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NEW RESULTS ON THE EXISTENCE OF THE GENERALIZED PARETO EQUILIBRIUM

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Abstract. In this paper, we state a new fixed point theorem for correspondences defined on Hausdorff locally convex spaces and we use it to prove the existence of the generalized weighted Nash equilibrium and the generalized Pareto equilibrium of a constrained multi-criteria game.

Key Words and Phrases: fixed point theorem, generalized weighted Nash equilibrium, generalized Pareto equilibrium, generalized multiobjective game.
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1. Introduction

The pioneer work of Nash [12] first proved a theorem of equilibrium existence for strategic games, and since then, equilibrium problems have been extensively studied. It is important to mention some milestones in the development of the theory, for a better understanding of the novelty brought by this paper to the domain. Nash's ideas were extended by various authors in different ways. For a survey of the results on this topic, the interested reader is referred to [13]. One of the directions of research is the multi-criteria games and their importance is emphasized by its application in real-world situations. The existence of the Pareto equilibrium in this class of games has been proven by considering different approaches, such as fixed point techniques, Ky Fan minimax inequality, quasi-equilibrium theorems or quasi-variational inequalities. A considerable number of papers devotes to applications in the financial markets (see [3] or [7]) and other specialized economic fields. We mention the works of Borm, Megen and Tijs [1], who introduced the concept of perfectness for multi-criteria games and Voorneveld, Grahn and Dufwenberg [15], who studied the existence of ideal equilibria. Other authors, as Yu (see [17]), showed the existence of a solution for multiobjective games by using new concepts of continuity and convexity.

The existence of Pareto equilibria in game theory with vector payoffs has been considered in the past decades by Chebbi [2], Ding [4], Ding [5], Hesth and Ku [7], Kim [9], Kim and Ding [10], Patriche [14], Wang [16], Yu [17], Yu and Yuan [18], Yuan and Tarafdar [20]. A reference work is the paper of Zeleny [21].

This paper provides sufficient conditions for showing the existence of Pareto equilibria. By using an approximation technique, we prove a fixed point theorem for correspondences defined on Hausdorff locally convex spaces and we use it to prove the existence of the generalized weighted Nash equilibrium and the generalized Pareto equilibrium of a constrained multi-criteria game.

The rest of the paper is organized in the following way: Section 2 contains preliminaries and notation. The fixed point theorem is presented in Section 3. Section 4 contains the model of a constrained multiobjective game and a Pareto equilibrium existence result.

2. Preliminaries and notation

We shall denote by $\mathbb{R}^m_+ := \{u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m : u_j \geq 0\ \forall j = 1, 2, ..., m\}$ and $\text{int}\mathbb{R}^m_+ := \{u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m : u_j > 0\ \forall j = 1, 2, ..., m\}$ the non-negative orthant of $\mathbb{R}^m$ and respective the non-empty interior of $\mathbb{R}^m_+$ with the topology induced in terms of convergence of vector with respect to the Euclidian metric. For each $u, v \in \mathbb{R}^m$, $u \cdot v$ denotes the standard Euclidian inner product.

Now, we present some notations and results concerning the theory of correspondences.

Let A be a subset of a topological space X. $\mathcal{F}(A)$ denotes the family of all nonempty finite subsets of A. 2^A denotes the family of all subsets of A. cl$A$ denotes the closure of A in X. If A is a subset of a vector space, co$A$ denotes the convex hull of A. Let Y be a real topological vector space. If F, $G : X \to 2^Y$ are correspondences, then co$G$, cl$G$, $G \cap F : X \to 2^Y$ are correspondences defined by $(\text{co}G)(x) = \text{co}G(x)$, $(\text{cl}G)(x) = \text{cl}G(x)$ and $(G \cap F)(x) = G(x) \cap F(x)$ for each $x \in X$, respectively. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$.

The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X\times Y}\text{Gr}(T)\}$ (the set $\text{cl}_{X\times Y}\text{Gr}(T)$ is called the adherence of the graph of T). It is easy to see that $\text{cl}T(x) \subset \overline{T}(x)$ for each $x \in X$.

