

$$L^p[0, 1] = \left\{ x : [0, 1] \rightarrow \mathbb{R} : \int_0^1 |x(t)|^p dt < \infty \right\}$$

and $D(x, y) = \left(\int_0^1 |x(t) - y(t)|^p \right)^{\frac{1}{p}}$ for all $x, y \in L^p[0, 1]$. Then D is a b -metric with the coefficient $K = 2^{\frac{1}{p}}$ [8, Example 1.4].

By a similar argument as in Example 3.2, we get $(L^p[0, 1], D, K)$ is metrizable by the metric d defined by $d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p \right)^{\frac{1}{p+1}}$ for all $x, y \in L^p[0, 1]$.

Two following b -metric spaces play an important role in showing some different properties of b -metric spaces [3]. By using Theorem 1.8, we can also get corresponding metrics induced by these b -metrics as follows.

Example 3.4. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ and

$$D(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0, 1\} \\ |x - y| & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n \in \mathbb{N}\} \\ 4 & \text{otherwise.} \end{cases}$$

Then D is a b -metric on X . Note that, in [28, Example 13] and also in [3, Example 3.9], the coefficient $K = \frac{8}{3}$ but this fact is not true since for all n ,

$$4 = D\left(1, \frac{1}{2n}\right) \leq K \left[D(1, 0) + D\left(0, \frac{1}{2n}\right) \right] = K \left(1 + \frac{1}{2n}\right).$$

This implies $K \geq 4$. Reconsidering the calculation in [28, Example 13] we find that D is exactly a b -metric with $K = 4$. Define $p = \frac{1}{3}$. Then $(2K)^p = 2$. By using (1.5), we get the corresponding metric d defined by

$$d(x, y) = d(y, x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0, 1\} \\ |x - y|^{\frac{1}{3}} & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n \in \mathbb{N}\} \\ 1 + \sqrt[3]{\frac{1}{2n}} & \text{if } x = 1, y \in \{\frac{1}{2n} : n \in \mathbb{N}\} \\ \sqrt[3]{4} & \text{otherwise.} \end{cases}$$

Example 3.5. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ and

$$D(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0, 1\} \\ |x - y| & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n \in \mathbb{N}\} \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Then D is a b -metric on X with $K = 4$ [3, Example 3.10]. By using (1.5), we get the corresponding metric d defined by

$$d(x, y) = d(y, x) = \begin{cases} 0 & \text{if } x = y \\ \sqrt[3]{\frac{1}{4}} & \text{if } x \neq y \in \{0, 1\} \\ |x - y|^{\frac{1}{3}} & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n \in \mathbb{N}\} \\ \sqrt[3]{\frac{1}{4}} & \text{otherwise.} \end{cases}$$

Next, we calculate the corresponding metric induced by the b -metric in Example 2.1.

Example 3.6. Let $X = \mathbb{R}$, and $D(x, y) = |x - y|^2$ for all $x, y \in X$ as in Example 2.1. Then D is a b -metric with the coefficient $K = 2$. Define $p = \frac{1}{2}$. Then $(2K)^p = 2$. It follows from (1.5) that for any $x, y \in \mathbb{R}$,

$$\begin{aligned} d(x, y) &= \inf \left\{ \sum_{i=1}^n D^{\frac{1}{2}}(x_i, x_{i+1}) : x_1 = x, x_2, \dots, x_{n+1} = y \in X, n \in \mathbb{N} \right\} \\ &= \inf \left\{ \sum_{i=1}^n |x_i - x_{i+1}| : x_1 = x, x_2, \dots, x_{n+1} = y \in X, n \in \mathbb{N} \right\} \\ &= |x - y|. \end{aligned}$$

Then d is again the usual metric in \mathbb{R} .

Remark 3.7. From above examples, authors should be very carefully to work with fixed point theorems in b -metric spaces. Note that ℓ^p , $0 < p < 1$, with the quasi-norm defined by $\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$ for all $x = \{x_n\} \in \ell^p$ is a quasi-Banach space that is not normable [19, page 1102]. The similar result also holds for $L^p[0, 1]$. So authors may study the fixed point theory in quasi-Banach spaces. For interesting ways to extend fixed point theory in quasi-Banach spaces, the reader may refer to and use the ideas in [33], [34] and references given there.

4. APPLICATIONS TO ANSWERING BERINDE-CHOBAN'S QUESTIONS

In this section, we give answers to Question 1.1 and Question 1.2 mentioned in Section 1. First, by using the technique in the proof of Corollary 2.3, we give an affirmative answer to Question 1.1 as follows.

