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VECTOR EQUILIBRIUM PROBLEMS FOR MULTIFUNCTIONS IN TOPOLOGICAL SEMILATTICE SPACES

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Dedicated to Professor Do Hong Tan on the occasion of his 80th birthday

Abstract. Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals. In this paper, under new assumptions, we establish some existence theorems of $x \in K$ such that $\mathcal{F}(A) \cap VEP(f) \neq \emptyset$, where $\mathcal{F}(A)$ is the set of all fixed points of the multifunction $A : K \to 2^K$ and $VEP(f)$ is the set of all solutions for the vector equilibrium problems of the multifunction f from $K \times K$ to a topological vector space Y. These results generalize and improve the recent ones in the literature. Some examples are given to illustrate our results.
Key Words and Phrases: KKM lemma, Ky Fan inequality, Browder-Fan fixed point theorem, multifunction, topological semilattice, C_Δ-quasiconvex (quasiconcave), C-upper (lower) semicontinuous, vector equilibrium problem.
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1. Introduction

Let K be a nonempty subset of a topological vector space, $f : K \times K \to \mathbb{R}$ a real-valued bifunction, where $\mathbb{R}$ denotes the set of real numbers. Consider the following inequality which is known as an equilibrium problem (see Blum and Oettli [3]):

$$\text{Find } x^* \in K \text{ such that } f(x^*, y) \geq 0 \quad \forall y \in K. \tag{1.1}$$

In 1972, Ky Fan [6] first established the existence of solutions of the inequality (1.1) (here, we write its dual form).

Theorem 1.1. *Let X be a Hausdorff topological vector space, and let K be a nonempty compact convex subset of X. Suppose that $f : K \times K \to \mathbb{R}$ satisfies the following conditions:*

(1) $f(x, x) \geq 0 \ \forall x \in K$*;*
(2) $\forall x \in K$*,* $f(x, .)$ *is quasiconvex;*

(3) $\forall y \in K$, $f(., y)$ *is upper semicontinuous.*

Then there exists $x^* \in K$ *such that* $f(x^*, y) \geq 0 \; \forall y \in K$.

The above result now has been called Ky Fan inequality. It plays a very important role in many fields, such as variational inequalities, game theory, mathematical economics, optimization theory, and fixed point theory. Because of wide applications, this inequality has been generalized in a number ways (e.g., see Allen [1], Aubin and Ekeland [2], Chang [4], Ding and Tan [5], Georgiev and Tanaka [7], Giannessi [8], Hadjisavvas et al. [9], Horvath [12], Tian [20], Yen [25], Yuan [26], and Zhou and Chen [27]). Topological vector spaces provide the usual mathematical framework in the study of many problems. To avoid the linear feature, semilattices may be good choices. In 1996, Horvath and Llinares Ciscar [13] first established an order theoretical version of the classical result of Knaster-Kuratowski-Mazurkiewicz, as well as fixed point theorems for multivalued mappings in the framework of topological semilattices.

In 2001, by using Horvath and Llinares Ciscar's results, Luo [15] proved a similar result to Theorem 1.1 in topological semilattices. In 2006, Luo [16] studied Ky Fan inequalities for vector multivalued mappings in topological semilattices. Recently, Song and Wang [18], Song [19] proved an extension of Ky Fan inequality but only for vector single-valued mappings in topological semilattices.

Let M be a topological semilattice, $K \subset M$ a nonempty Δ-convex subset, Y a topological vector space, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$, $A : K \to 2^K, f : K \times K \to 2^Y$. The set of fixed points of A is denoted by $\mathcal{F}(A)$, i.e., $\mathcal{F}(A) = \{x \in K : x \in A(x)\}$.

In 2006, Luo [16] studied some generalized vector quasi-equilibrium problems (**GVQEP**): Find $x^* \in K$ such that

$$x^* \in \mathcal{F}(A), \quad f(x^*, y)\rho C \quad \forall y \in A(x^*),$$

where $f(x, y)\rho C$ represents one of the following relations

$$f(x, y) \subset -C, \quad f(x, y) \cap \operatorname{int} C = \emptyset, \quad f(x, y) \not\subset \operatorname{int} C.$$

Luo proved the existence of solutions for these problems by using either upper semicontinuous or lower semicontinuous multifunctions in the first argument. Since in the scalar case, these functions are continuous, so his results are weaker than the original form. In 2008, Vinh [22] improved Luo's results and presented some genuine generalizations of scalar Ky Fan minimax inequality in topological semilattices.