Remark 2.1. *$\overline{T}(x) = clT(x)$ for each $x \in X$ if T has a closed graph in $X \times Y$ (by Theorem 7.1.15 in* [11]*, it follows that in particular, T has a closed graph when Y is regular and clT is upper semicontinuous with closed values).*

Remark 2.2. *$\overline{T}$ may not have convex values, even if T is convex valued.*

The next example shows the validity of the above statement.

Example 2.1. Let $D = [1, 2]$ and $T : [0, 2] \to 2^{[0,4)}$ be the correspondence defined by

$$T(x) = \begin{cases} [0,1], & \text{if} \quad x \in [0,1); \\ \phi, & \text{if} \quad x = 1; \\ [2,3], & \text{if} \quad x \in (1,2]. \end{cases}$$

Then,

$$\overline{T}(x) = \begin{cases} [0,1], & \text{if} \quad x \in [0,1); \\ [0,1] \cup [2,3], & \text{if} \quad x = 1; \\ [2,3], & \text{if} \quad x \in (1,2]. \end{cases}$$

Further, we shall use the following notation.

Notation 2.1. Let X be a topological space, Y be a topological vector space and let $T, R, S : X \to 2^Y$ be correspondences. We denote $T^{S,R}$ the correspondence $T^{S,R} : X \to 2^Y$ defined by $T^{S,R}(x) = (T(x) + S(x)) \cap R(x)$ for each $x \in X$.

Lemma 2.1. *Let X be a topological space, Y be a nonempty subset of a topological vector space E and $T, R : X \to 2^Y$ be correspondences. Let $\mathcal{S}$ be the family of all open valued correspondences $S : X \to 2^Y$ such that $0 \in S(x)$ for each $x \in X$. Then,*

$$\bigcap_{S \in \mathcal{S}} \overline{T^{S,R}}(x) \subseteq \overline{T}(x) \cap \overline{R}(x) \text{ for every } x \in X.$$

Proof. Let be x and y be such that $y \in \bigcap_{S \in \mathcal{S}} \overline{T^{S,R}}(x)$. Obviously, $y \in \overline{R}(x)$ and suppose, by way of contradiction, that $y \notin \overline{T}(x)$. This means that $(x, y) \notin \mathrm{clGr}(T)$, so that there exists an open neighborhood U of x and V an open neighborhood of zero in E such that:

$$(U \times (y + V)) \cap Gr(T) = \emptyset. \tag{2.1}$$

Choose $S \in \mathcal{S}$ and U_1 an open neighbourhood of x such that $S(x) - S(x') \subseteq V$ for each $x' \in U_1$.

Since $y \in \overline{T^{S,R}}(x)$, then $(x, y) \in \mathrm{clGr}(T^{S,R})$, so that

$$(U_1 \times (y + S(x))) \cap \mathrm{Gr}(T^{S,R}) \neq \emptyset.$$

Take any $x' \in U_1$ and $w' \in S(x)$ such that $(x', y + w') \in \mathrm{Gr}(T^{S,R})$, i.e. $y + w' \in T^S(x')$. Then, $y + w' \in R(x')$ and $y + w' = y' + w''$ for some $y' \in T(x')$ and $w'' \in S(x')$. Hence, $y' = y + (w' - w'') \in y + (S(x) - S(x')) \subseteq y + V$, so that $T(x') \cap (y + V) \neq \emptyset$. Since $x' \in U_1$, this means that $(U_1 \times (y + V)) \cap \mathrm{Gr}(T) \neq \emptyset$, and, furthermore, $((U_1 \cap U) \times (y + V)) \cap \mathrm{Gr}(T) \neq \emptyset$, contradicting (2.1).

As a particular case, we obtain the following result.

Lemma 2.2. *Let X be a topological space, Y be a nonempty subset of a topological space E and $T : X \to 2^Y$ be a correspondence. Let ß be a basis of open neighbourhoods of 0 in E and let D be a compact subset of Y. If for each $V \in$ß, the correspondence $T^V : X \to 2^Y$ is defined by $T^V(x) = (T(x) + V) \cap D$ for each $x \in X$, then $\cap_{V \in ß} \overline{T^V}(x) = \overline{T}(x) \cap D$ for every $x \in X$.*

3. New fixed point theorems

The next theorem provides sufficient conditions for the existence of fixed points for correspondences defined on Hausdorff locally convex spaces. An approximation technique is used to prove our result. Section 4 will set the main contribution of this paper, related to the existence of Pareto equilibria of multi-criteria games and the fixed point approach is based on Theorem 1.

