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STATIONARY POINTS OF SET-VALUED CONTRACTIVE AND NONEXPANSIVE MAPPINGS ON ULTRAMETRIC SPACES

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Abstract. In this paper we show that contractive set-valued mappings on spherically complete ultrametric spaces have stationary (or end) points if they have the approximate stationary point property. We also extend some known fixed point results to nonexpansive set-valued mappings.
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1. Introduction

An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has nonempty intersection. Let $CB(X)$ be the set of all closed and bounded subsets of X, $T : X \to CB(X)$ be a set-valued mapping, and $x, y \in X$ be two distinct points. If

$$H(Tx, Ty) < d(x, y),$$

then T is called contractive, and if

$$H(Tx, Ty) \leq d(x, y),$$

then T is called nonexpansive; here $H(\cdot, \cdot)$ is Hausdorff distance. A point $x \in X$ is said to be a fixed point of T if $x \in Tx$ and a stationary point (also known as end point or strict fixed point, see e.g. [6, 8]) of T if $Tx = \{x\}$. For any $x \in X$ and any nonempty subset A of X, the radius of A relative to x is defined as

$$r_x(A) = \sup_{y \in A} d(x, y).$$

The set-valued mapping T is said to have the approximate stationary point property if

$$\inf_{x \in X} r_x(Tx) = 0,$$

and to have the strong approximate stationary point property if for each ball of the form $B = B(x, r_x(Tx))$,

$$\inf_{z \in B} r_z(Tz) = 0.$$

Some papers in the literature discuss the existence and uniqueness of endpoints of set-valued mappings in metric and uniform spaces (see [1], [10], [9] and the references therein). In [1] Amini extended the Boyd-Wong contraction [2] to end-points of set-valued mappings. It is known that every contractive mapping in a spherically complete ultrametric space has a fixed point [5]. In this paper we show that a contractive mapping on an ultrametric space has a stationary point if and only if it has the approximate stationary point property. As a result we recover Amini's and Petalas and Vidalis's results (resp. [1] and [5]). We also extend some known result, due to Petalas and Vidalis [5] to set-valued nonexpansive mappings. We show that a set-valued nonexpansive mapping has a stationary point if and only if it has the strong approximate stationary point property.

2. Stationary points of contractive mappings

It is natural to ask what conditions may be imposed to have stationary points on spherecially complete ultrametric spaces. Let $B(x, r)$ denote the closed ball centered at x with the radius $r > 0$.

Theorem 2.1. *Let (X, d) be a spherically complete ultrametric space and $T : X \to CB(X)$ be a contractive mapping. Then T has a unique stationary point if and only if T has the approximate stationary point property.*

Proof. The 'if' part is obvious so we just prove the 'only if' part. Let T have the approximate stationary point property.

For each $w \in X$ note $B = B(w, r_w(Tw))$ is T-invariant. To see this, let $x \in B$, $z \in Tx$, and $\epsilon > 0$ be given. Let $v \in Tw$ be such that $d(z, v) \le d(z, Tw) + \epsilon$. Then

$$\begin{aligned} d(w, z) &\le \max\{d(w, v), d(z, v)\} \\ &\le \max\{r_w(Tw), d(z, Tw) + \epsilon\} \\ &\le \max\{r_w(Tw), H(Tx, Tw) + \epsilon\} \\ &\le \max\{r_w(Tw), d(x, w) + \epsilon\} \\ &\le \max\{r_w(Tw), r_w(Tw) + \epsilon\} \\ &= r_w(Tw) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we have that $z \in B$. This shows that $Tx \subseteq B$ for $x \in B$, so B is T-invariant.