Al-Homidan et al. [10, 11] considered and studied the system of the generalized vector quasi-equilibrium problems in topological semilattices. Their results extended the ones of Luo [16] and Vinh [22]. Very recently, Vinh and Hoai [24] used the cone semicontinuity and cone convexity of multivalued mappings to study the solvability of the (GVQEP). However, the results in [10, 11, 16, 18, 22, 24] require that the set $\mathcal{F}(A)$ is closed, since proof techniques in [10, 11, 16, 18, 22, 24] require this assumption.

Motivated and inspired by research works mentioned above, in this paper, we will study the vector equilibrium problems (VEP1)-(VEP4) in which the set $\mathcal{F}(A)$ is assumed to be open.

(**VEP1**) Find $x^* \in K$ such that

$$x^* \in A(x^*), \quad f(x^*, y) \not\subset \operatorname{int} C \quad \forall y \in K.$$

(**VEP2**) Find $x^* \in K$ such that

$$x^* \in A(x^*), \quad f(x^*, y) \cap \operatorname{int} C = \emptyset \quad \forall y \in K.$$

(**VEP3**) Find $x^* \in K$ such that

$$x^* \in A(x^*), \quad f(x^*, y) \cap (-C) \neq \emptyset \quad \forall y \in K.$$

(**VEP4**) Find $x^* \in K$ such that

$$x^* \in A(x^*), \quad f(x^*, y) \subset -C \quad \forall y \in K.$$

We remark that these problems include the corresponding generalized vector quasi-equilibrium problems as special cases. The main purpose of this paper, we provide sufficient conditions and prove the existence of solutions for the problems (VEP1)-(VEP4). Our results and our proof techniques are different from those given in [10, 11, 16, 18, 22, 24].

The rest of the paper is organized as follows. In Section 2, we introduce about topological semilattices and recall some concepts of cone semicontinuity and cone convexity. In Section 3, under some new assumptions, we prove the existence of solutions for vector equilibrium problems with multifunctions by using KKM lemma in the setting of topological semilattices. Our results generalize and improve the ones in [15, 18, 19, 21]. We also give some examples to illustrate our results.

2. Preliminaries

Definition 2.1. ([13]) A partially ordered set $(M, \leqslant)$ is called a sup-semilattice if any two elements x, y of M have a least upper bound, denoted by $\sup\{x, y\}$. The partially ordered set $(M, \leqslant)$ is a topological semilattice if M is a sup-semilattice equipped with a topology such that the mapping

$$\begin{aligned} M \times M &\to M \\ (x, y) &\mapsto \sup\{x, y\} \end{aligned}$$

is continuous.

We have given the definition of a sup-semilattice, we could obviously also consider inf-semilattices. When no confusion can arise we will simply use the word semilattice. It is also evident that each nonempty finite set A of M will have a least upper bound, denoted by $\sup A$.

In a partially ordered set $(M, \leq)$, two arbitrary elements x and x' do not have to be comparable but, in the case where $x \leq x'$, the set

$$[x, x'] = \{y \in M : x \leq y \leq x'\}$$

is called an order interval or simply, an interval. Now assume that $(M, \leq)$ is a semilattice and A is a nonempty finite subset; then the set

$$\Delta(A) = \bigcup_{a \in A} [a, \sup A]$$

is well defined and it has the following properties:

(1) $A \subseteq \Delta(A)$;
(2) if $A \subset A'$, then $\Delta(A) \subseteq \Delta(A')$.

We say that a subset $E \subseteq M$ is Δ-convex if for any nonempty finite subset $A \subseteq E$ we have $\Delta(A) \subseteq E$.

Example 2.2. We consider $\mathbb{R}^2$ with usual order defined by

$$x^1, x^2 \in \mathbb{R}^2,\ x^1 \leq x^2 \Longleftrightarrow x^2 \in x^1 + \mathbb{R}^2_+,$$

where $\mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 \geq 0\}$.

It is obvious that $(\mathbb{R}^2, \leq)$ is a topological semilattice, in which

$$x^1 \vee x^2 = (\max(x^1_1, x^2_1), \max(x^1_2, x^2_2)),\ \forall x^i = (x^i_1, x^i_2) \in \mathbb{R}^2,\ i = 1, 2.$$

We see that

(1) The subset $K = \{(x, 1) : 0 \leq x \leq 1\} \cup \{(1, y) : 0 \leq y \leq 1\}$ is Δ-convex but not convex in the usual sense.
(2) The subset $K = \{(x, y) : 0 \leq x \leq 1;\ y = 1 - x\}$ is convex in the usual sense but not Δ-convex.

