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TOPOLOGICAL DEGREE AND ATYPICAL BIFURCATION RESULTS FOR A CLASS OF MULTIVALUED PERTURBATIONS OF FREDHOLM MAPS IN BANACH SPACES

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Abstract. We obtain a global (atypical) bifurcation result for a semilinear inclusion problem in Banach spaces. The approach is topological, making use of a topological degree introduced by the second author for locally compact multivalued perturbations of nonlinerar Fredholm maps. We also study some specific aspects of the construction of this concept of degree.
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1. Introduction

The purpose of this paper is twofold: the first one is the study of an atypical bifurcation problem for a semilinear operator inclusion, the second one is the discussion of some aspects of the construction of a topological degree for a class of multivalued perturbations of Fredholm maps in Banach spaces.

The semilinear operator inclusion we consider is of the type

$$Lx + \lambda h(\lambda, x) \in \lambda \mathcal{H}(\lambda, x), \tag{1.1}$$

where, given two real Banach spaces E and F and a simply connected open subset Ω of $\mathbb{R} \times E$,

(1) $L : E \to F$ is a Fredholm linear operator of index zero,

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(2) $h : \Omega \to F$ is a continuous map such that for any $\lambda \in \mathbb{R}$ the partial map $x \mapsto Lx + \lambda h(\lambda, x)$ is a nonlinear Fredholm map of index zero on the (possibly empty) section

$$\Omega_\lambda = \{x \in E : (\lambda, x) \in \Omega\}.$$

In addition we assume that

$$(\lambda, x) \mapsto \partial_2 h(\lambda, x)$$

is continuous, where $\partial_2 h(\lambda, x)$ stands for the Fréchet partial derivative of h with respect to the second variable at the point (λ, x).

(3) $\mathcal{H} : \Omega \multimap F$ is a CJ and locally compact multimap (see section 3 for the notion of CJ multimap).

We call *trivial solutions* of (1.1) the pairs $(0, x)$ of Ω such that $x \in \operatorname{Ker} L$. A point p in $\operatorname{Ker} L \cap \Omega_0$ is said to be an (*atypical*) *bifurcation point* if $(0, p)$ lies in the closure of the set of nontrivial solutions. One of the problems related to inclusion (1.1) is to establish under what conditions the set of nontrivial solutions is not empty and bifurcation points exist.

In the literature the expression "bifurcation problem" is often related to the case when $f(\lambda, 0) = 0$ for every λ (as in the classical works of Krasnoselski, Rabinowitz, Crandall and Rabinowitz, for instance) and the elements $(\lambda, 0)$ are the trivial solutions. To distinguish our different case, we used previously and in the title the term "atypical bifurcation", as in [26, 3] (while in [11] one can find the term "co-bifurcation"). However, in order to simplify the language, throughout the paper we will use the term "bifurcation".

In [11, 21] the following semilinear equation

$$Lx + \lambda h(\lambda, x) = 0 \tag{1.2}$$

is studied in the case when h is a compact function and is proved the existence of a connected bifurcating branch of nontrivial solutions that either is unbounded or contains in its closure at least two bifurcation points. In [3, 4] an analogous result is obtained removing the compactness assumption on h. In this last case the proof is based on a degree theory developed by Benevieri and Furi [1, 2, 4] for a special class of locally compact perturbations of Fredholm maps of index zero in Banach spaces.

In this paper we prove an extension to the multivalued case of the results given in [3] (and extended in [4]). The bifurcation results are here obtained following the general spirit of the two above cited papers, i.e., applying a topological degree for a special class of CJ and locally compact multivalued perturbations of nonlinear Fredholm maps of index zero between Banach spaces. Such a topological degree we refer to has been introduced by Obukhovskii, Zecca and Zvyagin in [24]. They found the construction on a notion of orientation for Banach spaces given by Elworty and Tromba in [9, 10] and defined the degree thanks to a finite-dimensional reduction approach using the Brouwer degree for maps in Euclidean manifolds.

Recently, in very general setting, an extended version of this degree (including other classes of multivalued perturbations of nonlinear Fredholm maps) has been given by Vaeth in [27], where the reader can find a large number of references on the topic. In his construction, Vaeth uses a concept of orientation defined by Benevieri and Furi in [1, 2], which is more recent and easier than that of Elworthy-Tromba.

In this paper, beside the bifurcation results, we redefine this topological degree following part of the strategy in [24], but using the concept of orientation defined in [1, 2]. In other words we give the definition of the degree for CJ locally compact multivalued perturbations of Fredholm maps as a particular case of the more general construction of Vaeth. In addition, some properties of the degree are discussed. More precisely, we discuss some aspects of the notion of orientation in infinite dimension which supports the construction of the degree.

2. Preliminaries

2.1. **Remarks on the Brouwer degree.** This subsection is devoted to a quick review of the reduction property of the Brouwer degree that will be used later. Slightly extending the approach of Nirenberg [23], the degree is an integer number assigned to any triple (f, U, y), where f, U and y are as follows. Given two oriented C^1 real manifolds M and N of the same finite dimension, U is an open subset of M, $f : M \to N$ is a continuous map and y is an element of N such that $f^{-1}(y) \cap U$ compact.

We stress that the construction of Nirenberg includes the two classical approaches to the finite dimensional degree: one regarding maps defined on the closure of bounded open subsets of $\mathbb{R}^n$ and the other one concerning maps between compact manifolds.

The classical properties of the Brouwer degree still hold in this extended version. The proof can be easily obtained by a straightforward generalization of the same properties given in [23].