Theorem 4.1. *Let (X, σ) be an F -distance space and $T : X \rightarrow X$ be a map such that $\sigma(Tx, Ty) \leq \lambda \sigma(x, y)$ for some $\lambda \in [0, 1)$ and all $x, y \in X$. Then*

- (1) The sequence $\{x_n\}$ is Cauchy, where $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ and some $x_1 \in X$.
- (2) There exists a unique fixed point of T if the space (X, σ) is complete.

Proof. (1). For all $x, y \in X$, put $\rho(x, y) = \max\{\sigma(x, y), \sigma(y, x)\}$. Then (X, ρ) is a space satisfying (I), (II) and (V) and ρ is equivalent to σ . By Corollary 2.3, (X, ρ) is metrizable and so is (X, σ) . We also find that for all $x, y \in X$,

$$\rho(Tx, Ty) = \max\{\sigma(Tx, Ty), \sigma(Ty, Tx)\} \leq \max\{\lambda\sigma(x, y), \lambda\sigma(y, x)\} = \lambda\rho(x, y).$$

Now, for all $n \in \mathbb{N}$, we have

$$\rho(x_{n+1}, x_n) = \rho(Tx_n, Tx_{n-1}) \leq \lambda\rho(x_n, x_{n-1}) \leq \dots \leq \lambda^{n-1}\rho(x_2, x_1).$$

This implies $\lim_{n \rightarrow \infty} \rho(x_{n+1}, x_n) = 0$. So there exists n_0 such that $\rho(x_{n+1}, x_n) < 1$ for all $n \geq n_0$. By using notations d and D in the proof of Corollary 2.3 again, we find that for each $n \geq n_0$ there exists k_n such that $r_{k_n} > \rho(x_{n+1}, x_n) \geq r_{k_n+1}$. This implies $D(x_{n+1}, x_n) \leq \frac{1}{2^{k_n}}$. By using (1.3) we have $d(x_{n+1}, x_n) \leq D(x_{n+1}, x_n) \leq \frac{1}{2^{k_n}}$. So, for $m \geq n \geq n_0$,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \dots + d(x_{m-1}, x_m) \leq \frac{1}{2^{k_n}} + \dots + \frac{1}{2^{k_{m-1}}} \leq \sum_{i=k_n}^{\infty} \frac{1}{2^i}.$$

This implies $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$. Then $\{x_n\}$ is Cauchy in (X, d) . By Theorem 1.4.(3), $\{x_n\}$ is Cauchy in (X, D) . Now, for each $\varepsilon > 0$, there exists n_0 such that $r_{n_0} \leq \varepsilon$. Since $\{x_n\}$ is Cauchy in (X, D) , there exists n_1 such that $D(x_n, x_m) < \frac{1}{2^{n_0}}$ for all $n, m \geq n_1$. Therefore $\rho(x_n, x_m) \leq r_{n_0} < \varepsilon$ for all $n, m \geq n_1$. This implies that $\{x_n\}$ is Cauchy in (X, ρ) . Since ρ is equivalent to σ , we get that $\{x_n\}$ is Cauchy in (X, σ) .

(2). If (X, σ) is complete then there exists x^* such that $\lim_{n \rightarrow \infty} x_n = x^*$ in (X, σ) . Therefore $\lim_{n \rightarrow \infty} \sigma(x_n, x^*) = 0$. Note that for all n ,

$$\sigma(Tx^*, x_{n+1}) = \sigma(Tx^*, Tx_n) \leq \lambda\sigma(x^*, x_n).$$

Letting $n \rightarrow \infty$ yields $\lim_{n \rightarrow \infty} \sigma(Tx^*, x_{n+1}) = 0$. Then $\lim_{n \rightarrow \infty} x_{n+1} = Tx^*$ in (X, σ) . Since (X, σ) is metrizable by d , the limit of a convergent sequence in (X, σ) is unique. This implies that $Tx^* = x^*$ and then T has a fixed point. It is easy to see that the fixed point of T is unique. \square

Recall that a *symmetric distance* ρ on a topological space X is a function $\rho : X \times X \rightarrow [0, \infty)$ satisfying (I), (II) and $A = \bar{A}$ if and only if $\rho(x, A) > 0$ for any $x \notin A$, where \bar{A} is the closure of A and $\rho(x, A) = \inf\{\rho(x, y) : y \in A\}$ [4, page 125]. On Question 1.2 Berinde and Choban asserted that any Hausdorff compact space with a symmetric distance is metrizable [5, page 29], also see the details proof at [4, pages 126-127]. The following example shows that there exists a Hausdorff compact space with a symmetric distance that is not coherent.