Theorem 3.1. *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex compact subset of a Hausdorff locally convex topological vector space E_i, $X := \prod_{i \in I} X_i$, D_i be a nonempty compact convex subset of X_i and S_i be the family of all open convex valued correspondences $S_i : X \to 2^{X_i}$ such that $0 \in S_i(x)$ for each $x \in X$. Let $T_i, R_i : X \to 2^{X_i}$ be correspondences with the following conditions:*

1) for each $x \in X$, $\overline{T}_i(x) \subset Q_i(x)$ and $R_i(x) \subseteq D_i$;

2) for each $S_i \in S_i$, $\overline{T_i^{S_i,R_i}}$ is convex and nonempty valued on D;

Then, there exists $x^* \in D := \prod_{i \in I} D_i$ *such that* $x_i^* \in Q_i(x^*)$ *for each* $i \in I$.

Proof. Since D_i is compact, $D := \prod_{i \in I} D_i$ is also compact in X. By assumption 2), for each $i \in I$, $\overline{T_i^{S_i,R_i}}$ is nonempty closed convex valued. Since $\overline{T_i^{S_i,R_i}}$ has a closed graph, it is upper semicontinuous. Let's define $T^{S,R} : D \to 2^D$ by $T^{S,R}(x) = \prod_{i \in I} \overline{T_i^{S_i,R_i}}(x)$ for each $x \in D$. The correspondence $T^{S,R}$ is upper semicontinuous with nonempty closed convex values. Therefore, according to Himmelberg's fixed point theorem [8], there exists $x_S^* = \prod_{i \in I} x_{S_i}^* \in D$ such that $x^* \in T^{S,R}(x^*)$. It follows that $x_{S_i}^* \in \overline{T_i^{S_i,R_i}}(x_S^*)$ for each $i \in I$.

For each $S = (S_i)_{i \in I} \subset \mathcal{S}$, let's define $Q_S = \bigcap_{i \in I} \{x \in D : x_i \in \overline{T_i^{S_i,R_i}}(x)\}$.

Q_S is nonempty since $x_S^* \in Q_S$, then Q_S is nonempty and closed.

We prove that the family $\{Q_S : S \in \mathcal{S}\}$ has the finite intersection property.

Let $\{S^{(1)}, S^{(2)}, ..., S^{(n)}\}$ be any finite set and let $S^{(k)} = (S_i^{(k)})_{i \in I}$, $k = 1, ..., n$. For each $i \in I$ and $x \in X$, let $S_i(x) = \bigcap_{k=1}^{n} S_i^{(k)}(x)$, then $S_i \in \mathcal{S}$; If $S = (S_i)_{i \in I}$, clearly $Q_S \subset \bigcap_{k=1}^{n} Q_{S^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{S^{(k)}} \neq \emptyset$.

Since D is compact and the family $\{Q_S : S \in \mathcal{S}\}$ has the finite intersection property, we have that $\cap\{Q_S : S \in \mathcal{S}\} \neq \emptyset$. Take any $x^* \in \cap\{Q_S : S \in \mathcal{S}\}$, then for each $S_i \in \mathcal{S}$, $x_i^* \in \overline{T_i^{S_i,R_i}}(x^*)$. According to Lemma 2.1, we have that $x_i^* \in \overline{T_i}(x^*)$, for each $i \in I$, therefore $x_i^* \in Q(x^*)$.

Theorem 3.2 is obtained as a particular case of Theorem 3.1.

Theorem 3.2. *Let* I *be an index set. For each* $i \in I$, *let* X_i *be a nonempty convex subset of a Hausdorff locally convex topological vector space* E_i, D_i *be a nonempty compact convex subset of* X_i *and* $T_i, Q_i : X := \prod_{i \in I} X_i \to 2^{X_i}$ *be two correspondences with the following conditions:*

1) for each $x \in X$, $\overline{T}_i(x) \subset Q_i(x)$.

2) $\overline{T_i} \cap D_i$ *is convex nonempty valued on* D.

