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MÖNCH SETS AND FIXED POINT THEOREMS FOR MULTIMAPS IN LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

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Abstract. We present a variety of fixed point theorems for multimaps having weakly closed graph. We state in turn Sadovskii, Mönch and Daher type theorems which improve recent results in the literature. With this in mind, we introduce the definition of Mönch-set.
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1. Introduction

In this paper we present fixed point theory in topological vector spaces for multimaps having weakly closed graph.

After some preliminaries, in Section 4 we state a multivalued version of the Sadovskii fixed point theorem in Hausdorff locally convex topological vector spaces satisfying the Krein-Smulian property (see (X1)). The key hypothesis of the Sadovskii theorem is the condensivity of the operator. In our setting the definition of condensing multimap is given in terms of an abstract measure of noncompactness which is well defined thanks to property (X1), as we note in Section 3. Our Sadovskii theorem improves some results in the literature (see, e.g. [7, Corollary 3.3.1], [2, Theorem 2.1], [5, Theorem 5.1]).

In Section 5 we introduce the definition of a Mönch-set for a multimap (see Definition 5.1) which leads to a Mönch type fixed point theorem where the Mönch hypothesis is weaker than the others in the literature (see Remark 5.2). Moreover we establish a sufficient condition for the existence of Mönch-sets (see Theorem 5.1) and a proposition on their relative compactness (see Theorem 5.3).

As a consequence of this study we derive a Daher type fixed point theorem for multimaps (see Theorem 5.4).

2. Preliminaries

Let X be a Hausdorff locally convex topological vector space (HLCTVS, for short) and $\mathcal{P}(X)$ be the family of all nonempty subsets of X. Following [3], we recall the next notion

Definition 2.1. *Let D be a nonempty subset of X. A map $F : D \to \mathcal{P}(X)$ is said to have* weakly closed graph *in $D \times X$ if for every net $(x_\delta)_\delta$ in D, $x_\delta \to x$, $x \in D$, and for every net $(y_\delta)_\delta$, $y_\delta \in F(x_\delta)$, $y_\delta \to y$, we have $S(x,y) \cap F(x) \neq \emptyset$, where $S(x,y) = \{x + \lambda(y-x) : \lambda \in [0,1]\}$.*

We recall that in a locally convex topological vector space a set H is said to be *bounded* if for every neighborhood U of 0_X there exists a positive number ρ such that $H \subset \rho U$. We will use the following notations:
$\mathcal{P}_b(X) = \{H \subset X : H \neq \emptyset, H \textit{ bounded}\}$; $\mathcal{P}_k(X) = \{H \subset X : H \neq \emptyset, H \textit{ compact}\}$;
$\mathcal{P}_{kc}(X) = \{H \subset X : H \neq \emptyset, H \textit{ compact and convex}\}$;
$\mathcal{P}_{fc}(X) = \{H \subset X : H \neq \emptyset, H \textit{ closed and convex}\}$.

3. On the measure of noncompactness

Let X be a HLCTVS satisfying the Krein-Smulian property

(X1) if A is a compact subset of X, then $\overline{co}(A)$ is compact.

Remark 3.1. *If X is a quasi-complete locally convex topological vector space, then (X1) holds (cf.* [8, §20.6 (3)]*).*

Definition 3.1. *A function $\gamma : \mathcal{P}_b(X) \to I\!R_0^+$ is said to be a* measure of noncompactness *(MNC, for short) if, for every $\Omega \in \mathcal{P}_b(X)$, the following properties are satisfied:*

(γ_1) $\gamma(\Omega) = 0$ *if and only if* $\overline{\Omega}$ *is compact;*
(γ_2) $\gamma(\overline{co}(\Omega)) = \gamma(\Omega)$.

In the sequel, we will need also the next property:

(γ_3) *set additivity:* $\gamma(\Omega_1 \cup \Omega_2) = \max\{\gamma(\Omega_1), \gamma(\Omega_2)\}$, where $\Omega_1, \Omega_2 \in \mathcal{P}_b(X)$.