Example 4.2. There exists a Hausdorff compact space with a symmetric distance that is not coherent.

Proof. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and

$$\rho(x, y) = \rho(y, x) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2n} & \text{if } (x, y) = (0, \frac{1}{2n}) \\ \frac{1}{2n+1} & \text{if } (x, y) = (1, \frac{1}{2n+1}) \\ 1 & \text{if } (x, y) = (0, \frac{1}{2n-1}) \text{ or } (x, y) = (1, \frac{1}{2n}) \\ |x - y| & \text{otherwise.} \end{cases}$$

Then (X, ρ) is a Hausdorff compact space with the symmetric distance ρ , where the topology on X is induced by its convergence with respect to ρ . We find that

$$\lim_{n \rightarrow \infty} \rho\left(\frac{1}{2n}, \frac{1}{2n+1}\right) = \lim_{n \rightarrow \infty} \left| \frac{1}{2n} - \frac{1}{2n+1} \right| = 0$$

and $\lim_{n \rightarrow \infty} \rho\left(\frac{1}{2n}, 0\right) = 0$.

However, $\lim_{n \rightarrow \infty} \rho\left(\frac{1}{2n+1}, 0\right) = 1 \neq 0$. So (X, ρ) is not coherent. \square

The following theorem is a partial answer to Question 1.2.

Theorem 4.3. *Let (X, ρ) be a symmetric distance space and $T : X \rightarrow X$ be a map such that $(X, \mathcal{T}(\rho))$ is Hausdorff compact, ρ is coherent, and $\rho(Tx, Ty) \leq \lambda\rho(x, y)$ for some $\lambda \in [0, 1)$ and all $x, y \in X$. Then T has a unique fixed point.*

Proof. For each $x \in X$, since $(X, \mathcal{T}(\rho))$ is sequentially compact, there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} T^{k_n}x = x^*$ in $(X, \mathcal{T}(\rho))$. Then $\lim_{n \rightarrow \infty} T^{k_n}x = x^*$ in (X, ρ) and thus

$$\lim_{n \rightarrow \infty} \rho(T^{k_n}x, x^*) = 0. \quad (4.1)$$

We find that $\rho(TT^{k_n}x, Tx^*) \leq \lambda\rho(T^{k_n}x, x^*)$ for all n . Letting $n \rightarrow \infty$ and using (4.1) we obtain

$$\lim_{n \rightarrow \infty} \rho(TT^{k_n}x, Tx^*) = 0. \quad (4.2)$$

We also find that for all n ,

$$\rho(T^{n+1}x, T^n x) \leq \lambda\rho(T^n x, T^{n-1}x) \leq \dots \leq \lambda^n \rho(Tx, x).$$

This implies $\lim_{n \rightarrow \infty} \rho(TT^n x, T^n x) = 0$. Therefore

$$\lim_{n \rightarrow \infty} \rho(TT^{k_n}x, T^{k_n}x) = 0. \quad (4.3)$$

By (4.2) and (4.3), since ρ is coherent, we get

$$\lim_{n \rightarrow \infty} \rho(T^{k_n}x, Tx^*) = 0. \quad (4.4)$$

From (4.1) and (4.4), since (X, ρ) is Hausdorff, we get $Tx^* = x^*$. So T has a fixed point. It is easy to see that the fixed point of T is unique. \square

One may conjecture that Theorem 4.3 holds without the condition that ρ being coherent. However, the following question, that is inspired by Nemytzki-Edelstein theorem in metric spaces and also by [5, Conjecture 6.2], is still open.

Question 4.4. Let (X, ρ) be a symmetric distance space, $(X, \mathcal{T}(\rho))$ be Hausdorff compact, and $T : X \rightarrow X$ be a map such that $\rho(Tx, Ty) < \rho(x, y)$ for all distinct $x, y \in X$. Does there exist a unique fixed point of T ?

5. APPLICATIONS TO 2-GENERALIZED METRIC SPACES

In this section, by using the idea in the proof of [17, Theorem 2], we prove a metrization theorem for 2-generalized metric spaces. Here the main difference is that assumptions of [17, Theorem 2] hold for all elements while the assumptions relating to 2-generalized metric spaces in our result hold only for distinct elements. We also show that the Banach contraction principle in 2-generalized metric spaces can be deduced from the Banach contraction principle in metric spaces.

We first show that there exists a 2-generalized metric space that is not metrizable.