We explicitly recall here some facts regarding the notions of orientation and transversality. Consider a real manifold M, a real vector space F of the same finite dimension and a C^1 map $g : M \to F$. Let F_1 be a subspace of F, transverse to g. Thus $M_1 = g^{-1}(F_1)$ is a submanifold of M of the same dimension as F_1. Assume now that M and F are oriented. One can prove that any orientation of F_1 induces an orientation on M_1. Let us sketch how this can be done. Suppose F_1 oriented and let $x \in M_1$ be given. By the transversality assumption, the tangent space to M_1 at x, denoted by T_xM_1, coincides with $Dg(x)^{-1}(F_1)$. Let E_0 be any direct complement of T_xM_1 in T_xM and let $F_0 = Dg(x)(E_0)$. Observe that $Dg(x)$ maps isomorphically E_0 onto F_0 and that $F = F_0 \oplus F_1$. Let F_0 be endowed with the orientation such that a positively oriented basis of F_0 and a positively oriented basis of F_1, in this order, form a positively oriented basis of F. Then, orient E_0 in such a way that $Dg(x)|_{E_0} : E_0 \to F_0$ is orientation preserving. Finally, orient T_xM_1 in such a way that a positively oriented basis of E_0 and a positively oriented basis of T_xM_1, in this order, form a positively oriented basis of T_xM. One can prove that this pointwise choice induces a global orientation on M_1 (see e.g. [16, pages 100-101] for the details).

Definition 2.1. We will call *oriented g-preimage* of F_1 the submanifold M_1, oriented as above.

Let now $f : M \to F$ be continuous and let $y \in F$ be such that $f^{-1}(y)$ is compact. Consider a C^1 map $g : M \to F$ and a subspace F_1 of F such that

(a) F_1 contains y and $(f - g)(M)$,

(b) g is transverse to F_1.

Assumption (a) implies that $f^{-1}(y)$ coincides with $f_1^{-1}(y)$, where f_1 stands for the restriction $f|_{M_1} : M_1 \to F_1$. Therefore, the Brouwer degree of the triple (f_1, M_1, y) is well defined. We can now state the following reduction property of the degree. The proof of this result can be obtained following the outline of the analogous result given for maps between Euclidean spaces, where the rôle of g is played by the identity of $\mathbb{R}^n$ (see e.g. [20, Lemma 4.2.3]).

Proposition 2.2 (reduction). *Let M be an oriented manifold and F an oriented vector space of the same finite dimension as M. Let $f : M \to F$ be continuous and $y \in F$ such that $f^{-1}(y)$ is compact. Consider an oriented subspace F_1 of F and a C^1 map $g : M \to F$ such that*

(1) *F_1 contains y and $(f - g)(M)$,*

(2) *g is transverse to F_1.*

Let M_1 denote the oriented g-preimage of F_1. Then,

$$\deg_B(f, M, y) = \deg_B(f_1, M_1, y),$$

where f_1 is the restriction of f to M_1 as domain and to F_1 as codomain.

2.2. **Orientability for Fredholm maps.** In this subsection we summarize the notion, introduced in [1, 2], of orientability and orientation for nonlinear Fredholm maps of index zero between Banach spaces. The starting point is a concept of orientation for Fredholm linear operators of index zero between real Banach spaces. Recall that, given two real Banach spaces E and F, a continuous linear operator $L : E \to F$ is said to be *Fredholm* if $\operatorname{Ker} L$ and $\operatorname{coKer} L$ are finite-dimensional. The *index* of L is defined as

$$\operatorname{ind} L = \dim \operatorname{Ker} L - \dim \operatorname{coKer} L.$$

Given a Fredholm operator of index zero $L : E \to E$, a continuous linear operator $A : E \to F$ is called a *corrector* of L if the following conditions hold:

i) the image of A is finite-dimensional,

ii) $L + A$ is an isomorphism.

On the set $\mathcal{C}(L)$ of correctors of L, which is nonempty, we define an equivalence relation as follows. Let $A, B \in \mathcal{C}(L)$ be given and consider the following automorphism of E:

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A).$$

The operator $K = (L + B)^{-1}(B - A)$ has clearly finite-dimensional image. Hence, given any nontrivial finite dimensional subspace E_0 of E containing the image of K, the restriction of T to E_0 is an automorphism. Therefore, its determinant is well defined, nonzero and, as it is easy to check, independent of the choice of E_0. Thus, one can define the *determinant* of T as the determinant of the restriction of T to any nontrivial finite-dimensional subspace of E containing the image of K.

Remark 2.3. This extension to infinite dimension of the notion of determinant is a well known concept in Functional Analysis. It can be found, for instance, in the classical textbook of Kato (see [18, § III-4]).

We say that A is *equivalent* to B or, more precisely, A is *L-equivalent* to B if

$$\det\big((L+B)^{-1}(L+A)\big) > 0.$$

As shown in [1], this is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes. This provides a concept of orientation for Fredholm linear operators of index zero.

Definition 2.4. Let $L : E \to F$ be a Fredholm linear operator of index zero. An *orientation* of L is the choice of one of the two equivalence classes of $\mathcal{C}(L)$, and L is *oriented* when an orientation is chosen. Any of the two orientations of L is called *opposite* to the other. If L is oriented, the elements of its orientation are called *positive correctors* of L.

Denote by $L(E, F)$ the Banach space of bounded linear operators of E into F and by $\Phi_0(E, F)$ the open subset of $L(E, F)$ of the Fredholm operators of index zero. The orientation of an operator of $\Phi_0(E, F)$ induces an orientation to any sufficiently close operator. Precisely, consider an operator $L \in \Phi_0(E, F)$ and a corrector A of L. Suppose that L is oriented with A positive corrector. Since the set of the isomorphisms of E into F is open in $L(E, F)$, then A is a corrector of every T in a suitable neighborhood W of L in $\Phi_0(E, F)$. Thus, any $T \in W$ can be oriented by taking A as a positive corrector. This fact allows us to give the following definition.

