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NOTE ON THE FIXED POINT PROPERTY

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Abstract. In this paper it is proved that absolute approximative retracts and absolute multiretracts spaces have the fixed point property both for singlevalued continuous mappings and multivalued upper semi continuous mappings with R_δ-values.
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1. Introduction

There are two significant branches of methods in the fixed point theory (see [2]). The first are called homological methods and the second are approximation methods. Let us add that approximation methods are simpler then homological methods, but they are sufficient for applications to nonlinear analysis and several others sections of mathematics. In this paper we will use approximation methods.

Problem of fixed points was very popular and strongly developed. There is no possibility to name essential publications taking this theme. We are study the problem of fixed point property for:

(i) absolute approximative retracts in particular for absolute retracts;
(ii) absolute multiretracts;

both for singlevalued and multivalued mappings. Some of similar problems were studied in [6], [4].

2. Preliminaries

In this paper, we assumed that all topological spaces are metric. Let $(\mathbb{X}, d)$ be a space and let A be a subset of $\mathbb{X}$.

Definition 2.1. We recall that a continuous mapping $r : \mathbb{X} \to A$ is called *retraction* if $d(r(a), a) = 0$ for every $a \in A$, i.e. a retraction map is a continuous extension of the identity map on A onto $\mathbb{X}$.

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Definition 2.2. Let $\varepsilon > 0$, a continuous mapping $r_\varepsilon : \mathbb{X} \to A$ is called an *ε-retraction*, provided for every $x \in A$ we have $d(x, r_\varepsilon(x)) \leq \varepsilon$. Obviously any retraction map $r : \mathbb{X} \to A$ is an ε-retraction for every $\varepsilon > 0$.

Definition 2.3. A subset $A \subset \mathbb{X}$ is called an *approximative retract* of $\mathbb{X}$ (a *retract* of $\mathbb{X}$), if for every $\varepsilon > 0$ there exists an ε-retraction $r_\varepsilon : \mathbb{X} \to A$ (a retraction $r : \mathbb{X} \to A$).

Evidently any retract A of the space $\mathbb{X}$ is a closed subset of $\mathbb{X}$. The following question is an open problem:

Problem 2.4. Is it true that any approximative retract A of a space $\mathbb{X}$ is a closed subset of $\mathbb{X}$?

Definition 2.5. A compact space $\mathbb{X}$ is called an *absolute approximative retract* (written $\mathbb{X} \in AAR$), provided for every homeomorphic embedding $h : \mathbb{X} \to \mathbb{Y}$ the set $h(\mathbb{X})$ is an approximative retract of $\mathbb{Y}$.

If $h(\mathbb{X})$ is a retract of $\mathbb{Y}$ then we shall say that $\mathbb{X}$ is an *absolute retract* (written $\mathbb{X} \in AR$). Then we have:

$$AR \subseteq AAR.$$

Example 2.6. Consider the set:

$$B = \bigcup_{k=1}^{\infty} \left\{ \frac{1}{k} \right\} \times [0,1] \cup [0,1] \times \{0\} \subset \mathbb{R}^2.$$

The set B is an approximative retract of $\mathbb{R}^2$ but it is not retract of $\mathbb{R}^2$. So the inclusion $AR \subsetneq AAR$ is proper (see [5]).

We will need this well known theorem:

Fact 2.7. Let $\mathbb{E}$ be a normed space and let C be any convex subset of $\mathbb{E}$, then $C \in AR$. For proof see [1].

In what follows we will use Uryshon embedding theorem (see [3]):

Theorem 2.8. *For every compact metric space there exists a homeomorphic embedding into the Hilbert Cube $I^\infty = [0,1]^\infty$. Note that I^∞ can be regarded as a compact convex subset of the Banach space $l_2 = \{\{x_n\} \subset R^\infty \mid \sum_{n=1}^{+\infty} {x_n}^2 < +\infty\}$ because $I^\infty = \{\{x_n\} \in l_2 \mid |x_n| \leq \frac{1}{n},\ for\ every\ n = 1, 2, \dots\}$.*

Using Definition 2.3, Corollary 2.7, and Theorem 2.8 we obtain:

Proposition 2.9.