Example 5.1. Let (X, ρ) be the 2-generalized metric space in [27, Example 2.13]. Then there exists a convergent sequence having two limits. This implies that (X, ρ) is not metrizable.

Recently Suzuki [36, Example 7] constructed an example of a 2-generalized metric space (X, ρ) that does not have any topology being compatible with ρ . Therefore, that 2-generalized metric space (X, ρ) is also not metrizable in the sense that the induced metric and given 2-generalized metric having the same convergence of nets. Suzuki *et al.* [38] proved that every 3-generalized metric space is metrizable and for any $\nu \geq 4$, and not every ν -generalized metric space has a compatible symmetric topology. Note that if the 2-generalized metric ρ is continuous in its variables then the 2-generalized metric space (X, ρ) is metrizable [27, Corollary 2.6.(1)]. The following example shows that there exists a 2-generalized metric space (X, ρ) that is metrizable but ρ is not continuous in its variables.

Example 5.2. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ and

$$\rho(x, y) = \rho(y, x) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x = 0, y = \frac{1}{n} \\ 2 & \text{otherwise.} \end{cases}$$

We will show that ρ is a 2-generalized metric. For all $x, y \in X$ it is clear that $\rho(x, y) \geq 0$, $\rho(x, y) = \rho(y, x)$; and $\rho(x, y) = 0$ if and only if $x = y$. For all $x, y \in X$ and $u \neq v \in X \setminus \{x, y\}$ we consider the following three cases.

Case 1. $x = y$. Then $\rho(x, u) + \rho(u, v) + \rho(v, y) \geq 0 = \rho(x, y)$.

Case 2. $x = 0$ and $y = \frac{1}{n}$. Then $v \neq 0$ and thus

$$\rho(x, u) + \rho(u, v) + \rho(v, y) \geq \rho(v, y) = 2 \geq \frac{1}{n} = \rho(x, y).$$

Case 3. $x = \frac{1}{n} \neq y = \frac{1}{m}$. Then $u \neq 0$ or $v \neq 0$. This implies that

$$\rho(x, u) + \rho(u, v) + \rho(v, y) \geq 2 = \rho(x, y).$$

By the above, ρ is a 2-generalized metric on X . We find that $\lim_{n \rightarrow \infty} \rho(\frac{1}{n}, 0) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ in \mathbb{R} . Then $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ in (X, ρ) . However, $\lim_{n \rightarrow \infty} \rho(\frac{1}{n}, 1) = 2 \neq 1 = \rho(0, 1)$ and thus ρ is not continuous in its variables.

Since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ in (X, ρ) and each point $\frac{1}{n}$ is isolated in (X, ρ) , we find that (X, ρ) is metrizable by the usual metric d on X .

We next give a condition for the metrization of a 2-generalized metric space.

Theorem 5.3. *Let (X, ρ) be a 2-generalized metric space such that the limit of a convergent sequence is unique. Then*

- (1) *There exists a metric d on X such that $\lim_{n \rightarrow \infty} x_n = x$ in (X, ρ) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, d) . In particular, (X, ρ) is metrizable by the metric d .*
- (2) *A sequence $\{x_n\}$ is Cauchy in (X, ρ) if and only if it is Cauchy in (X, d) . In particular, (X, ρ) is complete if and only if (X, d) is complete.*

Proof. (1). For any $a \in X$ and $n \geq 0$, define $U_n(a) = \{x \in X : \rho(a, x) < \frac{1}{3^n}\}$. Let $a, b \in X$. If for each $n \geq 0$ there exists $y_n \in X$ such that $\{a, b\} \subset U_n(y_n)$, then $\rho(y_n, a) < \frac{1}{3^n}$ and $\rho(y_n, b) < \frac{1}{3^n}$ for all $n \geq 0$. Letting $n \rightarrow \infty$ we find that $\lim_{n \rightarrow \infty} \rho(y, a) = \lim_{n \rightarrow \infty} \rho(y, b) = 0$. This implies $\lim_{n \rightarrow \infty} y_n = a$ and $\lim_{n \rightarrow \infty} y_n = b$, and thus $a = b$. So, for $a \neq b$, there exists n such that $\{a, b\} \not\subset U_n(y)$ for all $y \in X$. Moreover, if $n \leq m$ then $U_n(y) \supset U_m(y)$ for all $y \in X$. So we can define a function $D : X \times X \rightarrow [0, \infty)$ as follows

$$D(a, b) = \begin{cases} 0 & \text{if } a = b \\ \frac{1}{2^k} & \text{if } a \neq b, k = \min \{n : \{a, b\} \not\subset U_n(y) \text{ for all } y \in X, m \geq n\}. \end{cases}$$

It is clear that D satisfies (I) and (II). We shall prove that D satisfies (IV). By Remark 1.6, it is sufficient to show that for each $\varepsilon > 0$ and all distinct elements a, b, c , if $D(a, b) \leq \varepsilon$ and $D(b, c) \leq \varepsilon$ then $D(a, c) \leq 2\varepsilon$.