Note that a set additive MNC satisfies also the properties

(γ_4) *monotonicity:* $\Omega_1 \subset \Omega_2$ implies $\gamma(\Omega_1) \leq \gamma(\Omega_2)$;
(γ_5) *nonsingularity:* $\gamma(\{x\} \cup \Omega) = \gamma(\Omega)$, for every $x \in X$;
(γ_6) *closure invariance:* $\gamma(\overline{\Omega}) = \gamma(\Omega)$.

We observe that the above definition is well posed since X is a locally convex space and satisfies property $(X1)$.

Remark 3.2. *Let E be a Banach space endowed with the weak topology $\mathcal{T}_w$ and take $X = (E, \mathcal{T}_w)$. Thanks to the Krein-Smulian Theorem (see, e.g.* [6, Theorem 3.5.15]*), X satisfies property $(X1)$. Observe that an example of set additive MNC in X is the De Blasi measure of weak noncompactness in E.*

Definition 3.2. *Let D be a nonempty subset of X. We say that a map $F : D \to \mathcal{P}(X)$ is* (countably) condensing *with respect to a MNC γ if*

(I) *$F(D)$ is bounded;*
(II) *$\gamma(F(B)) < \gamma(B)$ for all (countable) bounded subsets B of D with $\gamma(B) > 0$.*

4. Sadovskii type theorem

The first result (a Sadovskii type theorem) improves some theorems in the literature (see, e.g. [7, Corollary 3.3.1], [2, Theorem 2.1], [5, Theorem 5.1]).

Theorem 4.1. *Let D be a closed convex subset of a HLCTVS X satisfying property (X1) and $F : D \to \mathcal{P}_{fc}(D)$ be a map such that*

(F1) *F has weakly closed graph in $D \times X$;*
(F2) *F is condensing with respect to a nonsingular MNC.*

Then there exists $x \in D$ with $x \in F(x)$.

Proof. Let us fix an arbitrary $\bar{x} \in D$ and take the family $\{H_\alpha\}_\alpha$ of all the subsets of X each of them satisfying the following properties:

(p_1) $\bar{x} \in H_\alpha$;
(p_2) H_α is closed and convex ;
(p_3) $F(D \cap H_\alpha) \subset H_\alpha$;
(p_4) $x \in \overline{co}(F(x) \cup H_\alpha) \;\Rightarrow\; x \in H_\alpha$.

Then, we consider the nonempty set $H = \cap_\alpha H_\alpha$, which is well-defined, belongs to the family $\{H_\alpha\}_\alpha$ and satisfies the identity

$$H = \overline{co}(F(H) \cup \{\bar{x}\}) \ . \tag{4.1}$$

Note that the above properties of H were proved in Step 1 of the proof of [5, Theorem 5.1] (in that part of the proof it did not matter that the space was a Banach space, but just that it was a HLCTVS).

The set H is compact. First, $F(H)$ is bounded; in fact, $D \in \{H_\alpha\}_\alpha$ so that $H \subset D$ and $F(H)$ is contained in the bounded set $F(D)$ (see (F2)). Moreover, by (4.1), we obtain that $H \subset \overline{co}(F(D) \cup \{\bar{x}\})$, therefore H is bounded too.
Now, suppose that $\gamma(H) > 0$. By means of (F2), (4.1) and properties (γ_2) and (γ_5) of MNC, we deduce that

$$\gamma(F(H)) < \gamma(H) = \gamma(\overline{co}(F(H) \cup \{\bar{x}\})) = \gamma(F(H))$$

which is a contradiction. Therefore we must have $\gamma(H) = 0$. Hence, from property (γ_1), the closed set H is compact.

Next, let $x \in H$. By (4.1) we have that $F(x) \subset H$, so $F(x) \cap H \neq \emptyset$.

Finally, since all the hypotheses of [3, Theorem III] are satisfied, we deduce the existence of a fixed point for F. □

5. Mönch-sets and fixed point theorems

Let X be a HLCTVS satisfying (X1) and the further property

(X2) for any relatively compact subset A of X, there exists a countable set $B \subset A$ such that $\overline{B} = \overline{A}$.