Definition 2.5. Let X be a topological space and $h : X \to \Phi_0(E, F)$ a continuous map. An *orientation* of h is a 'continuous choice' of an orientation $\alpha(x)$ of $h(x)$ for each $x \in X$, in the sense that, for any $x \in X$, there exists $A \in \alpha(x)$ which is a positive corrector of $h(x')$ for any x' in a neighborhood of x. A map is *orientable* when it admits an orientation and *oriented* when an orientation is chosen.

An important property of the notion of orientation is its continuous transport along a homotopy. Consider first an orientable map $H : X \times \Lambda \to \Phi_0(E, F)$, where Λ is any topological space. It is immediate to see that any partial map $H_\lambda := H(\cdot, \lambda)$ is orientable. If, in addition, H is oriented, H_λ inherits an orientation, induced from that of H. We will say from now on that the orientations of H and any H_λ are *associated*.

The following proposition, proven in [2] (see also [5]) by the use of the covering theory, shows that the converse of the above argument is true in the case when Λ is simply connected and locally path connected. Such a result is particularly interesting, for our purposes, when $\Lambda = [0, 1]$ and it can be seen as a sort of continuous transport of an orientation along a homotopy.

Proposition 2.6. *Consider two topological spaces X and Λ, with Λ simply connected and locally path connected, and a continuous map $H : X \times \Lambda \to \Phi_0(E, F)$. Assume that, for some $\lambda \in \Lambda$, the partial map $H_\lambda := H(\cdot, \lambda)$ is oriented. Then, there exists and is unique the orientation of H which is associated with that of H_λ.*

Let us now give the notion of orientability for Fredholm maps of index zero between Banach spaces. Recall that, given an open subset Ω of E, a map $g : \Omega \to F$ is a *Fredholm map* if it is C^1 and its Fréchet derivative, $Dg(x)$, is a Fredholm operator for all $x \in \Omega$. The *index* of g at x is the index of $Dg(x)$ and g is said to be of *index* n if it is of index n at any point of its domain.

Definition 2.7. An *orientation* of a Fredholm map of index zero $g : \Omega \to F$ is an orientation of the continuous map $Dg : x \mapsto Dg(x)$; g is *orientable*, or *oriented*, if so is Dg according to Definition 2.5.

The notion of orientability of Fredholm maps of index zero is mainly discussed in [1, 2], where the reader can find examples of orientable and nonorientable maps. It is worthwhile here to recall the following result (see [1]).

Proposition 2.8. *Let $g : \Omega \to F$ be a Fredholm map of index zero. If g is orientable and Ω is connected, then g admits exactly two orientations. If Ω is simply connected, then g is orientable.*

Theorem 2.10 below is the analogue for nonlinear Fredholm maps between Banach spaces of Proposition 2.6 above. We need first the following definition.

Definition 2.9. Let $H : \Omega \times \Lambda \to F$ be a continuous map verifying the following conditions:

(1) any partial map $H_\lambda := H(\cdot, \lambda)$ is Fredholm of index zero;
(2) the partial derivative $\partial_1 : \Omega \times \Lambda \to \Phi_0(E, F)$, given by $\partial_1(x, \lambda) = D(H_\lambda)(x)$, is continuous.

H will be called a *Fredholm homotopy.* An *orientation* of H is an orientation of ∂_1 according to Definition 2.5, and H is *orientable*, or *oriented*, if so is ∂_1.

Theorem 2.10. *Let $H : \Omega \times \Lambda \to F$ be a Fredholm homotopy, with Λ connected and locally path connected. If a given H_λ is orientable, then H is orientable. If, in addition, H_λ is oriented, there exists a unique orientation of H which is associated with that of H_λ.*

We conclude this section by showing that the orientation of a Fredholm map g is related to the orientations of domain and codomain of suitable restrictions of g. This argument will be crucial in the definition of the degree in section 5.

Let $g : \Omega \to F$ be a Fredholm map of index zero and Z a finite dimensional subspace of F, transverse to g. We recall that Z is said to be *transverse* to *g at $x \in \Omega$* if $\operatorname{Im} Dg(x) + Z = F$. The space Z is *transverse* to g if it is transverse at any point of the domain of g.

By classical transversality results, $M = g^{-1}(Z)$ is a differentiable manifold of the same dimension as Z. Assume that g is orientable. It is possible to prove that M is orientable. The proof can be found in [1, Remark 2.5 and Lemma 3.1]. Here, let us show how, given any $x \in M$, an orientation of g and an orientation of Z induce an orientation on the tangent space T_xM of M at x.

Assume that g is oriented and let Z be oriented too. Consider $x \in M$ and a positive corrector A of $Dg(x)$ with image contained in Z (the existence of such a corrector is ensured by the transversality of Z to g). Then, orient T_xM in such a way that the isomorphism

$$(Dg(x) + A)|_{T_xM} : T_xM \to Z$$

is orientation preserving. As proved in [4], the orientation of T_xM does not depend on the choice of the positive corrector A, but only on the orientations of Z and $Dg(x)$. With this orientation, we call M the *oriented Fredholm g-preimage* of Z.

As pointed out in [4], this notion seems very similar to that of oriented preimage given in subsection 2.1. Actually, the two definitions of induced orientation are formally different but strictly related, as the following lemma shows (see [4, Lemma 3.10] for the proof). This will be crucial for the construction of the degree of locally compact multivalued perturbations of Fredholm maps of index zero.