Indeed, if $\varepsilon > \frac{1}{2}$ then $D(a, c) \leq 1 < 2\varepsilon$. If $\varepsilon \leq \frac{1}{2}$ then there exists $n \in \mathbb{N}$ such that $D(a, b) \leq \frac{1}{2^n} \leq \varepsilon$ and $D(b, c) \leq \frac{1}{2^n} \leq \varepsilon$. Then there exist $x, y \in X$ such that $\{a, b\} \subset U_n(x)$ and $\{b, c\} \subset U_n(y)$. If $x = y$ then $\{a, c\} \subset U_n(x)$. This implies $D(a, c) \leq \frac{1}{2^n} \leq \varepsilon \leq 2\varepsilon$. If $x \neq y$ and $a = y$ or $c = x$ then $\{a, c\} \subset U_n(a)$. This implies $D(a, c) \leq \frac{1}{2^n} \leq \varepsilon < 2\varepsilon$.

If $x \neq y$ and $a \neq y, c \neq x$, then we consider the following four cases.

Case 1. $a = x$ and $c = y$. Then $b \in U_n(a)$ and $b \in U_n(c)$. This implies $\{a, c\} \subset U_n(b)$. So $D(a, c) \leq \frac{1}{2^n} \leq \varepsilon < 2\varepsilon$.

Case 2. $a = x$ and $c \neq y$. If $b \neq y$ then

$$\rho(a, c) \leq \rho(a, b) + \rho(b, y) + \rho(y, c) < \frac{1}{3^n} + \frac{1}{3^n} + \frac{1}{3^n} = \frac{3}{3^n} = \frac{1}{3^{n-1}}.$$

So $\{a, c\} \subset U_{n-1}(c)$. This implies $D(a, c) \leq \frac{1}{2^{n-1}} \leq 2\varepsilon$. If $b = y$ then $\{a, c\} \subset U_n(b)$. This implies $D(a, c) \leq \frac{1}{2^n} \leq \varepsilon < 2\varepsilon$.

Case 3. $a \neq x$ and $c = y$. If $b \neq x$ then

$$\rho(a, c) \leq \rho(a, x) + \rho(x, b) + \rho(b, c) < \frac{1}{3^n} + \frac{1}{3^n} + \frac{1}{3^n} = \frac{3}{3^n} = \frac{1}{3^{n-1}}.$$

So $\{a, c\} \subset U_{n-1}(c)$. This implies $D(a, c) \leq \frac{1}{2^{n-1}} \leq 2\varepsilon$. If $b = x$ then $\{a, c\} \subset U_n(b)$. This implies $D(a, c) \leq \frac{1}{2^n} \leq \varepsilon < 2\varepsilon$.

Case 4. $a \neq x$ and $c \neq y$. If $b = x$ then

$$\rho(a, c) \leq \rho(a, b) + \rho(b, y) + \rho(y, c) < \frac{1}{3^n} + \frac{1}{3^n} + \frac{1}{3^n} = \frac{3}{3^n} = \frac{1}{3^{n-1}}.$$

So $\{a, c\} \subset U_{n-1}(c)$. This implies $D(a, c) \leq \frac{1}{2^{n-1}} \leq 2\varepsilon$. If $b = y$ then

$$\rho(a, c) \leq \rho(a, x) + \rho(x, b) + \rho(b, c) < \frac{1}{3^n} + \frac{1}{3^n} + \frac{1}{3^n} = \frac{3}{3^n} = \frac{1}{3^{n-1}}.$$

So $\{a, c\} \subset U_{n-1}(c)$. This implies $D(a, c) \leq \frac{1}{2^{n-1}} \leq 2\varepsilon$. If $b \neq x$ and $b \neq y$ then a, b, c, x, y are distinct. So

$$\rho(a, y) \leq \rho(a, x) + \rho(x, b) + \rho(b, y) < \frac{1}{3^n} + \frac{1}{3^n} + \frac{1}{3^n} = \frac{3}{3^n} = \frac{1}{3^{n-1}}.$$

Therefore $\rho(a, y) < \frac{1}{3^{n-1}}$. Note that $\rho(c, y) < \frac{1}{3^n} < \frac{1}{3^{n-1}}$. This implies $\{a, c\} \subset U_{n-1}(y)$. Then $D(a, c) \leq \frac{1}{2^{n-1}} \leq 2\varepsilon$.