In this paper, we will consider partial orders on vector spaces induced by cones. We agree that any cone contains the origin, according to the following definition.

Definition 2.3. Let C be a nonempty subset of a vector space Y. The set C is called a cone if $\lambda x \in C$ for all $x \in C$ and $\lambda \geq 0$. The cone C is pointed if $C \cap (-C) = \{0\}$.

Lemma 2.4. ([24], Lemma 2.3) *Let Y be a topological vector space and C a closed, convex and pointed cone of Y with* $\operatorname{int} C \neq \emptyset$*, where* $\operatorname{int} C$ *denotes the interior of C. Then we have* $\operatorname{int} C + C \subset \operatorname{int} C$.

We now recall some concepts of generalized convexity of multivalued mappings. Let X be a nonempty convex subset of a vector space E, C be a convex cone of a vector space Y, and $F : X \to 2^Y$ be a multivalued mapping with nonempty values.

The mapping F is called C-quasiconvex if for all $x_i \in X$, $i = 1, 2$ and $x \in conv\{x_1, x_2\}$, either $F(x) \subset F(x_1) - C$ or $F(x) \subset F(x_2) - C$.

The mapping F is called C-quasiconcave if for all $x_i \in X$, $i = 1, 2$, and $x \in conv\{x_1, x_2\}$, either $F(x_1) \subset F(x) + C$ or $F(x_2) \subset F(x) + C$.

Similarly, in the setting of topological semilattices, we introduce the following definition.

Definition 2.5. Let K be a Δ-convex subset of a topological semilattice, Y be a topological vector space, $C \subset Y$ be a convex cone. Let $F : K \to 2^Y$ be a multivalued mapping with nonempty values.

(1) F is called C_Δ-quasiconvex mapping if, for any pair $x_1, x_2 \in K$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either

$$F(x) \subset F(x_1) - C$$

or

$$F(x) \subset F(x_2) - C.$$

(2) F is called C_Δ-quasiconcave mapping if, for any pair $x_1, x_2 \in K$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either

$$F(x_1) \subset F(x) + C,$$

or

$$F(x_2) \subset F(x) + C.$$

We use $\in$ instead of $\subset$ when F is single-valued.

Remark 2.6. If $Y = \mathbb{R} = (-\infty, +\infty)$ and $C = [0, +\infty)$, and F is a real function, then the C_Δ-quasiconvexity of F is equivalent to the Δ-quasiconvexity of F (see [15]).

Example 2.7. We consider topological semilattice $(\mathbb{R}^2, \leq)$ as in Example 2.2 and $K = [0, 1] \times [0, 1]$ is a Δ-convex subset of $(\mathbb{R}^2, \leq)$.

(1) Let $F : K \to 2^{\mathbb{R}}$ and $C = \mathbb{R}_+$ such that

$$F(x) = [(1 - x_1)(1 - x_2), +\infty), \ \forall x = (x_1, x_2) \in K.$$

It is clear that F is C_Δ-quasiconcave mapping but not C-quasiconcave. Indeed, for $x^1 = (0, 1)$, $x^2 = (1, 0)$, $x = \frac{1}{2}x^1 + \frac{1}{2}x^2 = (\frac{1}{2}, \frac{1}{2})$, we see that

$$F(x^1) = F(x^2) = [0, +\infty), \ F(x) = \left[\frac{1}{4}, +\infty\right)$$

while

$$F(x^1) = F(x^2) = [0, +\infty) \not\subset F(x) + C = \left[\frac{1}{4}, +\infty\right).$$

(2) Let $F : K \to 2^{\mathbb{R}}$ and $C = \mathbb{R}_+$ such that

$$F(x) = \{x_1^2 + x_2^2\}, \ \ \forall x = (x_1, x_2) \in K.$$

It is easy to see that F is C-quasiconvex but not C_Δ-quasiconvex.

Now, we recall the semicontinuous properties of multivalued mappings (see [2]). Let $F : X \to 2^Y$ be a multivalued mapping between topological spaces X and Y. The domain of F is defined to be the set $domF = \{x \in X : F(x) \neq \emptyset\}$.