Then, there exists $x^* \in D := \prod_{i \in I} D_i$ *such that* $x_i^* \in Q_i(x^*)$ *for each* $i \in I$.

A simpler proof will be provided below for this particular result.

Proof. Since D_i is compact, $D := \prod_{i \in I} D_i$ is also compact in X. According to assumption 2) each $\overline{T_i} \cap D_i : X \to 2^{D_i}$ is nonempty closed convex valued on D. Since $\overline{T_i}$ has a closed graph, it is upper semicontinuous. Let's define $T : D \to 2^D$ by

$T(x) = \prod_{i \in I}(\overline{T_i}(x) \cap D_i)$ for each $x \in D$. The correspondence T is upper semicontinuous with nonempty closed convex values. Therefore, according to Himmelberg's fixed point theorem [8], there exists $x^* = \prod_{i \in I} x_i^* \in D$ such that $x^* \in T(x^*)$. It follows that $x_i^* \in \overline{T_i}(x^*) \subset Q_i(x)$ for each $i \in I$.

If $|I| = 1$ we get the following result.

Corollary 3.1. *Let X be a nonempty convex subset of a Hausdorff locally convex topological vector space F, D be a nonempty compact convex subset of X and $T, Q : X \to 2^X$ be two correspondences with the following conditions:*

1) *for each $x \in X$, $\overline{T}(x) \subset Q(x)$ and $T(x) \neq \emptyset$,*
2) *$\overline{T} \cap D$ is convex nonempty valued on D.*

Then, there exists a point $x^* \in D$ such that $x^* \in Q(x^*)$.

In the particular case that the correspondence $T = Q$, the following result stands.

Corollary 3.2. Let X be a nonempty convex subset of a *Hausdorff locally convex topological vector* space F, D be a nonempty compact convex subset of X and $T : X \to 2^X$ be a correspondence such that $\overline{T} \cap D$ *is convex nonempty valued.* Then, there exists a point $x^* \in D$ such that $x^* \in \overline{T}(x^*)$.

4. The model of a generalized multiobjective game and the existence of generalized Pareto equilibrium

The purpose of this section is to make a preliminary unitary presentation of the model of a constrained multi-criteria game in its strategic form and of the solution concepts for this type of game, and also to state an existence result for generalized Pareto equilibria.

Let I be a finite set (the set of players). For each $i \in I$, let X_i be the set of strategies and define $X = \prod_{i \in I} X_i$. Let $T^i : X \to 2^{\mathbb{R}^{k_i}}$, where $k_i \in \mathbb{N}$, be the multi-criteria payoff function and let $A^i : X \to 2^{X_i}$ be a constraint correspondence.

Definition 4.1. (see [10]) The family $G = (X_i, A^i, T^i)_{i \in I}$ is called a *generalized multi-criteria (multiobjective) game.*

Any n-tuple of strategies can be regarded as a point in the product space of sets of players' strategies: $x = (x_1, x_2, ...x_n) \in X$. For each player $i \in I$, the vector of the $n-1$ strategies of the other ones will be denoted by $x_{-i} = (x_1, ...x_{i-1}, x_{i+1}, ...x_n) \in X_{-i} = \prod_{j \in I \setminus \{i\}} X_i$. We note that $x = (x_{-i}, x_i)$.

We assume that each player is trying to minimize his/her own payoff according to his/her preferences, where for each player $i \in I$, the preference " $\gtrsim_i$ " over the outcome space $\mathbb{R}^{k_i}$ is the following:

$z^1 \gtrsim_i z^2$ if only if $z_j^1 \geq z_j^2$ for each $j = 1, 2, ...k_i$ and $z^1, z^2 \in \mathbb{R}^{k_i}$. The following preference can be defined on X for each player i (see [10]):

$x \gtrsim_i y$ whenever $F^i(x) \gtrsim_i F^i(y)$ and $x, y \in X$.

Let $x^* = (x_1^*, x_2^*, ..., x_n^*) \in X$.

We introduce slight generalizations of the equilibrium concepts defined by Kim and Ding in [10].