Since $\inf_{x \in X} r_x(Tx) = 0$, there exists a sequence (x_n) in X such that

$$\lim_n r_{x_n}(Tx_n) = 0, \text{ and } r_{x_{n+1}}(Tx_{n+1}) \le r_{x_n}(Tx_n)$$

for each $n \in \mathbb{N}$. We claim

$$B(x_{n+1}, r_{x_{n+1}}(Tx_{n+1})) \subseteq B(x_n, r_{x_n}(Tx_n)) \quad \text{for} \quad n \in \mathbb{N}. \tag{2.1}$$

If there exists a $n \in \mathbb{N}$ with $x_n = x_{n+1}$ then (2.1) is trivially true for this n. It remains to consider the case when $x_n \neq x_{n+1}$ for $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, $\epsilon > 0$ be given. Let $z \in Tx_n$ be arbitrary, and $w \in Tx_{n+1}$ be such that

$$d(z, w) \leq H(Tx_n, Tx_{n+1}) + \epsilon.$$

Then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \max\{d(x_n, z), d(z, x_{n+1})\} \\ &\leq \max\{r_{x_n}(Tx_n), d(z, w), d(w, x_{n+1})\} \\ &\leq \max\{r_{x_n}(Tx_n), H(Tx_n, Tx_{n+1}) + \epsilon, r_{x_{n+1}}(Tx_{n+1})\} \\ &\leq \max\{r_{x_n}(Tx_n), H(Tx_n, Tx_{n+1}) + \epsilon\}. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have

$$d(x_n, x_{n+1}) \leq \max\{r_{x_n}(Tx_n), H(Tx_n, Tx_{n+1})\}. \tag{2.2}$$

If $r_{x_n}(Tx_n) \leq H(Tx_n, Tx_{n+1})$, then (note $x_n \neq x_{n+1}$),

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_n, Tx_{n+1}) \\ &< d(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Therefore

$$H(Tx_n, Tx_{n+1}) < r_{x_n}(Tx_n),$$

so from (2.2), we get

$$d(x_n, x_{n+1}) \leq r_{x_n}(Tx_n).$$

Thus

$$B(x_{n+1}, r_{x_{n+1}}(Tx_{n+1})) \subseteq B(x_n, r_{x_n}(Tx_n));$$

to see this let $z \in B(x_{n+1}, r_{x_{n+1}}(Tx_{n+1}))$ and note

$$d(z, x_{n+1}) \leq r_{x_{n+1}}(Tx_{n+1}) \leq r_{x_n}(Tx_n)$$

so

$$d(z, x_n) \leq \max\{d(z, x_{n+1}), d(x_n, x_{n+1})\} \leq r_{x_n}(Tx_n).$$

Thus (2.1) is true.

Therefore $\{B(x_n, r_{x_n}(Tx_n))\}$ is a shrinking collection of balls. From the spherically completeness of X we get

$$\cap_{n\in\mathbb{N}} B(x_n, r_{x_n}(Tx_n)) \neq \emptyset.$$

Let $z \in \cap_{n\in\mathbb{N}} B(x_n, r_{x_n}(Tx_n))$. For each $n \in \mathbb{N}$ note $B(x_n, r_{x_n}(Tx_n))$ is T-invariant, so

$$Tz \subseteq B(x_n, r_{x_n}(Tx_n)).$$

This implies (note $d(z, y) \leq \max\{d(z, x_n), d(x_n, y)\} \leq r_{x_n}(Tx_n)$ for all $y \in T\,z$) that

$$r_z(Tz) \leq r_{x_n}(Tx_n),\ n \in \mathbb{N},$$

which leads to

$$r_z(Tz) \leq \lim_n r_{x_n}(Tx_n) = 0.$$

Hence $Tz = \{z\}$ and we are finished. □

As an immediate consequence of Theorem 1 we obtain an ultrametric version of Theorem 2.1. in [1].

Corollary 2.1. *([1], Theorem 2.1) Let (X, d) be a spherically complete ultrametric space. Let $F : X \to CB(X)$ be a set-valued map satisfying*

$$H(Fx, Fy) \leq \psi(d(x, y)) \quad (x, y \in X),$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous, $\psi(t) < t$ for each $t > 0$ and satisfies $\liminf_{t\to\infty}(t - \psi(t)) > 0$. Then F has a unique stationary point if and only if F has the approximate stationary point property.

A slight modification of the argument in Theorem 2.1 will yield Theorem 1 in [5].