Lemma 2.11. *Let $g : \Omega \to F$ be an oriented map and let F_1 and F_2 be two oriented finite dimensional subspaces of F, both transverse to g. Suppose that F_2 contains F_1. Let M_2 be the oriented Fredholm g-preimage of F_2 and put*

$$M_1 = (g|_{M_2})^{-1}(F_1) = g^{-1}(F_1).$$

Then, M_1 is the oriented $g|_{M_2}$-preimage of F_1 if and only if it is the oriented Fredholm g-preimage of F_1.

Remark 2.12. The reader can also see the interesting and well written survey [28] for a discussion on the finite-dimensional reduction approach to the construction of degree (for one valued functions).

3. Multivalued maps

We describe in this section some known notions of the theory of multivalued maps that will be used in the sequel (details can be found e.g. in [6, 7, 13, 19]). We start by pointing out the following assumption we will carry on through the rest of the paper.

Standing assumption. Given two metric spaces X and Z, any multimap $\Sigma : X \multimap Z$ we will consider in this paper is such that $\Sigma(x)$ is compact for any $x \in X$.

Let X be a metric space. Given a subset A of X and $\varepsilon > 0$, we denote by $O_\varepsilon(A)$ the open ε-neighborhood of A, that is, the union of the open balls of radius ε, centered at the points of X.

Definition 3.1. Let $\Sigma : X \multimap Z$ be a given multimap. Given a positive ε, a continuous map $f_\varepsilon : X \to Z$ is said to be an *ε-approximation* of Σ if for every $x \in X$ there exists $x' \in O_\varepsilon(x)$ such that $f_\varepsilon(x) \in O_\varepsilon(\Sigma(x'))$.

The reader can verify (we omit the details) that such a notion can be equivalently expressed saying that

$$f_\varepsilon(x) \in O_\varepsilon(\Sigma(O_\varepsilon(x)))$$

for all $x \in X$, or

$$\Gamma_{f_\varepsilon} \subseteq O_\varepsilon(\Gamma_\Sigma),$$

where Γ_{f_ε} and Γ_Σ denote the graphs of f_ε and Σ respectively, and the distance in $X \times Z$ is defined in a natural way as

$$d\left((x, z), (x', z')\right) = \max\left\{d_X(x, x'), d_Z(z, z')\right\}$$

(d_X and d_Z stand for the distances in X and Z, respectively). The family of the ε-approximations of a multimap Σ will be denoted $a(\Sigma, \varepsilon)$.

Definition 3.2. A multimap $\Sigma : X \multimap Z$ is said to be *upper semicontinuous* (*u.s.c.* in symbols) if for every open set $V \subseteq Z$ the set $\Sigma_+^{-1}(V) = \{x \in X : \Sigma(x) \subseteq V\}$ is open in X.

In the following proposition we summarize some properties of ε-approximations of u.s.c. multimaps (see e.g. [13]).

Proposition 3.3. *Let $\Sigma : X \multimap Z$ be an u.s.c. multimap. The following conditions hold.*

i) *Let X_1 be a compact subset of X. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $f \in a(\Sigma, \delta)$ implies $f_{|X_1} \in a\left(\Sigma_{|X_1}, \varepsilon\right)$.*
ii) *Suppose that X is compact. Consider a metric space Z_1 and a continuos map $\varphi : Z \to Z_1$. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $f \in a(\Sigma, \delta)$ implies $\varphi \circ f \in a(\varphi \circ \Sigma, \varepsilon)$.*
iii) *Suppose that X is compact and consider an u.s.c. multimap $\Sigma_* : X \times [0,1] \multimap Z$. Then, for every $\lambda \in [0,1]$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $f_* \in a(\Sigma_*, \delta)$ implies that $f_*(\cdot, \lambda) \in a(\Sigma_*(\cdot, \lambda), \varepsilon)$.*
iv) *Let Z_1 be a metric space and $\Sigma_1 : X \multimap Z_1$ an u.s.c. multimap. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $f \in a(\Sigma, \delta)$ and $f_1 \in a(\Sigma_1, \delta)$ imply that $f \times f_1 \in a(\Sigma \times \Sigma_1, \varepsilon)$, where $(f \times f_1) : X \to Z \times Z_1$ is given by*

$$(f \times f_1)(x) = (f(x), f_1(x)),$$

and analogously is defined $\Sigma \times \Sigma_1$.

To introduce the class of multimaps we will consider in the construction of the degree, we recall some other facts. In the following definition, for any natural n, S^n and B^{n+1} stand respectively for the unit sphere and the closed unit ball in $\mathbb{R}^{n+1}$.

Definition 3.4 (see e.g. [6, 13, 14, 22, 24])**.** A nonempty compact subset A of a metric space Z is said to be *aspheric* (or UV^∞, or ∞-*proximally connected*) if for every $\varepsilon > 0$ there exists δ, with $0 < \delta < \varepsilon$, such that for each $n \in \mathbb{N}$ every continuous map $g : S^n \to O_\delta(A)$ can be extended to a continuous map $\widetilde{g} : B^{n+1} \to O_\varepsilon(A)$.

Definition 3.5 (see [13, 17])**.** A nonempty compact metric space is said to be an *R_δ-set* if it can be represented as the intersection of a decreasing sequence of compact and contractible spaces.

Remark 3.6. The reader can see [13, Example 2.12] for an example of a noncontractible R_δ-set.

The following proposition shows a sufficient condition ensuring the equivalence of the above concepts in the case of ANR-spaces (see e.g. [13, 17, 25]).