Definition 5.1. *Let D be a convex subset of X and $F : D \to \mathcal{P}(X)$ be a given map. We say that a set $M \subset D$ is a* Mönch-set *for F if there exists $x_0 \in D$ such that $M = co(\{x_0\} \cup F(M))$ and there exists a countable set $C \subset M$ with $\overline{M} = \overline{C}$.*

Theorem 5.1. *Let X be a HLCTVS satisfying (X1) and (X2). Let D be a closed convex subset of X and $F : D \to \mathcal{P}_k(D)$ be a map such that*

(F3) *F maps compact sets into relatively compact sets.*

Then, there exists a Mönch-set for F.

Proof. Fix $x_0 \in D$ and consider the iterative sequence $(M_n)_{n\in I\!N}$ of sets:

$$M_0 = \{x_0\} \ ; \quad M_n = co(\{x_0\} \cup F(M_{n-1})) \ , \ n \in I\!N^+ \ .$$

Clearly,

$$\overline{M_n} \subset D \ , \quad n \in I\!N \ . \tag{5.1}$$

Let us prove by induction that M_n, $n \in I\!N^+$, is relatively compact.
First, assumption (X1) implies that $\overline{co}(\{x_0\} \cup F(M_0))$ is compact. Thus, M_1 is relatively compact.
Now, suppose that M_{n-1} is relatively compact, $n \geq 2$. Of course $\overline{M_n} \subset \overline{co}(\{x_0\} \cup F(\overline{M_{n-1}}))$ (see (5.1)). By (F3) and (X1), we can also see that M_n is relatively compact.
By induction again, we see that

$$M_{n-1} \subset M_n \ , \quad n \in I\!N^+ \ . \tag{5.2}$$

Now, for every $n \in I\!N$, by (X2) we have that there exists a countable set $C_n \subset M_n$ such that

$$\overline{C_n} = \overline{M_n} \ . \tag{5.3}$$

If we consider the subset of D defined as

$$M = \cup_{n\in I\!N} M_n \tag{5.4}$$

and its countable subset

$$C = \cup_{n\in I\!N} C_n \ , \tag{5.5}$$

then we have

$$\overline{M} = \overline{\cup_{n\in I\!N}\overline{M_n}} = \overline{\cup_{n\in I\!N}\overline{C_n}} = \overline{C} \ . \tag{5.6}$$

Furthermore using (5.2) and (5.4) we have

$$M = co(\{x_0\} \cup F(M)) \ , \tag{5.7}$$

so M is a Mönch-set. □

Theorem 5.2. *Let D be a closed convex subset of a HLCTVS X satisfying properties (X1) and (X2). Suppose that $F : D \to \mathcal{P}_{kc}(D)$ is a map such that*

(F1) *F has weakly closed graph;*
(F3) *F maps compact sets into relatively compact sets;*
(F4) *there exists a Mönch-set for F which is relatively compact.*

Then there exists $x \in D$ such that $x \in F(x)$.

Remark 5.1. *We underline that hypothesis (F4) is well posed since the Theorem works in the setting of Theorem 5.1.*

Proof. Let M be a relatively compact Mönch-set for F (see (F4)).

We prove that $F(x) \cap \overline{M} \neq \emptyset$, $x \in \overline{M}$. Fix $x \in \overline{M}$. Then there exists a net $(x_\delta)_\delta$ in M such that $x_\delta \to x$. Let us consider a net $(y_\delta)_\delta$ with $y_\delta \in F(x_\delta)$. By (5.7), $\{y_\delta\}_\delta$ is included in the compact set $\overline{M}$.

By proceeding as in the end of the proof of Theorem 4.1 (in this paper), w.l.o.g. we may assume that $y_\delta \to y \in \overline{M}$ and therefore $F(x) \cap \overline{M} \neq \emptyset$.
Now from [3, Theorem III] we deduce that there exists $x \in D$ such that $x \in F(x)$. □

Remark 5.2. *Theorem 5.2 improves all the theorems in the literature (see, e.g.* [9, Theorem 3.1]*,* [4, Theorem 3.1]*,* [5, Theorem 3.1]*) where the following Mönch hypothesis is assumed:*

(M) *there exists $x_0 \in D$ such that*

$$\left.\begin{array}{l} M \subset D,\ M = co(\{x_0\} \cup F(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{array}\right\} \quad \Rightarrow \quad \overline{M} \text{ is compact,}$$

that is **every** *Mönch-set for F must be relatively compact.*
To see this consider the following example.