(2.9.1) *If $\mathbb{X} \in AAR$, then $\mathbb{X}$ is homeomorphic to an approximative retract of I^∞;*
(2.9.2) *If $\mathbb{X} \in AR$, then $\mathbb{X}$ is homeomorphic to a retract of I^∞.*

We recall the Schauder approximation theorem in the following form:

Theorem 2.10. [3, p. 55] *Let $f : \mathbb{X} \to C$ be a continuous mapping of a metric space $\mathbb{X}$ to a compact convex subset C of some normed space $\mathbb{E}$. Then for each $\varepsilon > 0$ there exists a continuous mapping $f_\varepsilon : \mathbb{X} \to C$ such that $\|f_\varepsilon(x) - f(x)\| < \varepsilon$ for each $x \in \mathbb{X}$ and $f_\varepsilon(\mathbb{X}) \subset \mathbb{E}^{n(\varepsilon)} \subset \mathbb{E}$, where $\mathbb{E}^{n(\varepsilon)}$ is a $n(\varepsilon)$ dimensional subspace of $\mathbb{E}$.*

3. Fixed Point Property – the case of singlevalued mappings

Definition 3.1. We shall say that a compact space $\mathbb{X}$ has the Fixed Point Property if any continuous mapping $f : \mathbb{X} \to \mathbb{X}$ has a fixed point (written $X \in FPP$), i.e., there exists $x_0 \in \mathbb{X}$ such that $f(x_0) = x_0$.

The set of all fixed points of f we will denote by $fix\ f$.

We will prove the following proposition:

Proposition 3.2. *If $\mathbb{X} \in FPP$ and $\mathbb{X}$ is homeomorphic to $\mathbb{Y}$, then $\mathbb{Y} \in FPP$.*

Proof. Let $g : \mathbb{Y} \to \mathbb{Y}$ be a continuous map. By the assumption there exists a homeomorphism $h : \mathbb{X} \to \mathbb{Y}$. Consider the following superposition

$$f = h^{-1} \circ g \circ h : \mathbb{X} \to \mathbb{X},$$

then f is continuous so there exists a point $x_0 \in fix\ f$. We have

$$f(x_0) = h(g(h^{-1}(x_0))) = x_0.$$

Thus $g(h^{-1}(x_0)) = h^{-1}(x_0)$ so the proof is completed. □

Proposition 3.3. *If $\mathbb{X} \in FPP$ and $A \subset \mathbb{X}$ is a closed approximative retract of $\mathbb{X}$, then $A \in FPP$.*

Proof. Let $f : A \to A$ is a continuous mapping. Put $\varepsilon_n = \frac{1}{n}$ and consider superposition $g_n = i \circ f \circ r_n : \mathbb{X} \to \mathbb{X}$, where $r_n : \mathbb{X} \to A$ is an approximative retraction for ε_n. By assumption for each n, there exists a point $x_n \in fix\ g_n \subset A$. Without loosing of generality we can assume that $\lim\limits_{n\to+\infty} x_n = x_0$ (A is compact). Obviously $\lim\limits_{n\to\infty} r_n(x_n) = x_0$. Then we have:

$$\begin{array}{ccc} f(r_n(x_n)) & \longrightarrow & f(x_0) \\ \| & & \\ r_n(x_n) & \longrightarrow & x_0 \end{array} \qquad => \qquad \begin{array}{ccc} f(r_n(x_n)) & \longrightarrow & f(x_0) \\ \| & & \| \\ r_n(x_n) & \longrightarrow & x_0 \end{array}$$

So the proof is completed. □

We shall recall the Brouwer fixed point theorem:

Theorem 3.4. [Brouwer] *If $K = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ then $K \in FPP$.*

Thus we are ready to prove the main result of this section:

Theorem 3.5. *If $\mathbb{X}$ is a compact AAR, then $\mathbb{X} \in FPP$.*

Proof of Theorem 3.5 will be given in 4 steps.