By the above four cases, we get that D satisfies (IV). So D satisfies (I), (II) and (IV). By Theorem 1.4, there exists a metric d on X such that $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, d) . We check at once that $\lim_{n \rightarrow \infty} x_n = x$ in (X, ρ) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) . Therefore $\lim_{n \rightarrow \infty} x_n = x$ in (X, D) if and only if $\lim_{n \rightarrow \infty} x_n = x$ in (X, d) . In particular, (X, ρ) is metrizable by the metric d which is defined by (1.1).

(2). We will check that $\{x_n\}$ is Cauchy in (X, ρ) if and only if $\{x_n\}$ is Cauchy in (X, D) . Indeed, if $\{x_n\}$ is Cauchy in (X, ρ) , then $\lim_{n, m \rightarrow \infty} \rho(x_n, x_m) = 0$. For

each $\varepsilon > 0$, there exists n_0 such that $\frac{1}{2^{n_0}} < \varepsilon$. There also exists n_1 such that

$\rho(x_n, x_m) < \frac{1}{3^{n_0}}$ for all $n, m \geq n_1$. Since $\rho(x_n, x_m) < \frac{1}{3^{n_0}}$, $\{x_n, x_m\} \subset U_{n_0}(x_m)$. So

$D(x_n, x_m) \leq \frac{1}{2^{n_0}} < \varepsilon$ for all $n, m \geq n_1$. Therefore $\lim_{n \rightarrow \infty} D(x_n, x_m) = 0$. This implies that $\{x_n\}$ is Cauchy in (X, D) .

Next, if $\{x_n\}$ is Cauchy in (X, D) , then $\lim_{n, m \rightarrow \infty} D(x_n, x_m) = 0$. For each $\varepsilon > 0$,

there exists n_0 such that $\frac{1}{3^{n_0}} < \varepsilon$. There also exists n_1 such that $D(x_n, x_m) \leq \frac{1}{2^{n_0}}$ for all $n, m \geq n_1$. So $\{x_n, x_m\} \subset U_{n_0}(x_m)$ for all $n, m \geq n_1$. Therefore $\rho(x_n, x_m) \leq \frac{1}{3^{n_0}} < \varepsilon$ for all $n, m \geq n_1$. Thus $\lim_{n \rightarrow \infty} \rho(x_n, x_m) = 0$. This implies that $\{x_n\}$ is Cauchy in (X, ρ) .

By the above, $\{x_n\}$ is Cauchy in (X, ρ) if and only if $\{x_n\}$ is Cauchy in (X, D) . By Theorem 1.4.(3), $\{x_n\}$ is Cauchy in (X, D) if and only if $\{x_n\}$ is Cauchy in (X, d) . So $\{x_n\}$ is Cauchy in (X, ρ) if and only if $\{x_n\}$ is Cauchy in (X, d) . By (1) we get that (X, ρ) is complete if and only if (X, d) is complete. \square

Corollary 5.4. *Let (X, ρ) be a 2-generalized metric space. Then*

(1) *If ρ is continuous in its variables then (X, ρ) is metrizable.*

(2) (X, ρ) is metrizable if and only if the limits of a convergent sequence in (X, ρ) is unique.

Proof. (2) is a direct consequence of Theorem 5.3. We only need to prove (1). Since ρ is continuous in its variables, the limit of a convergent sequence in (X, ρ) is unique. By Theorem 5.3.(1), (X, ρ) is metrizable. \square

By using the metrization technique of a 2-generalized metric space presented in the proof of Theorem 5.3 we reprove the Banach contraction principle on 2-generalized metric spaces as follows. Note that Suzuki *et al.* [37] also studied Banach contraction principle and some other fixed point results in ν -generalized metric spaces. Moreover, the Hausdorff property of a 2-generalized metric space was used in the proof of [9, Theorem 2.1] though it is a confusion, see also [27, Remark 2.12].