Proposition 3.7. *A compact subset of an ANR-space is aspheric if and only if is an R_δ-set.*

Definition 3.8 (see [13])**.** An u.s.c. multimap $\Sigma : X \multimap Z$ is said to be a *J-multimap* if, for every $x \in X$, $\Sigma(x)$ is an aspheric set. The set of J-multimaps of X to Z will be denoted by $J(X,Z)$.

Proposition 3.9 (see [13, 24])**.** *Let Z be an ANR-space and $\Sigma : X \multimap Z$ an u.s.c. multimap. Then, Σ is a J-multimap in each of the following cases: for every $x \in X$, $\Sigma(x)$ is*

a) *a convex set;*
b) *a contractible set;*
c) *an R_δ-set;*
d) *an AR-space.*

Remark 3.10. Actually, one can observe that a) and b) are particular cases of c).

The next statement describes some approximation properties of J-multimaps.

Proposition 3.11 (see [13, 14, 22])**.** *Let X be a compact ANR-space and Z a metric space. Consider a J-multimap $\Sigma : X \multimap Z$. Then:*

i) *Σ is approximable, i.e. for every $\varepsilon > 0$ there exists $f_\varepsilon \in a(\Sigma,\varepsilon)$;*
ii) *for each $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for every δ $(0 < \delta < \delta_0)$ and for every two δ-approximations $f_\delta, f_\delta' \in a(\Sigma,\delta)$ there exists a continuous homotopy $f_* : X \times [0,1] \to Z$ such that*
   (a) $f_*(\cdot,0) = f_\delta$, $f_*(\cdot,1) = f_\delta'$;
   (b) $f_*(\cdot,\lambda) \in a(\Sigma,\varepsilon)$ for all $\lambda \in [0,1]$.

Definition 3.12. Given X, X' metric spaces, by $CJ(X,X')$ we denote the collection of all multimaps $\Sigma : X \multimap X'$ of the form $\Sigma = \varphi \circ \widetilde{\Sigma}$, where $\widetilde{\Sigma} \in J(X,Z)$ for some metric space Z and $\varphi : Z \to X'$ is a continuous map. The composition $\varphi \circ \widetilde{\Sigma}$ will be called a *representation* (or *decomposition*, see [13]) of Σ.

4. Orientability for quasi-Fredholm multimaps

In this section we define the concepts of orientability and orientation for *quasi-Fredholm multimaps*, i.e. locally compact CJ multivalued perturbations of Fredholm maps of index zero in Banach spaces.

As in the final part of the above section, also in this one E and F will stand for real Banach spaces, while Ω will denote an open subset of E. The notions of orientability and orientation introduced in this section generalize the analogous notions for quasi-Fredholm maps, introduced in [4].

Definition 4.1. A multimap $f\colon \Omega \multimap F$ is called a *quasi-Fredholm multimap* if it can be written as $f = g - K$, where $g\colon \Omega \to F$ is a Fredholm map of index zero and $K : \Omega \multimap F$ is a locally compact CJ-multimap. We will call g a *smoothing map* of f.

In what follows, unless otherwise stated, f will denote a quasi-Fredholm multimap of an open subset Ω of E to F, and $\mathcal{S}(f)$ will stand for the family of smoothing maps of f.

Remark 4.2. If g_0 is a given smoothing map of quasi-Fredholm multimap $f\colon \Omega \multimap F$ and $h\colon \Omega \to F$ is an arbitrary C^1 locally compact map, then $g_0 - h$ is in $\mathcal{S}(f)$ as it is immediate to verify. The converse is also true. To see this, consider two smoothing maps g_1 and g_2 of f, and write

$$f = g_1 - K_1 = g_2 - K_2,$$

where K_1 and K_2 are locally compact CJ-multimaps. One has

$$g_1(x) - g_2(x) \in K_1(x) - K_2(x), \qquad \forall x \in \Omega,$$

where $K_1(x) - K_2(x)$ is the set defined as

$$\{z \in F : \exists\, p \in K_1(x) \text{ and } q \in K_2(x) \text{ with } z = p - q\}.$$

Let $x \in \Omega$ be given and O a neighborhood of x in Ω such that the two closed sets $\overline{K_1(O)}$ and $\overline{K_2(O)}$ are compact. It is easy to see that $\overline{K_1(O)} - \overline{K_2(O)}$ is compact. Thus $\overline{(g_1 - g_2)(O)}$ turns out be compact, being contained in $\overline{K_1(O)} - \overline{K_2(O)}$. Therefore, $g_1 - g_1$ is locally compact (and clearly C^1).

The following definition provides an extension to quasi-Fredholm multimaps of the concept of orientability given in section 2.2.

Definition 4.3. A quasi-Fredholm multimap $f\colon \Omega \multimap F$ is *orientable* if it has an orientable smoothing map.

If f is orientable, then any smoothing map of f is orientable. Indeed, given $g_0, g_1 \in \mathcal{S}(f)$, consider the homotopy $H\colon \Omega \times [0,1] \to F$ defined by

$$H(x,\lambda) = (1-\lambda)g_0(x) + \lambda g_1(x). \tag{4.1}$$

Since $\mathcal{S}(f)$ is a convex set (Remark 4.2), H is a Fredholm homotopy (recall Definition 2.9). Thus, because of Theorem 2.10, if g_0 is orientable, then g_1 is orientable as well. Applying again Theorem 2.10, if g_0 is oriented, g_1 can be oriented by transporting the orientation of g_0 up to g_1 along the line segment joining g_0 with g_1. By the convexity of $\mathcal{S}(f)$, any other map g in $\mathcal{S}(f)$ can be oriented with the unique orientation transported by g_0.