Corollary 2.2. *([5], Theorem 1) Let X be a spherically complete ultrametric space. If $T : X \to X$ is a contractive mapping, then T has an unique fixed point.*

Proof. Let $T : X \to X$ be a contractive mapping. We show that

$$\alpha = \inf_{x\in X} d(x, Tx) = 0.$$

A slight modification of the argument in Theorem 2.1 establishes that for each $x \in X$ we have that $B(x, d(x, Tx))$ is T-invariant; note also since T is single-valued so

$$r_x(Tx) = d(x, Tx).$$

Suppose, for contradiction, that $\alpha \neq 0$. Let $(x_n) \subseteq X$ be such that

$$\lim_n d(x_n, Tx_n) = \alpha, \text{ and } d(x_{n+1}, Tx_{n+1}) \leq d(x_n, Tx_n), \; n = 1, 2, ...$$

Note for $n \in \mathbb{N}$ that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \max\{d(x_n, Tx_n), d(Tx_n, Tx_{n+1}), d(x_{n+1}, Tx_{n+1})\} \\ &\leq \max\{d(x_n, Tx_n), d(Tx_n, Tx_{n+1})\}. \end{aligned}$$

Essentially the same reasoning as in the proof of Theorem 2.1 yields

$$B(x_{n+1}, d(x_{n+1}, Tx_{n+1})) \subseteq B(x_n, d(x_n, Tx_n)) \text{ for } n \in \mathbb{N};$$

note if $x_n \neq x_{n+1}$ we have that $d(Tx_n, Tx_{n+1}) \leq d(x_n, Tx_n)$ and so

$$d(x_n, x_{n+1}) \leq d(x_n, Tx_n).$$

From the spherically completeness of X we have

$$\cap_{n\in\mathbb{N}} B(x_n, d(x_n, Tx_n)) \neq \emptyset.$$

Let $z \in \cap_{n\in\mathbb{N}} B(x_n, d(x_n, Tx_n))$. For each $n \in \mathbb{N}$ note $B(x_n, d(x_n, Tx_n))$ is T-invariant, so

$$Tz \in B(x_n, d(x_n, Tx_n)).$$

Thus for each $n \in \mathbb{N}$ we have $d(z, Tz) \leq \max\{d(z, x_n), d(x_n, Tz)\} \leq d(x_n, Tx_n)$ so (note $\alpha = \lim_n d(x_n, Tx_n)$) we have

$$d(z, Tz) \leq \alpha.$$

Now T is contractive so (note $z \neq Tz$ since $\alpha \neq 0$) we have

$$d(T^2 z, Tz) < d(z, Tz) \leq \alpha = \lim_n d(x_n, Tx_n),$$

which is a contradiction because $\alpha = \inf_{x \in X} d(x, Tx)$. Thus $\alpha = 0$ and the result follows from Theorem 1. □

3. Stationary points of nonexpansive mappings

A ball $B(x, r)$ is said to be T-invariant if for each $z \in B(x, r)$ we have $Tz \subseteq B(x, r)$. A T-invariant ball is said to be minimal T-invariant if it is T-invariant and does not contain any T-invariant ball except itself.

Theorem 3.1. *([4], Theorem 4) Suppose (X, d) is a spherically complete ultrametric space and $T : X \to X$ is a (single-valued) nonexpansive mapping. Then every ball of the form*

$$B(x, d(x, T(x)))$$

contains either a fixed point of T or a minimal T-invariant ball.

In the following we extend this result to set-valued mappings.

Theorem 3.2. *Let (X, d) be a spherically complete ultrametric space and $T : X \to CB(X)$ be a nonexpansive mapping. Then every ball of the form*

$$B(x, r_x(Tx))$$

contains either a stationary point of T or a minimal T-invariant ball.

Proof. Essentially the same reasoning as in the proof of Theorem 2.1 establishes that for each $x \in X$ we have that $B_x = B(x, r_x(Tx))$ is T-invariant.