The mapping F is upper semicontinuous (shortly, usc) at $x_0 \in domF$ if, for any open set V of Y with $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$.

The mapping F is lower semicontinuous (shortly, lsc) at $x_0 \in domF$ if, for any open set V of Y with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

The mapping F is continuous at $x_0 \in domF$ if it is both usc and lsc at x_0. The mapping F is continuous (resp. usc, lsc) if $domF = X$ and if F is continuous (resp. usc, lsc) at each point $x \in X$.

If Y is a partially ordered topological vector space, then the above definitions of semicontinuous can be weakened. More precisely, we can introduce the following definitions taken from [14, 17].

Definition 2.8. Let X be a topological space, Y be a topological vector space with a cone C. Let $F : X \to 2^Y$. We say that

(1) F is C-upper semicontinuous (shortly, C-usc) at $x_0 \in domF$ if for any open set V of Y with $F(x_0) \subset V$ there exists a neighborhood U of x_0 such that

$$F(x) \subset V + C \quad \text{for each } x \in domF \cap U.$$

(2) F is C-lower semicontinuous (shortly, C-lsc) at $x_0 \in domF$ if for any open set V of Y with $F(x_0) \cap V \neq \emptyset$ there exists a neighborhood U of x_0 such that

$$F(x) \cap [V - C] \neq \emptyset \quad \text{for each } x \in domF \cap U.$$

(3) F is C-usc (resp. C-lsc) if $domF = X$ and if F is C-usc (resp. C-lsc) at each point of $domF$.

Remark 2.9. If $Y = \mathbb{R}$ and $C = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ (resp. $C = -\mathbb{R}_+$), F is single-valued and C-usc at x_0, then F is lower semicontinuous (resp. upper semicontinuous) at x_0 in the usual sense.

Remark 2.10. The upper (resp. lower) semicontinuity of F implies the C-upper (resp. C-lower) semicontinuity of F. Example 3.5 in Section 3 will show that the converse statement is no longer true.

It is interesting to note that the following KKM lemma plays a crucial role to prove existence results for equilibrium problems.

Lemma 2.11. (Horvath and Ciscar [13]) *Let X be a topological semilattice with path-connected intervals, $C \subset X$ a nonempty subset of X, and $T : C \to 2^X$ be such that:*

(1) *T has closed values;*
(2) *T is a KKM mapping, i.e., for each nonempty finite subset A of X,*

$$\Delta(A) \subset \bigcup_{x \in A} T(x);$$

(3) *There exists $x_0 \in C$ such that the set $T(x_0)$ is compact.*

Then we have the set $\cap_{x \in C} T(x)$ is not empty.

Definition 2.12. Let X be a topological space and M be a topological semilattice. A mapping $F : X \to 2^M$ is called a Browder-Fan mapping if the following conditions are satisfied:

(1) For each $x \in X$, $F(x)$ is nonempty and Δ-convex;
(2) For each $y \in M$, $F^{-1}(y)$ is open in X.

The following lemma is a special case of [13, Corollary 1, pp. 298].

Lemma 2.13. (Browder-Fan fixed point theorem) *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M and $F : K \to 2^K$ be a Browder-Fan mapping. Then F has a fixed point.*

3. Vector equilibrium problems for vector-valued multifunctions

In this section, under some new assumptions, we prove the existence of solutions for vector equilibrium problems with multifunctions by using KKM lemma in the setting of topological semilattices. Any of our Theorems 3.1-3.4 is a genuine generalization of scalar Ky Fan inequality in topological semilattices. Our results generalize and improve the ones in [15, 18, 19, 21].

Theorem 3.1. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $f : K \times K \to 2^Y$ a multifunction with nonempty values, C a pointed closed convex cone in Y with $\operatorname{int} C \neq \emptyset$ and let $A : K \to 2^K$ be a Browder-Fan mapping and the set*

$$\mathcal{F}(A) = \{x \in K : x \in A(x)\}$$

is open in K. Assume that

(1) $f(x, x) \not\subset \operatorname{int} C,\ \forall x \in K$;
(2) $\forall x \in K,\ f(x, .)$ *is* $-C_\Delta$*-quasiconvex;*
(3) $\forall y \in K,\ f(., y)$ *is* C*-upper semicontinuous;*
(4) $\forall x \in K \setminus \mathcal{F}(A),\ A(x) \cap \{y \in K : f(x, y) \subset \operatorname{int} C\} \neq \emptyset$.