Definition 4.2. A strategy $x_i^* \in X_i$ of player i is said to be a *generalized Pareto efficient strategy (respectively, a weak Pareto efficient strategy) with respect to* x if $x_i^* \in \overline{A^i}(x^*)$ and there is no strategy $x_i \in A^i(x^*)$ such that

$T^i(x^*) - T^i(x^*_{i-1}, x_i) \in \mathbb{R}_+^{k_i} \backslash \{0\}$ (respectively, $T^i(x^*) - T^i(x^*_{i-1}, x_i) \in \text{int}\mathbb{R}_+^{k_i} \backslash \{0\}$).

Definition 4.3. A strategy $x^* \in X$ is said to be a *generalized Pareto equilibrium (respectively, a weak Pareto equilibrium) of a game* $G = (X_i, A^i, T^i)_{i\in I}$, if for each player $i \in I$, $x_i^* \in X_i$ is a Pareto efficient strategy against x^* (respective, a generalized weak Pareto efficient strategy against x^*).

The following notion contains the idea of a game equilibrium defined by using a scalarization function. In this case, the scalarization method uses weighted coefficients W_i, so that each player i has his own vector of weights $W_i \in \mathbb{R}_+^{k_i} \backslash \{0\}$.

Definition 4.4. A strategy $x^* \in X$ is said to be a *generalized weighted Nash equilibrium* with respect to the weighted vector $W = (W_i)_{i\in I}$ with $W_i = (W_{i,1}, W_{i,2}, ..., W_{i,k_i}) \in \mathbb{R}_+^{k_i}$ of the multiobjective game $G = ((X_i, A^i, T^i)_{i\in I}$, if for each player $i \in I$, we have:

1) $x_i^* \in \overline{A^i}(x^*)$;

2) $W_i \in \mathbb{R}_+^{k_i} \backslash \{0\}$;

3) for all $x_i \in A^i(x^*)$, $W_i \cdot T^i(x^*) \leq W_i \cdot T^i(x^*_{-i}, x_i)$, where $\cdot$ denotes the inner product in $\mathbb{R}^{k_i}$.

Remark 4.1. In particular, if $W_i \in \Delta^{k_i} = \left\{ u_i \in \mathbb{R}_+^{k_i} \text{ with } \sum_{j=1}^{k_i} u_{i,j} = 1 \right\}$ for each $i \in I$, then the strategy $x^* \in X$ is said to be a normalized generalized weighted Nash equilibrium with respect to W.

Remark 4.2. If for each $i \in I$, $\overline{A^i}$ has closed values and a closed graph in $X \times X_i$, the notions of equilibrium introduced above coincide with the equilibrium notions defined by Kim and Ding in [10].

The relationship between the two types of equilibrium notions is given by the following result.

Lemma 4.1. *Each normalized generalized weighted Nash equilibrium* $x^* \in X$ *with a weight* $W = (W_1, ...W_n) \in \Delta^{k_1} \times ... \times \Delta^{k_n}$ *(respectively,* $W = (W_1, ..., W_n) \in int\Delta^{k_1} \times ... \times int\Delta^{k_n}$*) is a weak Pareto equilibrium (respectively, a Pareto equilibrium) of the game* $G = ((X_i, A^i, T^i)_{i\in I}$.

The proof follows the same line as the proof of Lemma 7 in [10].

Remark 4.3. As in [10], the above lemma remains true when $W = (W_1, ..., W_n)$ satisfies $W_i \in \mathbb{R}_+^{k_i}$ (resp. $W_i \in \text{int}\mathbb{R}_+^{k_i}$).

In order to prove the existence result for generalized weighted Nash equilibrium of generalized multiobjective games, first we prove the following lemma.

Lemma 4.2. *Let* X *be a nonempty convex compact of a Hausdorff locally convex topological vector space* E, D *be a nonempty compact convex subset of* X, $A : X \to 2^X$ *be a correspondence with non-empty convex values and* $f : X \times X \to \overline{\mathbb{R}}$ *be a function such that:*

1) $\overline{A}$ *is nonempty convex valued;*

2) *The correspondence* $F : X \to 2^X$, $F(x) = \{y \in X : f(x,x) - f(y,x) > 0\}$ *is such that* $\overline{F}$ *is nonempty convex valued on* $K = \{x \in X : x \in \overline{A}(x)\}$;

3) $x \notin \overline{F}(x)$ *for each* $x \in K$.