Theorem 5.5 ([9], Theorem 2.1). *Let (X, ρ) be a Hausdorff complete 2-generalized metric space and $T : X \rightarrow X$ be a map such that $\rho(Tx, Ty) \leq \lambda\rho(x, y)$ for all $x, y \in X$ and some $\lambda \in [0, 1)$. Then T has a unique fixed point x^* , and $\lim_{n \rightarrow \infty} T^n x = x^*$ for all $x \in X$.*

Proof. Let $x = x_0 \in X$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. We find that

$$\rho(x_{n+1}, x_n) = \rho(Tx_n, Tx_{n-1}) \leq \lambda\rho(x_n, x_{n-1}) \leq \dots \leq \lambda^n \rho(x_1, x_0).$$

This implies $\lim_{n \rightarrow \infty} \rho(x_{n+1}, x_n) = 0$. So there exists n_0 such that $\rho(x_{n+1}, x_n) \leq \frac{1}{3}$ for all $n \geq n_0$. By using again notations in the proof of Theorem 5.3, we find that, for each $n \geq n_0$, there exist k_n and $a \in X$ such that $\{x_{n+1}, x_n\} \subset U_{k_n}(a)$. This implies $D(x_{n+1}, x_n) \leq \frac{1}{2^{k_n}}$. By (1.3) we have $d(x_{n+1}, x_n) \leq D(x_{n+1}, x_n) \leq \frac{1}{2^{k_n}}$. So, for $m \geq n \geq n_0$,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \dots + d(x_{m-1}, x_m) \leq \frac{1}{2^{k_n}} + \dots + \frac{1}{2^{k_{m-1}}} \leq \sum_{i=k_n}^{\infty} \frac{1}{2^i}.$$

This implies $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$. Then $\{x_n\}$ is Cauchy in (X, d) . By Theorem 5.3.(2), $\{x_n\}$ is Cauchy in (X, ρ) . Since (X, ρ) is complete, there exists x^* such that $\lim_{n \rightarrow \infty} x_n = x^*$ in (X, ρ) . Note that, for all n ,

$$\rho(Tx^*, x_{n+1}) = \rho(Tx^*, Tx_n) \leq \lambda\rho(x^*, x_n).$$

This implies $\lim_{n \rightarrow \infty} \rho(Tx^*, x_{n+1}) = 0$. So $\lim_{n \rightarrow \infty} x_{n+1} = Tx^*$ in (X, ρ) . By Theorem 5.3.(1), (X, ρ) is metrizable, so the limit of a convergent sequence in (X, ρ) is unique. Then $Tx^* = x^*$, that is, T has a fixed point. It is easy to see that the fixed point of T is unique. \square

Finally, by using the technique used in the proof of Theorem 5.3, we calculate the corresponding metric d induced by the first 2-generalized metric space [9, 3. Example] as follows.

Example 5.6. Let $X = \{a, b, c, e\}$ and

$$\rho(x, y) = \rho(y, x) = \begin{cases} 0 & \text{if } x = y \\ 3 & \text{if } (x, y) = (a, b) \\ 1 & \text{if } (x, y) \in \{(a, c), (b, c)\} \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, ρ) is a 2-generalized metric space and ρ is not a metric [9, 3. Example]. By using again notations in the proof of Theorem 5.3 we find that $x \notin U_0(y)$ for all $x \neq y$. Therefore

$$D(x, y) = D(y, x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Note that D is a metric on X . So $d = D$.

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REFERENCES

- [1] H. Aimar, B. Iaffei, and L. Nitti, *On the Macías-Segovia metrization of quasi-metric spaces*, Rev. Un. Mat. Argentina, **41**(1998), 67-75.
- [2] T.V. An, N.V. Dung, Z. Kadelburg, Stojan Radenović, *Various generalizations of metric spaces and fixed point theorems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **109**(2015), 175-198.
- [3] T.V. An, L.Q. Tuyen, N.V. Dung, *Stone-type theorem on b-metric spaces and applications*, Topology Appl., **185-186**(2015), 50-64.
- [4] A.V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys, **21**(1966), 115-162.
- [5] V. Berinde, M. Choban, *Generalized distances and their associate metrics. Impact on fixed point theory*, Creat. Math. Inform., **22**(2013), no. 1, 23-32.
- [6] M. Boriceanu, *Fixed point theory for multivalued contractions on a set with two b-metrics*, Creat. Math. Inform., **17**(2008), no. 3, 326-332.
- [7] M. Boriceanu, M. Bota, Adrian Petruşel, *Multivalued fractals in b-metric spaces*, Central Eur. J. Math., **8**(2010), no. 2, 367-377.
- [8] M. Bota, A. Molnár, and C. Varga, *On Ekeland's variational principle in b-metric spaces*, Fixed Point Theory, **2011**(2011), 21-28.
- [9] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57**(2000), no. 1-2, 31-37.
- [10] E. W. Chittenden, *On the equivalence of ecart and voisinage*, Trans. Amer. Math. Soc., **18**(1917), no. 2, 161-166.
- [11] E.W. Chittenden, *On the metrization problem and related problems in the theory of abstract sets*, Bull. Amer. Math. Soc., **33**(1927), 13-34.
- [12] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Univ. Ostrav., **1**(1993), no. 1, 5-11.
- [13] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Math. Fis. Univ. Modena, **46**(1998), 263-276.
- [14] P. Das, L.K. Dey, *Fixed point of contractive mappings in generalized metric spaces*, Math. Slovaca, **59**(2009), no. 4, 499-504.