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*, y) \not\subset \operatorname{int} C$ for all $y \in K$.

Proof. By Lemma 2.13, $\mathcal{F}(A)$ is a nonempty set. For $x, y \in K$, we define three multivalued mappings from K to K as follows.

$$P(x) = \{y \in K : f(x, y) \subset \operatorname{int} C\},$$

$$S(x) = \begin{cases} P(x), & \text{if } x \in \mathcal{F}(A), \\ P(x) \cap A(x), & \text{if } x \in K \setminus \mathcal{F}(A), \end{cases}$$

$$T(y) = K \setminus S^{-1}(y).$$

We split the proof into several steps.

Step 1. We show that for any $x \in K$, $P(x)$ is Δ-convex.

Suppose that there exists $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1, y_2\}) \not\subset P(x')$, i.e., there exists $z \in \Delta(\{y_1, y_2\})$ and $z \notin P(x')$; hence $f(x', z) \not\subset \operatorname{int} C$. By (2), we have either

$$f(x', z) \subset f(x', y_1) + C$$

or

$$f(x', z) \subset f(x', y_2) + C.$$

By Lemma 2.4, we have either

$$f(x', z) \subset f(x', y_1) + C \subset \operatorname{int} C + C \subset \operatorname{int} C$$

or

$$f(x', z) \subset f(x', y_2) + C \subset \operatorname{int} C + C \subset \operatorname{int} C$$

which is a contradiction. Therefore, for any $x \in K$, $P(x)$ is Δ-convex.

Step 2. We prove that $P^{-1}(y)$ is open for each $y \in K$.

We have

$$P^{-1}(y) = \{x \in K : f(x, y) \subset \operatorname{int} C\}.$$

For each $y \in K$ and each $x \in P^{-1}(y)$, we have $f(x, y) \subset \operatorname{int} C$. By (3), there exists a neighborhood $U(x)$ of x such that $f(x', y) \subset \operatorname{int} C + C \subset \operatorname{int} C$ whenever $x' \in U(x)$, which implies that $U(x) \subset P^{-1}(y)$, i.e., $P^{-1}(y)$ is open.

Step 3. We verify the closedness of $T(y)$, for every $y \in K$.

One has

$$\begin{aligned} S^{-1}(y) &= (P^{-1}(y) \cap \mathcal{F}(A)) \cup (P^{-1}(y) \cap A^{-1}(y) \cap (K \setminus \mathcal{F}(A))) \\ &= [P^{-1}(y) \cup (P^{-1}(y) \cap A^{-1}(y) \cap (K \setminus \mathcal{F}(A)))] \cap \\ &\quad \cap [\mathcal{F}(A) \cup (P^{-1}(y) \cap A^{-1}(y) \cap (K \setminus \mathcal{F}(A)))] \\ &= \{P^{-1}(y) \cap [P^{-1}(y) \cup (K \setminus \mathcal{F}(A))]\} \cap \{[\mathcal{F}(A) \cup (P^{-1}(y) \cap A^{-1}(y))] \cap K\} \\ &= P^{-1}(y) \cap [\mathcal{F}(A) \cup (P^{-1}(y) \cap A^{-1}(y))]. \end{aligned}$$

Since for any $y \in K$, $A^{-1}(y)$, $P^{-1}(y)$ and $\mathcal{F}(A)$ are open in K, we have $S^{-1}(y)$ is also open in K. It follows that $T(y)$ is closed in K for each $y \in K$.

Step 4. We show T is a KKM mapping in K.

Suppose that T is not a KKM mapping. Hence, there exists $A = \{y_1, y_2, ..., y_n\} \subset K$ such that $\Delta(A) \not\subset \cup_{y \in A} T(y)$. We infer that there exists $\bar{x} \in \Delta(A)$ such that $\bar{x} \notin T(y_i)$ for all $i = 1, 2, ..., n$, namely $y_i \in S(\bar{x})$ for all $i = 1, 2, ..., n$. If $\bar{x} \in \mathcal{F}(A)$, then $y_i \in P(\bar{x})$. That is $\bar{x} \in \Delta(A) \subset P(\bar{x})$ because $P(\bar{x})$ is Δ-convex, which is a contradiction to (1). On the other hand if $\bar{x} \in K \setminus \mathcal{F}(A)$ then $y_j \in P(\bar{x}) \cap A(\bar{x})$ for $i = 1, 2, ..., n$. So $\bar{x} \in \Delta(A) \subset P(\bar{x}) \cap A(\bar{x})$, which is another contradiction. Thus T is KKM.