Then, there exists $x^* \in X$ *such that* $x^* \in \overline{A}(x^*)$ *and* $f(x^*, x^*) \leq f(y, x^*)$ *for each* $y \in \overline{A}(x^*)$.

Proof. We notice first that the set $K = \{x \in X : x \in \overline{A}(x)\}$ is closed.

Assume that for each $x \in K$, $A(x) \cap F(x) \neq \phi$ and define the correspondence $G : X \to 2^X$ by

$$G(x) = \begin{cases} A(x) \cap F(x) & \text{if} \quad x \in K; \\ A(x) & \text{if} \quad x \notin K. \end{cases}$$

By 1) and 3), the correspondence $\overline{G} : X \to 2^X$ has nonempty convex closed values. By Corollary 3.2, there exists $x^* \in X$ such that $x^* \in \overline{G}(x^*)$. By definition of G and A, x^* must be in K. It follows that $x^* \in \overline{A \cap F}(x^*)$, and since clGr$(A \cap F)$ ⊂clGr(A)∩clGr(F), we have that $x^* \in \overline{A}(x^*) \cap \overline{F}(x^*)$, that is $x^* \in \overline{F}(x^*)$, which contradicts 3). Therefore, there exists $x^* \in K$ such that $\overline{A}(x^*) \cap \overline{F}(x^*) = \phi$ (this implies also $A(x^*) \cap F(x^*) = \phi$). Hence

$$x^* \in \overline{A}(x^*) \text{ and } f(x^*, x^*) \leq f(y, x^*) \text{ for each } y \in \overline{A}(x^*).$$

Example 4.1. Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$,

$$f(x,y) = \begin{cases} 1, & \text{if} \quad (x,y) = (-1,0); \\ 2, & \text{if} \quad (x,y) = (0,0); \\ 1, & \text{if} \quad x,y \in [0,1) \times [0,1] \backslash \{(0,0)\}; \\ 2, & \text{if} \quad (x,y) \in (\frac{1}{2}, 1] \times [-1,0); \\ 3, & \text{if} \quad (x,y) \in [-1, \frac{1}{2}] \times [-1,0) \cup \{(-1,0) \times \{0\}\}; \\ 4, & \text{if} \quad (x,y) \in [-1,0) \times (0,1]; \\ 0, & \text{if} \quad (x,y) \in \{1\} \times (0,1]. \end{cases}$$

Let $A : [-1,1] \to 2^{[-1,1]}$ defined by $A(x) = [-1,0]$ if $x \in [-1,1]$. $\overline{A^V}$ is nonempty and convex valued and $K = \{x \in [-1,1] : x \in \overline{A}(x)\} = [-1,0]$ is closed.

$$F : X \to 2^X, \; F(x) = \{y \in X : f(x,x) - f(y,x) > 0\}$$

$$F(x) = \begin{cases} (\frac{1}{2}, 1], & \text{if} \quad x \in [-1,0); \\ \{-1\}, & \text{if} \quad x = 0; \\ \{1\}, & \text{if} \quad x \in (0,1]. \end{cases}$$

F is neither lower semicontinuous, nor upper semicontinuous and $x \notin \overline{F}(x)$, $\forall x \in K = [-1,0]$, where

$$\overline{F}_{|K}(x) = \begin{cases} [\frac{1}{2}, 1], & \text{if} \quad x \in [-1,0); \\ \{-1\} \cup [\frac{1}{2}, 1], & \text{if} \quad x = 0. \end{cases}$$

$\overline{F}_{|K} \cap K$ is nonempty convex valued.

By Lemma 4.2, we have that there is $x^* \in \overline{A}(x^*)$ such that $A(x^*) \cap F(x^*) = \phi$.

For example, $x^* = -\frac{1}{2}$, $-\frac{1}{2} \in \overline{A}(-\frac{1}{2})$ and $-\frac{1}{2} \notin F(-\frac{1}{2})$, that is $3 = f(-\frac{1}{2}, -\frac{1}{2}) \geq f(y, -\frac{1}{2}) = 3$ for each $y \in \overline{A}(-\frac{1}{2}) = [-1,0]$.

