Fixed Point Theory, 19(2018), No. 1, 219-224
DOI 10.24193/fpt-ro.2018.1.17
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

RANDOM NASH EQUILIBRIUM

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Abstract. In this paper the notion of random n-persons game is defined. The most important problem of such games is the existence of Random Nash Equilibrium (RNE) which is defined in section 4. Since not every random game has RNE, we will consider here a subclass of random games called admissible random games. Finally, using a generalized version of Kakutani's theorem it is proved that for any admissible random game there exists a RNE.
Key Words and Phrases: Nash equilibrium, random fixed points, multivalued operators.
2010 Mathematics Subject Classification: 91A02, 47B80, 47H40, 47H04, 47H10.

1. Introduction

The game theory is a wide part of the whole mathematics and it interacts with all main currents. Certainly, J.Nash and his game equilibrium concept had a strong impact on game theory development [13]. This concept had much progress over the years and was modified in many ways f.e. see [2], [8], [11], [12].

As it is well known, equilibrium is a fixed point of a smart defined mapping (multivalued mapping). That is why, fixed point theorems are widely used in searching for the equilibria (compare f.e. [6]).

In our paper we are focused on n-person random games which are strongly linked with topology and fixed point theory. We improve the Nash Equilibrium by adding some randomness implemented by the random space Ω. We prove that a subclass of games have a Random Nash Equilibrium; however it does not have to be true generally.

The notion of Random Nash Equilibrium was considered in [3] and [16] but this equilibrium has nothing in common with that considered in this paper. Moreover, stochastic games are connected with randomness; however, these problems are different then studied in our paper, see [9],[14].

To the best of author's knowledge, there were not considered any similar games with Random Nash equilibrium.

The project was financed by funds of National Science Center granted by decision number DEC-2013/11/B/HS4/00857; carried out at Warsaw University .

2. Topological background

If it is not indicated otherwise, we assume that all spaces are compact metric spaces and all single-valued mappings are continuous. We would like to recall some needful definitions.

Definition 2.1. A space $\mathbb{X}$ is called contractible, if it is homotopically equivalent to the one-point space $\{p\}$; i.e. the following conditions are satisfied:

for the function $f : \mathbb{X} \to \{p\}$ and some function $g : \{p\} \to \mathbb{X}$

- $g \circ f \sim id_{\mathbb{X}}$,
- $f \circ g \sim id_{\{p\}}$,

where $f \sim g$, if there exists the homotopy linking f and g.

Definition 2.2. We shall say $\mathbb{X}$ is an absolute retract space, provided for any space $\mathbb{Y}$ and for any embedding $h : \mathbb{X} \to \mathbb{Y}$, a set $h(\mathbb{X})$ is a retract of $\mathbb{Y}$. We shall write $\mathbb{X} \in AR$.

Note that every AR-space is contractible. The converse theorem is not true; for example we propose the comb space (see [5]).

The following two facts are evident(see [5],[7]): the finite Cartesian product of contractible sets (AR-sets) is a contractible set (AR-set respectively).

We shall consider the notion of multivalued (set-valued) mappings.

Definition 2.3. Let $\mathbb{X}$ and $\mathbb{Y}$ be two spaces. We say that φ is the multivalued mapping (sign. $\varphi : \mathbb{X} \multimap \mathbb{Y}$), if for every $x \in \mathbb{X}$, the set $\varphi(x)$ is a non-empty, compact subset of $\mathbb{Y}$. With any multivalued mapping φ, we associate the graph $\Gamma_\varphi \subset \mathbb{X} \times \mathbb{Y}$ which fulfils: $(x, y) \in \Gamma_\varphi$, if and only if $y \in \varphi(x)$.

In order to avoid misunderstandings we recall the definition of upper semicontinous mapping:

Definition 2.4. A multivalued mapping $\varphi : \mathbb{X} \multimap \mathbb{Y}$ is called *upper semicontinous*, if for every open subset $U \subset \mathbb{Y}$, the set

$$\varphi^{-1}(U) = \{x \in \mathbb{X};\ \varphi(x) \subset U\}$$

is an open subset of $\mathbb{X}$ (sign. usc.).