Step 1. If $K(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ then $K(x_0, r) \in FPP$.
This is immediate consequence of Proposition 3.1.1.

Step 2. A compact convex subset $C \subset \mathbb{R}^n$ has the FPP.
Because C is a retract of some ball $K(x_0, r)$ so it follows from Step 1 and Proposition 3.3.

Step 3. If $C \subset \mathbb{E}$ is a compact convex subset of some normed space $\mathbb{E}$, then $C \in FPP$.

Proof. Let $f : C \to C$ be a continuous map. Put $\varepsilon_n = \frac{1}{n}$, by Theorem 2.10, there exists $f_n : C \cap E^{n(\varepsilon)} \to E^{n(\varepsilon)}$ such that $|f_n(x) - f(x)| < \varepsilon$. Since $C_\varepsilon = C \cap \mathbb{E}^{n(\varepsilon)} \subset C$ is a convex compact subset of $\mathbb{E}^{n(\varepsilon)}$ we can apply Step 2. So for every n there exist x_n such that $f_n(x_n) = x_n$. We can assume that $\lim\limits_{n\to+\infty} x_n = x_0$ (because of the compactness of set C).

$$\begin{array}{ccc} f_n(x_n) \longrightarrow f(x_0) & \quad => \quad & f_n(x_n) \longrightarrow f(x_0) \\ \| & & \| \qquad\qquad \| \\ x_n \xrightarrow{\qquad} x_0 & & x_n \xrightarrow{\qquad} x_0 \end{array}$$

So x_0 is a fixed point of the mapping f what ends proof. □

Step 4. If $\mathbb{X} \in AAR$, then $X \in FPP$.

Proof. The Hilbert Cube I^∞ is compact convex set so it has the fixed point property by [Step 3.]. Moreover if $\mathbb{X} \in AAR$ then it is an approximative retract of I^∞. So by Proposition 3.1.2 any compact AAR has the fixed point property. □

4. Fixed Point Property for some class of multivalued mappings of AAR-s

First we recall the notion of R_δ sets:

Definition 4.1. A subset $A \subset \mathbb{X}$ is called an R_δ-*set* if it is a countable intersection of contractible compact subsets of $\mathbb{X}$.

We shall formulate an obvious fact about R_δ sets:

Fact 4.2. Let $h : \mathbb{X} \to \mathbb{Y}$ be a homeomorphism and $A \subset \mathbb{X}$, if A is an R_δ set then $h(A)$ is R_δ set too.

We shall use the concept of ε-neighbourhood of a set:

Definition 4.3. Let U be a subset of $\mathbb{X}$, $\varepsilon > 0$. We put

$$O_\varepsilon(U) = \{x \in \mathbb{X} \mid \exists\ u \in U \text{ such that } d((x, u)) < \varepsilon\}.$$

We shall consider the following class of multivalued mappings:

$$\mathcal{M}(\mathbb{X}, \mathbb{Y}) = \{\varphi : \mathbb{X} \multimap \mathbb{Y} \mid \varphi \text{ is u.s.c. and for each } x \in \mathbb{X}, \text{ the set } \varphi(x) \text{ is } R_\delta\}.$$

The following theorem is well known:

Proposition 4.4. *If $\varphi \in \mathcal{M}(\mathbb{X}, \mathbb{Y})$, then the graph, $\Gamma_\varphi = \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid y \in \varphi(x)\}$ of φ, is a closed subset of $\mathbb{X} \times \mathbb{Y}$; where in $\mathbb{X} \times \mathbb{Y}$ we consider the max-metric.*

Note that if $\mathbb{X}$ is compact then we can formulate:

Proposition 4.4.1. *A mapping $\varphi : \mathbb{X} \multimap \mathbb{Y}$ is u.s.c. if and only if Γ_φ is a closed subset of $\mathbb{X} \times \mathbb{Y}$.*

To prove main result of this section we need the following approximation theorem:

Theorem 4.5. [2, p. 114] *If $\varphi \in \mathcal{M}(\mathbb{X}, \mathbb{X})$ and C is a compact convex subset of some normed space $\mathbb{E}$, then for each $\varepsilon > 0$ there exists a continuous mapping $f_\varepsilon : C \to C$ such that $\Gamma_f \subset O_\varepsilon(\Gamma_\varphi)$.*

We define:

Definition 4.6. We shall say that a compact space $\mathbb{X}$ has the multi fixed point property (written $\mathbb{X} \in MFPP$) if for every $\varphi \in \mathcal{M}(\mathbb{X}, \mathbb{X})$ there exists $x_0 \in fix\ \varphi = \{x \in \mathbb{X} \mid x \in \varphi(x)\}$.

We prove:

Proposition 4.7.

(4.7.1) *If $\mathbb{X} \in MFPP$ and $\mathbb{Y}$ is homeomorphic to $\mathbb{X}$, then $\mathbb{Y} \in MFPP$.*

(4.7.2) *If $\mathbb{X} \in MFPP$ and A is a closed approximative retract of $\mathbb{X}$, then $A \in MFPP$.*

Proof. (4.7.1) For the proof of this proposition it is sufficient to observe that for any $\varphi \in \mathcal{M}(Y,Y)$ and given homeomorphism $h : \mathbb{X} \to \mathbb{Y}$ we get $\Psi = h^{-1} \circ \varphi \circ h \in \mathcal{M}(X,X)$ (compare Proposition 4.2).

(4.7.2) Let $\varphi \in \mathcal{M}(A,A)$ and $r_\varepsilon : \mathbb{X} \to A$ be an approximative retraction for every $\varepsilon > 0$. We put $\varepsilon_n = \frac{1}{n}$ and define the following mappings:

$$\Psi_n = i \circ \varphi \circ r_n : \mathbb{X} \multimap \mathbb{X},\ n = 1,2,\ldots$$

Evidently $\Psi_n \in \mathcal{M}(\mathbb{X},\mathbb{X})$ for every $n = 1,2,\ldots$. By assumption $fix\Psi_n \neq \emptyset$ so for every n there exists $x_n \in \Psi_n$. Consequently:

$$x_n \in i \circ \varphi \circ r_n(x_n).$$

Since r_n is a $\frac{1}{n}$-retraction we have: $d(x_n, r_n(x_n)) < \frac{1}{n}$. Without loss of generality we can assume that $lim_{n\to+\infty} x_n = x_0$ (because of compactness of A). But that implies $lim_{n\to+\infty} r_n(x_n) = x_0$. We know that:

$$(x_n, r_n(x_n)) \in \Gamma_\varphi, \text{ for every } n = 1,2,\ldots \text{ and } \lim_{\infty_{n\to+\infty}} (x_n, r_n(x_n)) = (x_0, x_0).$$

Moreover by Proposition 4.4 we know that graph of φ is closed so $(x_0, x_0) \in \Gamma_\varphi$. □

Using Theorems 2.8, 3.3 and 4.5 we are able to prove:

Theorem 4.8. *If $\mathbb{X} \in AAR$, then $\mathbb{X} \in MFPP$.*

Step 1. $\mathbb{X} = I^\infty$

Proof. Let $\varphi \in \mathcal{M}(I^\infty, I^\infty)$. From Theorem 4.5 we know that for every $\varepsilon > 0$, there exists a mapping $f_\varepsilon : I^\infty \to I^\infty$, such that $\Gamma_{f_\varepsilon} \subset O_\varepsilon(\Gamma\varphi)$.
Put $\varepsilon_n = \frac{1}{n}$. Because I^∞ has FPP (see Theorem 3.3) then there exist $x_n = f_n(x_n)$ for every n. Since $(x_n, f_n(x_n)) \in \Gamma_f$ there exists $(\tilde{x}_n, \tilde{y}_n) \in \Gamma\varphi$ such that $d((x_n, f_n(x_n)), (\tilde{x}_n, \tilde{y}_n)) < \frac{1}{n}$, for every n. Consequently $d(x_n, \tilde{x}_n) < \frac{1}{n}$ and $d(f_n(x_n), \tilde{y}_n) < \frac{1}{n}$. But we can assume, without loss of generality that $\lim_{n\to+\infty} x_n = x_0$. Hence we get: $\lim_{n\to+\infty} \tilde{x}_n = x_0$ and $\lim_{n\to+\infty} f_n(x_n) = \lim_{n\to+\infty} x_n = x_0$.
Finally we obtain that $\lim_{n\to+\infty} (\tilde{x}_n, \tilde{y}_n) = (x_0, y_0) \in \Gamma_\varphi$. Because Γ_φ is a closed subset of $I^\infty \times I^\infty$, see 4.4. So $x_0 \in fix_\varphi$ and the proof is complete. □

Step 2. *Proof.* By Proposition 2.9.1 there exists homeomorphism $h : \mathbb{X} \to \mathbb{A} \subset I^\infty$, where subset A is a closed approximative retract of I^∞ (so it is a compact subset of I^∞). Thus our claim follows from Step 1 and Proposition 4.1.2. □

5. Some applications

The notion of multiretracts first were studied in [7]. We will apply Theorem 4.8 in order to prove that any multiretract of a space having MFP property has FPP property.

Definition 5.1. Let A be a subset of a space $\mathbb{X}$. A continuous mapping $r : \mathbb{X} \to A$ is called multiretraction if there exists a multivalued mapping $\varphi \in \mathcal{M}(A, \mathbb{X})$, such that $r \circ \varphi = Id_A$. Moreover the set A is called a multiretract of $\mathbb{X}$.

Definition 5.2. The compact space $\mathbb{X}$ is called an absolute multiretract provided it is homeomorphic to a compact subset $A \subset I^\infty$ which is a multiretract of I^∞ (written $\mathbb{X} \in AMR$).
For more details and examples concerning AMR-spaces see [7].

First we prove:

Proposition 5.3. *If $\mathbb{X} \in MFPP$ and A is multiretract of $\mathbb{X}$, then $A \in FPP$.*

Proof. Let $f : A \to A$ be a continuous map. According to (5.1), let $r : \mathbb{X} \to A$ be a multiretraction and $\varphi \in \mathcal{M}(A, X)$ such that $r \circ \varphi = Id_A$. For the proof we define a multivalued map $\Psi : \mathbb{X} \multimap \mathbb{X}$ given by the formula: $\Psi = \varphi \circ f \circ r$.

Evidently Ψ is an u.s.c. mapping with R_δ values. Since $\mathbb{X} \in MFPP$, we know there exists a point $x_0 \in \mathbb{X}$ such that $x_0 \in \Psi(x_0)$.

We have:

$$x_0 \in \Psi(x_0) = \varphi(f(r(x_0))),$$

it implies that: $r(x_0) \in (r \circ \varphi)(f(r(x_0)))$.

Since $(r \circ \varphi) = Id_A$ we have $r(x_0) \in fixf$. □

Problem 5.4. Is it true that assumptions of (5.3) imply that $A \in MFPP$?

Using Propositions 5.3, 3.2 and Theorem 4.8 we obtain:

Theorem 5.5. *If $\mathbb{X} \in AMR$, then $\mathbb{X} \in FPP$.*

There are many further questions connected with the notion of fixed point theory for multivalued mappings of AMR-spaces. Some of them we shall present in next works.

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