Let $z \in X$ and let

$$\Sigma = \{B_x : x \in B_z\}.$$

We order Σ, by

$$B_x \preceq B_y \Longleftrightarrow B_y \subseteq B_x.$$

Let $\{B_{x_\alpha}\}$ be an arbitrary chain in Σ. The spherically completeness of X implies that

$$\cap B_{x_\alpha} \neq \emptyset.$$

Suppose that $z \in \cap B_{x_\alpha}$. For each α we have $Tz \subseteq B_{x_\alpha}$, since B_{x_α} is T-invariant. Thus (note $d(z, y) \leq \max\{d(z, x_\alpha), d(x_\alpha, y)\} \leq r_{x_\alpha}(Tx_\alpha)$ for all $y \in Tz$) we have

$$r_z(Tz) \leq r_{x_\alpha}(Tx_\alpha).$$

As a result we have that

$$B_z \subseteq B_{x_\alpha};$$

to see this let $w \in B_z = B(z, r_z(Tz))$ and note $d(w, z) \leq r_z(Tz) \leq r_{x_\alpha}(Tx_\alpha)$ and so

$$d(w, x_\alpha) \leq \max\{d(w, z), d(z, x_\alpha)\} \leq r_{x_\alpha}(Tx_\alpha).$$

Thus

$$B_{x_\alpha} \preceq B_z,$$

which means B_z is an upper bound of the chain $\{B_{x_\alpha}\}$. By Zorn's Lemma, Σ possesses a maximal element, say B_w. If $r_w(Tw) = 0$, then w is a stationary point of T and we are finished. Suppose $r_w(Tw) \neq 0$. We claim that B_w is a minimal T-invariant ball. To see this, let $x \in X$ and $B(x,r)$ be an arbitrary T-invariant ball such that $B(x,r) \subseteq B_w$. Since $x \in B_w$ it follows from the T-invariancy of B_w that (note $d(x,y) \leq \max\{d(x,w), d(w,y)\} \leq r_w(Tw)$ for all $y \in Tx$) $r_x(Tx) \leq r_w(Tw)$. Thus we have

$$B_x \subseteq B_w;$$

to see this let $z \in B_x = B(x, r_x(Tx))$ and note $d(z,x) \leq r_x(Tx) \leq r_w(Tw)$ and so

$$d(z,w) \leq \max\{d(z,x), d(x,w)\} \leq r_w(Tw).$$

This means $B_w \preceq B_x$. Since $B_x \in \Sigma$ it follows from the maximality of B_w that

$$B_x = B_w.$$

Since $B(x,r)$ is T-invariant (note $d(x,y) \leq r$ for all $y \in Tx$) it follows that $r_x(Tx) \leq r$. Thus $B_x \subseteq B(x,r)$ since if $z \in B_x = B(x, r_x(Tx))$ then $d(z,x) \leq r_x(Tx) \leq r$. Thus

$$B_w = B_x \subseteq B(x,r) \subseteq B_w,$$

which means $B(x,r) = B_w$. Since $B(x,r) \subseteq B_w$ was an arbitrary T-invariant ball, we conclude that, B_w is a minimal T-invariant ball. □

The following result shows that there exists an equivalency between the strong approximate stationary point property and the existence of a stationary point for nonexpansive set-valued mappings.

Corollary 3.1. *Let (X,d) be a spherically complete ultrametric space and $T : X \to CB(X)$ be a nonexpansive mapping. Then every ball of the form*

$$B(x, r_x(Tx))$$

contains a stationary point of T if and only if T has the strong approximate stationary point property.

Proof. The 'if' part is obvious, so we just prove the 'only if' part. Let T have the strong approximate stationary point property and fix $z \in X$. Suppose, for contradiction, that T is stationary point free. Theorem 3.2 implies that B_z contains a minimal T-invariant ball, say $B(x,r)$. Now for each $w \in B(x,r)$, $B_w = B(w, r_w(Tw))$ is T-invariant so it follows from minimality of $B(x,r)$ that $B(x,r) = B_w$. Thus

$$r = r_w(Tw) \quad \text{for} \quad w \in B(x,r).$$

As a result

$$\inf_{w \in B(x,r)} r_w(Tw) \neq 0,$$

which contradicts the strong approximate stationary point property of T, since for each $w \in B(x,r), B_w = B(w, r_w(Tw)) = B(x,r)$. □

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