Step 5. We show that there exists a point $x^* \in K$ such that $S(x^*) = \emptyset$.

Indeed, by Lemma 2.11, we obtain a point $x^* \in K$ such that

$$x^* \in \bigcap_{y \in K} T(y) = K \setminus \bigcup_{y \in K} S^{-1}(y).$$

So, $x^* \notin S^{-1}(y)$ for every $y \in K$, that is $S(x^*) = \emptyset$. Since $P(x) \cap A(x)$ is nonempty for all $x \in K \setminus \mathcal{F}(A)$, hence $x^* \in \mathcal{F}(A)$, $S(x^*) = P(x^*) = \emptyset$, i.e., $x^* \in A(x^*)$ and we have

$$x^* \in A(x^*), \quad f(x^*, y) \not\subset \operatorname{int} C, \quad \text{for all } y \in K.$$

This completes the proof. □

Remark 3.2. Let us underline the following items.

(1) The set $\mathcal{F}(A) = \{x \in K : x \in A(x)\}$ is assumed to be open in K while in most of article appeared in the literature, the set $\mathcal{F}(A)$ is assumed to be closed in K (see [10, 11, 16, 18, 22, 24] and references therein). So, our existence results are obtained under new assumptions different from those of [10, 11, 16, 18, 22, 24].

(2) In the proof of Theorem 3.1 we used KKM lemma, while the authors of [10, 11, 16, 18, 22, 24] used Browder-Fan fixed point theorem. Hence, our proof techniques are different.
(3) This theorem improve and extend Corrolary 3.1 in [18], Corrolary 16 in [19].

In Theorem 3.1, when f is single-valued, we have the following corollary.

Corollary 3.3. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $f : K\times K \to Y$, C a pointed closed convex cone in Y with $\operatorname{int} C \neq \emptyset$ and let $A : K \to 2^K$ be a Browder-Fan mapping and the set*

$$\mathcal{F}(A) = \{x \in K : x \in A(x)\}$$

is open in K. Assume that

(1) $f(x, x) \notin \operatorname{int} C$, $\forall x \in K$;
(2) $\forall x \in K$, $f(x, .)$ *is* $-C_\Delta$*-quasiconvex;*
(3) $\forall y \in K$, $f(., y)$ *is* C*-upper semicontinuous;*
(4) $\forall x \in K \setminus \mathcal{F}(A)$, $A(x) \cap \{y \in K : f(x, y) \in \operatorname{int} C\} \neq \emptyset$.

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*, y) \notin \operatorname{int} C$ for all $y \in K$.

Remark 3.4. Corollary 3.3 improve and extend Theorem 5.2 in [21], Theorem 3.1 in [15].

Now we give an example to explain that Corollary 3.3 is applicable.

Example 3.5. Let $M = \mathbb{R}^2$. Arguing as in Example 2.2, $(M, \leq)$ is a topological semilattice. Then $K = [0, 1] \times [0, 1]$ is a nonempty compact Δ-convex subset of $(M, \leq)$. Let $Y = \mathbb{R}$ with $C = \mathbb{R}_+$. Define $A : K \to 2^K$ by

$$A(x) = 1 \times (0, 1] \cup (0, 1] \times 1 \quad \text{for each } x \in K.$$

Denote by L_1 the set $1 \times (0, 1] \cup (0, 1] \times 1$. Then we have:

(1) for each $x \in K$, $A(x)$ is nonempty and Δ-convex;
(2) for $y = (y_1, y_2) \in K$,

$$A^{-1}(y) = \begin{cases} K & \text{if } \ y \in L_1, \\ \emptyset & \text{if } \ y \in K \setminus L_1. \end{cases}$$

Therefore, for each $y \in K$, $A^{-1}(y)$ is open in K. It means that A is a Browder-Fan mapping.
(3) The set $\mathcal{F}(A) = \{x \in K : x \in A(x)\} = 1 \times (0, 1] \cup (0, 1] \times 1$ is open in K.
For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in K$, we define $f : K \times K \to Y$ by

$$f(x, y) = \begin{cases} -(1 - y_1)(1 - y_2) + 1 - \frac{1}{2}x_1 - \frac{1}{2}x_2 & \text{if } (x, y) \neq (0, 0), \\ -2 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then all the assumptions of Corollary 3.3 are satisfied. So Corollary 3.3 is applicable. We can see that $x^* = (1, 1)$ is the unique solution of (VEP1).