Keeping in mind this argument, to define a notion of orientation of f consider the set $\widehat{\mathcal{S}}(f)$ of the oriented smoothing maps of f. We introduce in $\widehat{\mathcal{S}}(f)$ the following equivalence relation. Given g_0, g_1 in $\widehat{\mathcal{S}}(f)$, consider, as in formula (4.1), the straight-line homotopy H joining g_0 and g_1. We say that g_0 is *equivalent* to g_1 if the unique orientation of H which is associated with g_0 (ensured by Theorem 2.10; recall also the definition given after Definition 2.5) is associated with g_1 too.

The proof of Proposition 4.4 below can be found in [5], where the result is used in the construction of the orientation for locally compact *single valued* perturbations of nonlinear Fredholm maps.

Proposition 4.4. *The above is an equivalence relation in* $\widehat{\mathcal{S}}(f)$.

The following definition provides an extension to multivalued quasi-Fredholm maps of the concept of orientation given in Definition 2.7.

Definition 4.5. Let $f\colon \Omega \multimap F$ be an orientable multivalued quasi-Fredholm map. An *orientation* of f is an equivalence class of $\widehat{\mathcal{S}}(f)$.

In the sequel, if a multivalued quasi-Fredholm map f is oriented, any element in the chosen class of $\widehat{\mathcal{S}}(f)$ will be called a *positively oriented smoothing map* of f.

Observe that, if two oriented smoothing maps of $f\colon \Omega \multimap F$ are equivalent and V is an open subset of Ω, then the oriented restrictions to V of these two smoothing maps are equivalent in the set $\widehat{\mathcal{S}}(f|_V)$. Thus, an orientation of f induces in a natural way an orientation of the restriction $f|_V$, which will be called the *oriented restriction* of f to V. Notice that, if V is empty, the oriented restriction of f to V is unique, and this does not depend on the orientation of f.

Below, we have the analogue for multivalued quasi-Fredholm maps of Proposition 2.8.

Proposition 4.6. *Let $f\colon \Omega \multimap F$ be a multivalued quasi-Fredholm map. If f is orientable and Ω is nonempty, then f admits at least two orientations. If, in addition, Ω is connected, then f admits exactly two orientations (one opposite to the other). If Ω is simply connected, then f is orientable.*

As for Fredholm maps of index zero, the orientability of multivalued quasi-Fredholm maps verifies a property of continuous transport along homotopies, Theorem 4.11 below states. The following construction extends to multivalued maps the analogous one given in [4] (see also [5]). We need first some definitions.

Definition 4.7. Let $H\colon \Omega \times [0,1] \multimap F$ be a multimap of the form

$$H(x,\lambda) = G(x,\lambda) - K(x,\lambda),$$

where G is continuous and verifies the following condition:

i) for any $\lambda \in [0,1]$ the partial map $x \mapsto G(x,\lambda)$ is a nonlinear Fredholm map of index zero in Ω,
ii) $(x,\lambda) \mapsto \partial_1 G(x,\lambda)$ is continuous, $\partial_1 G(x,\lambda)$ stands for the Fréchet partial derivative of G with respect to the first variable at the point (x,λ);

in addition K is assumed to be a locally compact CJ-multimap. We call H a *multivalued quasi-Fredholm homotopy* and G a *smoothing homotopy* of H.

Remark 4.8. This last term can be taken with pinch of salt: G is not necessarily smooth (with respect to λ), but any $G(\cdot,\lambda)$ is a smoothing map of $H(\cdot,\lambda)$.

Remark 4.9. The definition of orientability for multivalued quasi-Fredholm homotopies is analogous to that given for multivalued quasi-Fredholm maps. Let $H\colon \Omega \times [0,1] \multimap F$ be a multivalued quasi-Fredholm homotopy. Let $\widehat{\mathcal{S}}(H)$ be the set of oriented smoothing homotopies of H. Assume that $\widehat{\mathcal{S}}(H)$ is nonempty and define on this set an equivalence relation as follows. Given G_0 and G_1 in $\widehat{\mathcal{S}}(H)$, consider the map

$$\mathcal{G}\colon \Omega \times [0,1] \times [0,1] \to F,$$

defined as

$$\mathcal{G}(x,\lambda,\mu) = (1-\mu)G_0(x,\lambda) + \mu G_1(x,\lambda).$$

We say that G_0 is *equivalent* to G_1 if their orientations are associated with an orientation of the map

$$(x,\lambda,\mu) \mapsto \partial_1\mathcal{G}(x,\lambda,\mu).$$

As in Proposition 4.4, it is possible to prove (we omit the details) that this is actually an equivalence relation on $\widehat{\mathcal{S}}(H)$.

Definition 4.10. A multivalued quasi-Fredholm homotopy $H\colon \Omega\times[0,1] \multimap F$ is said to be *orientable* if $\widehat{\mathcal{S}}(H)$ is nonempty. An *orientation* of H is an equivalence class of $\widehat{\mathcal{S}}(H)$.

Theorem 4.11 (Orientation transport for quasi-Fredholm multimaps)**.** *Let $H\colon \Omega\times[0,1] \multimap F$ be a multivalued quasi-Fredholm homotopy. If a partial multimap H_λ is oriented, then there exists and is unique an orientation of H which is associated with H_λ.*

We conclude the section by showing an example of multivalued quasi-Fredholm homotopy.