Notice that a multivalued mapping $\varphi : \mathbb{X} \multimap \mathbb{Y}$ is usc. if and only if the graph Γ_φ of φ is a closed subset of $\mathbb{X} \times \mathbb{Y}$; however, it is not true for arbitrary metric spaces.

We define:

Definition 2.5. We shall say an usc. mapping $\varphi : \mathbb{X} \multimap \mathbb{Y}$ has contractible values, if for every point $x \in \mathbb{X}$ the set $\varphi(x)$ is contractible.

Proposition 2.6. *Let $\mathbb{X} = \mathbb{X}_1 \times \ldots \times \mathbb{X}_n$ and assume that $\varphi_i : \mathbb{X} \multimap \mathbb{X}_i$ is usc. for $i \in \{1, \ldots, n\}$. Then the mapping $\varphi : \mathbb{X} \multimap \mathbb{X}$ defined by:*

$$\varphi(x) := \varphi_1(x) \times \ldots \times \varphi_n(x)$$

is usc.

If $\varphi_i,\ i = 1, \ldots, n$ are mappings with contractible values, then φ has also contractible values. In further considerations we will need the following generalization of a Kakutani theorem:

Theorem 2.7. (Kakutani) [7][10] *If $\mathbb{X} \in AR$ and $\varphi : \mathbb{X} \multimap \mathbb{X}$ is an usc. with contractible values, then φ has a fixed point, i.e. there exists $x \in \mathbb{X}$, such that $x \in \varphi(x)$.*

For more details, see [5] or [7].

3. Random operators

In further considerations we will assume that: $\mathbb{X}$ and $\mathbb{Y}$ are two compact spaces, Ω is a measurable space and $\varphi, \psi : \Omega \times \mathbb{X} \multimap \mathbb{Y}$ are two multivalued mappings (these assumptions are analogous to section 2).

We are able to recall basic definitions connected with random operators:

Definition 3.1. A mapping φ is called a random operator provided φ is product measurable, i.e. for any open subset $U \subset \mathbb{Y}$, the set

$$\varphi^{-1}(U) = \{(\omega, x) \in \Omega, \mathbb{X};\ F(\omega, x) \subset U\}$$

is a measurable subset of $\Omega \times \mathbb{X}$ (in $\mathbb{X}$ we consider σ-field of Borel's sets). Where a mapping φ is called a random usc. operator if φ is a random operator and $\varphi(\omega, \cdot)$ is usc. for almost all parameters $\omega \in \Omega$.

Proposition 3.2. *Let us assume that $\phi : \Omega \times \mathbb{X} \multimap \mathbb{Z}$ and $\varphi : \Omega \times \mathbb{X} \multimap \mathbb{Y}$ are usc., random mappings. By product of two random mappings we understand a mapping $F : \Omega \times \mathbb{X} \multimap \mathbb{Y} \times \mathbb{Z}$ given by $F(\omega, x) = \varphi(\omega, x) \times \phi(\omega, x)$. We claim the mapping F is an usc. random mapping.*

Proposition 3.2 is a consequence of the properties of a counterimage of multivalued mappings and the definition of measurability (compare [7][19.3 and 14.8] for randomness and continuity respectively).

Definition 3.3. A measurable function $\zeta : \Omega \to \mathbb{X}$ is called a random fixed point of $\psi : \Omega \times \mathbb{X} \multimap \mathbb{X}$ if for almost all $\omega \in \Omega$, we have $\zeta(\omega) \in \psi(\omega, \zeta(\omega))$.

Theorem 3.4. [1] *Let $\varphi : \Omega \times \mathbb{X} \multimap \mathbb{X}$ be a random mapping. If for almost all $\omega \in \Omega$ the mapping $\varphi(\omega, \cdot)$ has a fixed point, then there exists a random fixed point of φ.*

Note that for an arbitrary metric space the above theorem is an open problem (comp. [1]).