Remark 3.6. For every fixed x, arguing as in Example 2.1 of [18], we see that $f(x,.)$ is not a usual quasiconcave function. Indeed, for $\bar{x} = (1,1)$, we have

$$f(\bar{x}, y) = \begin{cases} -(1-y_1)(1-y_2) & \text{if } (y_1, y_2) \neq (0,0), \\ -2 & \text{if } (y_1, y_2) = (0,0). \end{cases}$$

Clearly, for $y^1 = (1,0)$, $y^2 = (0,1)$, $y = \frac{1}{2}y^1 + \frac{1}{2}y^2 = (\frac{1}{2}, \frac{1}{2})$, we see that $f(\bar{x}, y^1) = 0$, $f(\bar{x}, y^2) = 0$, while $f(\bar{x}, y) = -\frac{1}{4}$.

Remark 3.7. Observe that Corrolary 3.3 fails to hold if the assumption that the set $\mathcal{F}(A)$ is open in K is violated.

This remark is illustrated by the following example.

Example 3.8. Let M, K, Y, C be given in Example 3.5 and $L_2 = 1 \times [0,1] \cup [0,1] \times 1$.

(1) For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in K$, the function f is defined by

$$f(x, y) = \begin{cases} 1 - x_1^2 - x_2^2 + y_1^2 + y_2^2 & \text{if } (x, y) \neq (0,0), \\ -1 & \text{if } (x, y) = (0,0). \end{cases}$$

It can be easily checked that for each $x \in K$, $f(x,.)$ is C_Δ-quasiconcave but not a usual quasiconcave function.

(2) We define the multivalued mapping $A : K \to 2^K$ as in Example 17 of [19]:

$$A(x) = \begin{cases} (x_1, 1] \times [0,1] \cup [0,1] \times (x_2, 1] & \text{if } x \in K \setminus L_2, \\ (1,1) & \text{if } x \in L_2. \end{cases}$$

for each $x = (x_1, x_2) \in K$.

Then, A is a Browder-Fan mapping. Obviously, in this example, the set $\mathcal{F}(A)$ is not open in K, because $\mathcal{F}(A) = \{(1,1)\}$. It is easy to see that each of conditions (1), (2), (3), (4) of Corrolary 3.3 is satisfied. However, (VEP1) has no solution. Indeed, if x^* is a solution of (VEP1) then $x^* = (1,1)$ and

$$f(x^*, y) = -1 + y_1^2 + y_2^2 \leq 0 \quad \text{for all } y \in K.$$

That means that $y_1^2 + y_2^2 \leq 1$ for all $y \in K$, which is impossible.

When $Y = (-\infty, +\infty)$, $C = [0, +\infty)$ and $A(x) = K$, $\forall x \in K$, from Corollary 3.3, we get scalar Ky Fan inequality for real-valued functions in topological semilattices (see, for instance, [15, 21]).

Corollary 3.9. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M and let $f : K \times K \to \mathbb{R}$ be such that*

(1) $f(x,x) \leq 0$, $\forall x \in K$*;*
(2) $\forall x \in K$, $f(x,.)$ *is Δ-quasiconcave;*
(3) $\forall y \in K$, $f(.,y)$ *is lower semicontinuous.*

Then there exists $x^* \in K$ such that $f(x^*, y) \leq 0$ $\forall y \in K$.

Theorem 3.10. *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, Y a topological vector space, $f : K \times K \to$*

2^Y *a multifunction with nonempty values,* C *a pointed closed convex cone in* Y *with* $\operatorname{int} C \neq \emptyset$ *and let* $A : K \to 2^K$ *be a Browder-Fan mapping and the set*

$$\mathcal{F}(A) = \{x \in K : x \in A(x)\}$$

is open in K*. Assume that*

(1) $f(x, x) \cap \operatorname{int} C = \emptyset,\ \forall x \in K$;
(2) $\forall x \in K,\ f(x, .)$ *is* C_Δ*-quasiconvex;*
(3) $\forall y \in K,\ f(., y)$ *is* $-C$*-lower semicontinuous;*
(4) $\forall x \in K \setminus \mathcal{F}(A),\ A(x) \cap \{y \in K : f(x, y) \cap \operatorname{int} C \neq \emptyset\} \neq \emptyset$.