Example 5.1. *Let X be a separable Banach space with $\dim X = +\infty$ and denote by $B(0,1)$ the closed unit ball in X. Let $F : B(0,1) \to \mathcal{P}(B(0,1))$ be the map defined by*

$$F(x) = \{x\} \ , \quad x \in B(0,1) \ .$$

The map F satisfies hypothesis (F4)*; just take* $\{0\}$ *as the Mönch-set for F. Moreover, of course, also all the other assumptions of Theorem 5.2 hold.*
On the other hand, F does not satisfy property (M). Indeed, for every $x_0 \in B(0,1)$ there exists a set M with $M = co(\{x_0\} \cup F(M))$ and for which there exists $C \subset M$ countable such that $\overline{M} = \overline{C}$ which is not relatively compact. It is enough to take $M = B(0,1)$.

Theorem 5.3. *Let D be a closed convex subset of a HLCTVS X satisfying properties (X1) and (X2). Suppose that $F : D \to \mathcal{P}_{kc}(D)$ is a map such that*

(F3) *F maps compact sets into relatively compact sets;*
(F5) *F is countably condensing with respect to a set additive MNC.*

Then every Mönch-set for F is relatively compact.

Proof. Fix M a Mönch-set for F, which exists thanks to Theorem 5.1. We recall that there exists $x_0 \in D$ such that $M = co(\{x_0\} \cup F(M))$ and

$$\overline{M} = \overline{C} \ , \tag{5.8}$$

with C a countable subset of M.
Since $C \subset co(\{x_0\} \cup F(M))$, every point of C can be written as a finite combination of points belonging to the set $\{x_0\} \cup F(M)$. Therefore, there exists a countable set $\mathcal{M} \subset M$ such that

$$C \subset co(\{x_0\} \cup F(\mathcal{M})) \ . \tag{5.9}$$

Note that, since $F(D)$ is bounded (see hypothesis (F5)) then the sets M, C, $\mathcal{M}$ are bounded.

Let us prove that $\gamma(C) = 0$. First of all, by using (5.9) and properties (γ_4), (γ_2), (γ_5) of γ, we have

$$\gamma(C) \leq \gamma(\overline{co}(\{x_0\} \cup F(\mathcal{M}))) = \gamma(\{x_0\} \cup F(\mathcal{M})) = \gamma(F(\mathcal{M})) . \tag{5.10}$$

Now, suppose that $\gamma(\mathcal{M}) > 0$. Then, hypothesis *(F5)* yields

$$\gamma(F(\mathcal{M})) < \gamma(\mathcal{M}) . \tag{5.11}$$

Combining (5.10) and (5.11), by properties (γ_4) and (γ_6) of γ and (5.8), we obtain

$$\gamma(C) < \gamma(\mathcal{M}) \leq \gamma(M) = \gamma(\overline{M}) = \gamma(\overline{C}) = \gamma(C) ,$$

and this is a contradiction.
Hence $\gamma(\mathcal{M}) = 0$. So, by (γ_1), $\overline{\mathcal{M}}$ is compact. Thus, assumption *(F3)* and (γ_1) imply

$$\gamma(F(\overline{\mathcal{M}})) = 0 ;$$

then, since $F(\mathcal{M}) \subset F(\overline{\mathcal{M}})$, by (γ_4) we have

$$\gamma(F(\mathcal{M})) = 0 . \tag{5.12}$$

Therefore, by (5.10) and (5.12), we can conclude

$$\gamma(C) = 0 .$$

Hence, thanks to (γ_6) and (5.8), we have

$$\gamma(M) = \gamma(\overline{M}) = \gamma(\overline{C}) = \gamma(C) = 0 ,$$

therefore, the set $\overline{M}$ is compact (see (γ_1)). □

From Theorem 5.3 and Theorem 5.2, we can deduce the following fixed point theorem.

Theorem 5.4. *Let D be a closed convex subset of a HLCTVS X satisfying properties (X1) and (X2). Suppose that $F : D \to \mathcal{P}_{kc}(D)$ is a map such that*

(F1) *F has weakly closed graph in $D \times X$;*
(F3) *F maps compact sets into relatively compact sets;*
(F5) *F is countably condensing with respect to a set additive MNC.*

Then there exists $x \in D$ with $x \in F(x)$.

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