The following theorem is indispensable to prove the existence of a random Nash equilibrium:

Theorem 3.5. [7] *If a space $\mathbb{X} \in AR$ and $\varphi : \Omega \times \mathbb{X} \multimap \mathbb{X}$ is random mapping with contractible values, then φ has the random fixed point.*

Since $\mathbb{X}$ is an AR-space by Theorem 2.7 we obtain that $\varphi(\omega, \cdot) : \mathbb{X} \multimap \mathbb{X}$ has a fixed point. Hence Theorem 3.5 follows from Theorem 3.4 immediately.

For more details concerning random operators see [1].

4. Random games

This section contains the definition of random Nash equilibrium, which is non-trivial property in class of random n-person game. Moreover, we proof that any admissible games has got Random Nash Equilibrium. Note that similar games were considered in [15]; however, the concept of random games was an improvement of games considered in [4].

We shall assume that $\mathbb{X}_1, \ldots, \mathbb{X}_n$ are compact AR-spaces and a space (Ω, μ) is measure space with μ as a measure. Moreover, we assume that for every $i \in \{1, \ldots, n\}$ $F_i, G_i : \Omega \times \mathbb{X} \multimap \mathbb{X}_i$ are random multivalued mappings (we allow one exception, the mapping G_i may have empty values).

We put $\mathbb{X} = \mathbb{X}_1 \times \ldots \mathbb{X}_n$. It is well known (see[5]), that $\mathbb{X}$ is an AR-space as well.

Definition 4.1. By the random n-players game we understand the set:

$$\kappa = \{\Omega, \mathbb{X}, \{F_i\}_{i=1}^n, \{G_i\}_{i=1}^n\}.$$

The random game is containing: the parameter space Ω - randomness of the game, the strategy space $\mathbb{X}$, the restriction of the strategy sets and the forbidden strategies sets, for any player i, are given by F_i and G_i respectively. The mappings F_i, G_i are dependent on random parameter $\omega \in \Omega$ and strategy $x \in \mathbb{X}$. Then mappings $F_i, G_i : \Omega \times \mathbb{X} \multimap \mathbb{X}_i$ describe some subsets of $\mathbb{X}$ for any ω and x we allow that $G_i(\omega, x)$ can be an empty set. However, we demand that mapping F fulfills:

$x_i \in F_i(x)$ for every $x = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{X}$

We interpret the game as follows: players are building 'some' algorithm dependent on random parameter ω, which is randomized as the game starts. So the strategy set is given by measurable function $\varsigma : \Omega \to \mathbb{X}$. For any $\omega \in \Omega$ we shall write

$$\varsigma(\omega) = (\varsigma_1(\omega), \ldots, \varsigma_n(\omega)),$$

where ς_i is an i-th coordinate function associated with ς.

Definition 4.2. (Random Nash Equilibrium) A measurable function $\zeta : \Omega \to \mathbb{X}$ is called the Random Nash Equilibrium of the game κ if following conditions are satisfied (sign. RNE):

4.2.1 $\varsigma_i(\omega) \in F_i(\omega, \zeta(\omega))$ for almost all $\omega \in \Omega$ and $i \in \{1, \ldots, n\}$

4.2.2 $F_i(\omega, \varsigma(\omega)) \cap G_i(\omega, \varsigma(\omega)) = \emptyset$ for almost all $\omega \in \Omega$ and $i \in \{1, \ldots, n\}$

Let us make an agreement for simplify notation: let $x \in \mathbb{X}, y_i \in \mathbb{X}_i$, then $(x_{-i}, y_i) := (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

The existence of Random Nash Equilibrium for given random game is most important question. It is easy to observe there are random games for which RNE do not exists (it is enough to construct the game in which $F_i(\omega, \varsigma(\omega)) \cap G_i(\omega, \varsigma(\omega)) \neq \emptyset$). That is why the assumption of game admissibility looks reasonable:

Admissible games

Let $f_i : \Omega \times \mathbb{X} \to \mathbb{R}_+ = [0, +\infty)$ for every $i = \{1, \ldots, n\}$. We say that f_i is payoff function if only it fulfills:

a. Let $x \in \mathbb{X}$, for every $y_i \in G_i(x) \subset \mathbb{X}_i$, for almost all $\omega \in \Omega$, we have

$$f_i(\omega, (x_{-i}, y_i)) > f_i(\omega, x).$$

b. Let $i \in \{1, \ldots, n\}$, we put $H_i : \Omega \times \mathbb{X} \multimap \mathbb{X}_i$, such that

$$H_i(\omega, x) = \{y_i \in \mathbb{X}_i \mid \text{for all } z_i \in F_i(\omega, x),\ f_i(\omega, (x_{-i}, y_i)) \geq f_i(\omega, (x_{-i}, z_i))\}$$

and we shall assume that H_i is random usc. mapping with contractible values for every $i \in \{1, \ldots, n\}$.

Definition 4.4. Any game κ, with payoff functions as above, we shall call *admissible game.*

Finally we are able to prove the existence of RNE:

Theorem 4.5. *Any admissible game κ has a Random Nash Equilibrium.*

Proof. Firstly, observe that if the game κ is admissible, then the mapping $H = H_1 \times \ldots \times H_n : \Omega \times \mathbb{X} \multimap \mathbb{X}$ is a random usc. mapping with contractible-values so H fulfils assumptions of Theorem 3.4.

By Theorem 3.4 we claim that there exists a random fixed point $\zeta : \Omega \to \mathbb{X}$ of mapping H, i.e., $\zeta(\omega) \in H(\omega, \zeta(\omega))$ for almost all $\omega \in \Omega$.

Then for every $i \in \{1, \ldots, n\}$ and almost all $\omega \in \Omega$, $\zeta_i(\omega) \in H_i(\omega, \zeta(\omega))$ and

$$f_i(\omega, \zeta(\omega))) \geq f_i(\omega, (\zeta(\omega)_{-i}, x_i),$$

for any $x_i \in F_i(\omega, \zeta(\omega))$ (compare condition b for payoff function).

To complete the proof it is enough to show that: $G_i(\omega, \zeta(\omega)) \cap F_i(\omega, \zeta(\omega)) = \emptyset$ for almost all $\omega \in \Omega$ and $i \in \{1, \ldots, n\}$.

We assume to the contrary that:

$$y_i \in G_i(\omega, \zeta(\omega)) \cap F_i(\omega, \zeta(\omega)),$$

for some $\omega \in \Omega$ and $i \in \{1, \ldots, n\}$.

Then by the first condition of payoff function, we get:

$$f_i(x, (\zeta(\omega)_{-i}, y_i)) > f_i(\omega, \zeta(\omega)).$$

However, from the following inequalities:

$$f_i(\omega, \zeta(\omega)) \geq f_i(x, (\zeta(\omega)_{-i}, y_i)) > f_i(\omega, \zeta(\omega)),$$

we obtain a contradiction, so $G_i(\omega, \zeta(\omega)) \cap F_i(\omega, \zeta(\omega)) = \emptyset$.

Finally we checked that both conditions of RNE are fulfilled by ζ so the mapping ζ is a RNE of game κ. □

5. Concluding remarks

We owe several remarks to the reader:

Remark 5.1. In a deterministic game usually the strategy sets are assumed to be compact convex (respectively multivalued mappings are usc. with compact convex values). The reason for considering this kind of games is the original Ky Fan fixed point theorem, proved in 1941 which was formulated for compact convex subsets of the n-dimensional Euclidean space.

Remark 5.2. All results presented in section 4 remains true if we change strategy sets to arbitrary separable absolute retracts and usc. mappings with compact values to compact acyclic mappings (comp. [1],[7]).

Remark 5.3. Moreover, it is possible to consider arbitrary separable absolute neighborhood retracts as the strategy sets, provided for almost all $\omega \in \Omega$ the Lefschetz number of the mapping $H_i : \Omega \times \mathbb{X} \to \mathbb{X}_i$ is different from zero for every $i \in \{1, \ldots, n\}$. For proof compare [7].

Open problem 5.4. Is Theorem 3.5 true if strategy space is not separable metric spaces ([7])?

We will continue working on this theme in the near future.

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Received: October 26, 2016; Accepted: November 28, 2016.