Example 4.12. Let $\phi : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ and $\psi : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be C^1 and continuous, respectively. Denote by $\mathcal{C}^1$ and $\mathcal{C}^0$ the Banach spaces $C^1([0,T],\mathbb{R}^n)$ and $C([0,T],\mathbb{R}^n)$, then consider

$$\widetilde{G} : \mathcal{C}^1\times\mathbb{R} \to \mathcal{C}^0, \qquad \widetilde{G}(x,\lambda)(t) = x'(t) + \lambda\phi(t,x(t),x'(t)),$$
$$\widetilde{K} : \mathcal{C}^1\times\mathbb{R} \to \mathcal{C}^0, \qquad \widetilde{K}(x,\lambda)(t) = \lambda\psi(t,x(t)).$$

Since ϕ is C^1, so is $\widetilde{G}$ and the Fréchet derivative $D\widetilde{G}_\lambda(x) : \mathcal{C}^1 \to \mathcal{C}^0$ of any partial map $\widetilde{G}_\lambda$ at any $x\in\mathcal{C}^1$ is given by

$$\big(D\widetilde{G}_\lambda(x)q\big)(t) = q'(t) + \lambda\partial_2\phi(t,x(t),x'(t))q(t) + \lambda\partial_3\phi(t,x(t),x'(t))q'(t), \tag{4.2}$$

where $\partial_2\phi$ and $\partial_3\phi$ denote the jacobian matrices of ϕ with respect to the second and third variable. Formula (4.2) can be rewritten as

$$\big(D\widetilde{G}_\lambda(x)q\big)(t) = (I+\lambda M_x(t))q'(t) + \lambda N_x(t)q(t)$$

where I is the $n\times n$ identity matrix and, given $x\in\mathcal{C}^1$, M_x and N_x are $n\times n$ matrices of continuous real functions defined in $[0,T]$. Clearly, if x and λ are such that

$$\det(I+\lambda M_x(t)) \neq 0, \quad \forall t\in[0,T], \tag{4.3}$$

then $D\widetilde{G}_\lambda(x) : \mathcal{C}^1 \to \mathcal{C}^0$ is a first order linear differential operator and, consequently, it is onto with n-dimensional kernel.

Consider now the boundary operator

$$B : \mathcal{C}^1 \to \mathbb{R}^n, \quad B(x) = x(T) - x(0).$$

Set $E = \operatorname{Ker} B$ and $F = \mathcal{C}^0$, and let $G, K : E\times\mathbb{R} \to F$ denote the restrictions of $\widetilde{G}$ and $\widetilde{K}$ to the space $E\times\mathbb{R}$. Observe that, as B is surjective, E is a closed subspace of $\mathcal{C}^1$ of codimension n and thus, for each $x\in E$ and $\lambda\in\mathbb{R}$ such that (4.3) is verified, $DG_\lambda(x)$ is Fredholm of index zero. In fact, $DG_\lambda(x)$ is the composition of the inclusion

$E \hookrightarrow \mathcal{C}^1$, which is Fredholm of index $-n$, with $D\widetilde{G}_\lambda(x)$. Since the inclusion $\mathcal{C}^1 \hookrightarrow \mathcal{C}^0$ is compact, the map K is locally compact (completely continuous, actually).

Consider a CJ-multimap $\gamma : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ and

$$\Gamma : E \times \mathbb{R} \multimap F, \quad \Gamma(x, \lambda)(t) = \lambda \int_0^t \gamma(s, x(s))\, ds,$$

recalling that $\int_0^t \gamma(s, x(s))\, ds$ is defined as $\left\{\int_0^t f(s)\, ds\right\}$, where f is a measurable selection, i.e., $f(s) \in \gamma(s, x(s))$ a.e. $s \in [0, T]$. The multimap Γ is actually a locally compact CJ-multimap (see [19, Chapter 1] for the details).

Thus, if condition (4.3) is satisfied for any $x \in E$ and $\lambda \geq 0$,

$$H : E \times [0, +\infty) \multimap F, \quad H(x, \lambda) = G(x, \lambda) + K(x, \lambda) + \Gamma(x, \lambda)$$

turns out to be a homotopy of quasi-Fredholm multimaps, which is orientable since $E \times [0, +\infty)$ is simply connected. This is the case if (and only if) for every

$$(t, a, b) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n,$$

the jacobian matrix $\partial_3\phi(t, a, b)$ has no negative eigenvalues. The reader can see [3, 4] for analogous examples with single valued maps.

5. Degree for multivalued maps

In this section we recall the construction of the degree for the class of the oriented multivalued quasi-Fredholm maps in Banach spaces.

Definition 5.1. Let Ω be an open subset of E and consider an oriented quasi-Fredholm multimap $f : \Omega \multimap F$. Given an open subset U of Ω, the pair (f, U) is called an *admissible pair* if the coincidence set

$$C(f, U) = \{x \in U : 0 \in f(x)\}$$

is compact.

The construction is divided in two steps. In the first one we consider pairs (f, U) such that f has a smoothing map g with $(f - g)(U)$ contained in a finite dimensional subspace of F. In the second step we remove this assumption, defining the degree for general admissible pairs.

Step 1. Let (f, U) be an admissible pair and let g be a positively oriented smoothing map of f such that $(f - g)(U)$ is contained in a finite dimensional subspace of F. Being $C(f, U)$ compact, by classical transversality results, there exist a finite dimensional subspace Z of F and an open neighborhood W of $C(f, U)$ in U, such that

i) g is transverse to Z in W;
ii) Z contains $(g - f)(U)$.

Let $M = g^{-1}(Z) \cap W$. One can easily verify that M contains $C(f, U)$. As seen in subsection 2.2, M is an orientable C^1 manifold of the same dimension as Z. Then, let Z be oriented and orient M in such a way that it is the oriented Fredholm $g|_W$-preimage of Z.