Now, as an application of Lemma 4.2, we have the following existence theorem of generalized weighted Nash equilibrium for generalized multiobjective games.

Theorem 4.1. *Let I be a finite set of indices, let $(X_i, A^i, T^i)_{i\in I}$ be a constrained multi-criteria game with for each $i \in I$, X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E^i and suppose that there is a nonempty compact convex subset D of $X = \prod_{i\in I} X_i$ and a weighted vector $W = (W_1, W_2, ..., W_n)$ with $W_i \in R_+^{k_i}\backslash\{0\}$ such that the following conditions are satisfied:*

1) for each $i \in I$, $\overline{A^i}$ is convex nonempty valued;

2) The set $K = \{x \in X : x \in \overline{A}(x)\}$, where $A(x) = \prod_{i\in I} A^i(x)$, is closed in X;

3) The correspondence $F : X \to 2^X$,

$$F(x) = \{y \in X : \sum_{i=1}^{n} W_i \cdot (T^i(x_{-i}, x_i) - T^i(x_{-i}, y_i)) > 0\}$$

is such that $\overline{F}$ is nonempty convex valued;

4) $x \notin \overline{F}(x)$ for each $x \in K$.

Then, there exists $x^* \in X$ such that x^* is a generalized weighted Nash equilibria with respect to W.

Proof. Define the function $f : X \times X \to \mathbb{R}$ by

$$f(x, y) = \sum_{i=1}^{n} W_i \cdot (T^i(x_{-i}, x_i) - T^i(x_{-i}, y_i)),\ (x, y) \in X \times X.$$

It is easy to see that f satisfies all the hypotheses of Lemma 4.2, hence there exists $x^* \in X$ such that $x^* \in \overline{A}(x^*)$ and $\sum_{i=1}^{n} W_i \cdot (T^i(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i) \leq 0$ for any $y \in \overline{A}(x^*)$. We use the fact that $\prod_{i\in I} A^i \subseteq \overline{\prod_{i\in I} A^i} \subseteq \prod_{i\in I} \overline{A^i}$. We obtain first $x^*_i \in \overline{A_i}(x^*)$ for each $i \in I$. For any given $i \in I$ and any given $y_i \in A^i(x^*)$, let $y = (x^*_{-i}, y_i)$. Then,

$$\begin{aligned} W_i \cdot (T^i(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) &= \sum_{j=1}^{n} W_j \cdot (T^j(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) \\ &- \sum_{j\neq i} W_j \cdot (T^j(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) \\ &= \sum_{j=1}^{n} W_j \cdot (T^j(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) \leq 0. \end{aligned}$$

Therefore, we have $W_i \cdot (T^i(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) \leq 0$ for each $i \in I$ and $y_i \in A^i(x^*)$. Hence, x^* is a generalized weighted Nash equilibrium of the game G with respect to W.

By using Lemma 4.2, we obtain the following existence theorem of generalized Pareto equilibrium as a consequence of Theorem 4.1.

Theorem 4.2. *Let I be a finite set of indices, let $(X_i, A^i, T^i)_{i\in I}$ be a constrained multi-criteria game with for each $i \in I$, X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E^i and suppose that there is a nonempty compact convex subset D of $X = \prod_{i\in I} X_i$ and a weighted vector $W = (W_1, W_2, ..., W_n)$ with $W_i \in R_+^{k_i} \backslash \{0\}$ such that the following conditions are satisfied:*

1) for each $i \in I$, $\overline{A^i}$ is convex nonempty valued;

2) The set $K = \{x \in X : x \in \overline{A}(x)\}$, where $A(x) = \prod_{i\in I} A^i(x)$, is closed in X;

3) The correspondence $F : X \to 2^X$,

$$F(x) = \{y \in X : \sum_{i=1}^{n} W_i \cdot (T^i(x_{-i}, x_i) - T^i(x_{-i}, y_i)) > 0\}$$

is such that $\overline{F}$ is nonempty convex valued;

4) $x \notin \overline{F}(x)$ for each $x \in K$.

Then, there exists $x^* \in X$ such that x^* is a generalized weak Pareto equilibrium.

Furthermore, if $W_i \in int R_+^{k_i} \backslash \{0\}$ for all $i \in I$, then x^* is a generalized Pareto equilibrium.

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