- [15] R. Fagin, R. Kumar, D. Sivakumar, *Comparing top k lists*, SIAM J. Discrete Math., **17**(2003), no. 1, 134-160.
- [16] S.P. Franklin, *Spaces in which sequences suffice*, Fund. Math., **57**(1965), 107-115.
- [17] A.H. Frink, *Distance functions and the metrization problem*, Bull. Amer. Math. Soc., **43**(1937), no. 2, 133-142.
- [18] M. Jovanović, Z. Kadelburg, S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl., **2010**(2010), 1-15.
- [19] N. Kalton, *Quasi-Banach spaces*, Handbook of the geometry of Banach spaces 2 (W. B. Johnson and J. Lindenstrauss - eds.), Elsevier, 2003, 1099-1130.
- [20] M.A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory Appl., **2010**(2010), 1-7.
- [21] M.A. Khamsi, *Generalized metric spaces: A survey*, J. Fixed Point Theory Appl., **17**(2015), no. 3, 455-475.
- [22] M.A. Khamsi, N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., **7**(2010), no. 9, 3123-3129.
- [23] L. Kikina, K. Kikina, *On fixed point of a Ljubomir Ćirić quasi-contraction mapping in generalized metric spaces*, Publ. Math. Debrecen, **83**(2013), no. 3, 1-6.
- [24] W. Kirk, N. Shahzad, *Fixed point theory in distance spaces*, Springer, Cham, 2014.
- [25] W.A. Kirk, N. Shahzad, *Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl., **2013:129** (2013), 1-9.
- [26] W.A. Kirk, N. Shahzad, *Correction: Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl., **2014**(2014), 1-3.
- [27] P. Kumam, N.V. Dung, *Remarks on generalized metric spaces in the Branciari's sense*, Sarajevo J. Math., **10**(2014), no. 2, 209-219.
- [28] P. Kumam, N.V. Dung, V.T.L. Hang, *Some equivalences between cone b-metric spaces and b-metric spaces*, Abstr. Appl. Anal., **2013**(2013), 1-8.
- [29] P. Lo'lo', S. M. Vaezpour, J. Esmaily, *Common best proximity points theorem for four mappings in metric-type spaces*, Fixed Point Theory Appl., **2015:47**(2015), 1-7.
- [30] R.A. Macias, C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math., **33**(1979), no. 3, 257-270.
- [31] M. Paluszynski, K. Stempak, *On quasi-metric and metric spaces*, Proc. Amer. Math. Soc., **137** (2009), no. 12, 4307-4312.
- [32] A. Petrusel, I.A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134**(2006), no. 2, 411-418.
- [33] A.B. Pilarska, T.D. Benavides, *The fixed point property for some generalized nonexpansive mappings and renormings*, J. Math. Anal. Appl., **429**(2015), no. 2, 800-813.
- [34] S. Reich, A. J. Zaslavski, *Genericity and porosity in fixed point theory: a survey of recent results*, Fixed Point Theory Appl., **2015:195**(2015), 1-21.
- [35] V. Schroeder, *Quasi-metric and metric spaces*, Conform. Geom. Dyn., **10**(2006), 355-360.
- [36] T. Suzuki, *Generalized metric spaces do not have the compatible topology*, Abstr. Appl. Anal., **2014**(2014), 1-5.
- [37] T. Suzuki, B. Alamri, L.A. Khan, *Some notes on fixed point theorems in ν -generalized metric spaces*, Bull. Kyushu Inst. Tech. Pure Appl. Math., **62**(2015), 15-23.
- [38] T. Suzuki, B. Alamri, M. Kikkawa, *Only 3-generalized metric spaces have a compatible symmetric topology*, Open Math., **13**(2015), no. 1, 510-517.
- [39] Q. Xia, *The geodesic problem in quasimetric spaces*, J. Geom. Anal., **19**(2009), 452-479.

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