Then there exists $x^* \in K$ *such that* $x^* \in A(x^*)$ *and* $f(x^*, y) \cap \operatorname{int} C = \emptyset$ *for all* $y \in K$.

Proof. By Lemma 2.13, $\mathcal{F}(A)$ is a nonempty set. For $x, y \in K$, we define three multivalued mappings from K to K as follows.

$$P(x) = \{y \in K : f(x, y) \cap \operatorname{int} C \neq \emptyset\},$$
$$S(x) = \begin{cases} P(x), & \text{if } x \in \mathcal{F}(A), \\ P(x) \cap A(x), & \text{if } x \in K \setminus \mathcal{F}(A), \end{cases}$$
$$T(y) = K \setminus S^{-1}(y).$$

The rest of the proof can be done as in proving Theorem 3.1, so it is omitted. □

Theorem 3.11. *Let* K *be a nonempty compact* Δ*-convex subset of a topological semilattice with path-connected intervals* M*,* Y *a topological vector space,* $f : K \times K \to 2^Y$ *a multifunction with nonempty values,* C *a pointed closed convex cone in* Y *with* $\operatorname{int} C \neq \emptyset$ *and let* $A : K \to 2^K$ *be a Browder-Fan mapping and the set*

$$\mathcal{F}(A) = \{x \in K : x \in A(x)\}$$

is open in K*. Assume that*

(1) $f(x, x) \cap (-C) \neq \emptyset,\ \forall x \in K$;
(2) $\forall x \in K,\ f(x, .)$ *is* $-C_\Delta$*-quasiconvex;*
(3) $\forall y \in K,\ f(., y)$ *is* C*-upper semicontinuous;*
(4) $\forall x \in K \setminus \mathcal{F}(A),\ A(x) \cap \{y \in K : f(x, y) \cap (-C) = \emptyset\} \neq \emptyset$.

Then there exists $x^* \in K$ *such that* $x^* \in A(x^*)$ *and* $f(x^*, y) \cap (-C) \neq \emptyset$ *for all* $y \in K$.

Proof. By Lemma 2.13, $\mathcal{F}(A)$ is a nonempty set. For $x, y \in K$, we define three multivalued mappings from K to K as follows.

$$P(x) = \{y \in K : f(x, y) \cap (-C) = \emptyset\},$$
$$S(x) = \begin{cases} P(x), & \text{if } x \in \mathcal{F}(A), \\ P(x) \cap A(x), & \text{if } x \in K \setminus \mathcal{F}(A), \end{cases}$$
$$T(y) = K \setminus S^{-1}(y).$$

The rest of the proof can be proceeded exactly as the one of Theorem 3.1, so it is omitted. □

Theorem 3.12 *Let* K *be a nonempty compact* Δ*-convex subset of a topological semilattice with path-connected intervals* M*,* Y *a topological vector space,* $f : K \times K \to 2^Y$

a multifunction with nonempty values, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$ *and let* $A : K \to 2^K$ *be a Browder-Fan mapping and the set*

$$\mathcal{F}(A) = \{x \in K : x \in A(x)\}$$

is open in K. Assume that

(1) $f(x,x) \subset -C,\ \forall x \in K$;
(2) $\forall x \in K,\ f(x,.)$ *is* $-C_\Delta$*-quasiconcave;*
(3) $\forall y \in K,\ f(.,y)$ *is* $-C$*-lower semicontinuous;*
(4) $\forall x \in K \setminus \mathcal{F}(A),\ A(x) \cap \{y \in K : f(x,y) \not\subset -C\} \neq \emptyset$.

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and $f(x^*,y) \subset -C$ for all $y \in K$.

Proof. By Lemma 2.13, $\mathcal{F}(A)$ is a nonempty set. For $x, y \in K$, we define three multivalued mappings from K to K as follows.

$$P(x) = \{y \in K : f(x,y) \not\subset -C\},$$

$$S(x) = \begin{cases} P(x), & \text{if } x \in \mathcal{F}(A), \\ P(x) \cap A(x), & \text{if } x \in K \setminus \mathcal{F}(A), \end{cases}$$

$$T(y) = K \setminus S^{-1}(y).$$

The rest of the proof is similar to that of Theorem 3.1, so it is omitted. □

Remark 3.13. Other interesting results on topological semilattices and vector equilibrium problems can be found in [13, 15, 16, 18, 19, 21, 22, 23, 24].

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