By the compactness of $C(f,U)$ (also as a subset of M) it follows that there exists a finite number of bounded open subsets $V_1, ..., V_k$ of M such that:

(1) the closure of V_j in E, $\overline{V}_j$, is contained in M, $j = 1, ..., k$;
(2) $V := \cup_{j=1}^k V_j$ contains $C(f,U)$;
(3) every $\overline{V}_j$ is diffeomorphic to a closed convex subset of $\mathbb{R}^m$, where m is the dimension of Z (and hence of M).

Therefore, by Proposition 3.11, the restriction of $K := g - f$ to $\overline{V}$ is approximable. In particular there exists a positive ε such that any ε-approximation $k : \overline{V} \to Z$ of the restriction of K to $\overline{V}$ verifies the condition

$$\operatorname{dist}(0, (g - k)(\partial V)) > 0.$$

Therefore, the triple $(g|_V - k, V, 0)$ is admissible for the Brouwer degree, and we define

$$\deg(f, U) = \deg_B(g|_V - k, V, 0), \tag{5.1}$$

where the right hand side denotes the Brouwer degree of the triple $(g|_V - k, V, 0)$, and $k : \overline{V} \to Z$ is an ε-approximation of the restriction of K to $\overline{V}$, where ε is sufficiently small.

In order to prove that the degree is well-defined, we have to check that the right hand side of (5.1) is independent of the choice of the smoothing map g, the open set W, the subspace Z, the open set V and the approximation k.

Concerning the independence of k, fix a positively oriented smoothing map g of f such that $(f - g)(U)$ is contained in a finite dimensional subspace of F. Let W, Z, V be given as in the above construction. As proven is [24, Lemma 3.4], if ε is sufficiently small and $k_1, k_2 : \overline{V} \to Z$ of K are two ε-approximations of $K|_{\overline{V}}$, we have the equality

$$\deg_B(g|_V - k_1, V, 0) = \deg_B(g|_V - k_2, V, 0),$$

which is obtained by the homotopy invariance property of the Bouwer degree.

Now, if Z and W are given as above, and $\overline{V_1}$, $\overline{V_2}$ are two ANR and compact neighborhoods of $C(f,U)$ in M, we can assume without loss of generality that $\overline{V_1} \subseteq \overline{V_2}$. Hence, item i) of Proposition 3.3 ensures that, if ε is suffciently small, we can find an ε-approximation $k : \overline{V_2} \to Z$ of the restriction of K to $\overline{V_2}$ such that $k|_{\overline{V_1}}$ is an ε-approximation of $K|_{\overline{V_1}}$ and that

$$(g|_{V_1} - k, V_1, 0) \quad \text{and} \quad (g|_{V_2} - k, V_2, 0)$$

are admissible for the Brouwer degree. Finally,

$$\deg_B(g|_{V_1} - k, V_1, 0) = \deg_B(g|_{V_2} - k, V_2, 0)$$

by the excision property of the Brouwer degree.

To conclude, the independence of the definition of the degree of W and Z is a straightforward application of the reduction property of the Brouwer degree (Proposition 2.2).

It remains to show the independence of the smoothing map g. To this purpose, consider two positively oriented smoothing maps g_0 and g_1 of f such that $(f - g_0)(U)$

and $(f - g_1)(U)$ are contained in a finite dimensional subspace of F. Consider the homotopy $G : \Omega \times [0,1] \to F$, defined by

$$G(x, \lambda) = (1 - \lambda)g_0(x) + \lambda g_1(x).$$

By the compactness of $C(f, U)$, there exist an open subset W of U, containing $C(f, U)$, and a finite dimensional subspace Z of F, containing $(f - g_0)(U)$ and $(f - g_1)(U)$, such that, for each $\lambda \in [0, 1]$, the partial map G_λ is transverse to Z in W. Hence, Z is transverse to G in $W \times [0, 1]$ and to the restriction of G to the boundary of $W \times [0, 1]$. Thus $G^{-1}(Z) \cap (W \times [0, 1])$ is a C^1 manifold with boundary of dimension equal to $1 + \dim Z$.

Since $(f - g_0)(U)$ and $(f - g_1)(U)$ are contained in Z, we get $G_\lambda^{-1}(Z) \cap W = G_s^{-1}(Z) \cap W$, for any $\lambda, s \in [0, 1]$. Therefore $G^{-1}(Z) \cap (W \times [0, 1])$ is actually a product manifold, denoted by $M \times [0, 1]$, where $M = G_\lambda^{-1}(Z) \cap W$, for any $\lambda \in [0, 1]$.

Let now Z be oriented and, for any $\lambda \in [0, 1]$, denote by M_λ the manifold M, which is oriented in such a way that it becomes the oriented Fredholm $G_\lambda|_W$-preimage of Z. The reader can imagine each M_λ as the set of pairs $(x, \alpha(x, \lambda))$, where $x \in M$ and $\alpha(x, \lambda)$ is the orientation of M at x induced by $G_\lambda|_W$ and Z.

We can prove that, for any $s, \lambda \in [0, 1]$, $M_s = M_\lambda$ (in other words, we can prove that the orientations of M_s and M_λ coincide). To see this, let $\lambda_0 \in [0, 1]$ and $(x, \alpha(x, \lambda_0)) \in M_{\lambda_0}$ be given. Since G is clearly oriented (with an orientation such that the orientations of g_0 and g_1 are inherited from that of G), a positive corrector A of $G'_{\lambda_0}(x)$ remains a positive corrector for $G'_\lambda(x)$, with λ in a suitable neighborhood of λ_0. Then, recalling the definition of oriented Fredholm preimage, $\alpha(x, \lambda_0) = \alpha(x, \lambda)$. By the connectedness of $[0, 1]$, the claim follows. Therefore,

$$\deg(f|_{M_0}, M_0) = \deg(f|_{M_1}, M_1), \tag{5.2}$$

and thus we can say that $\deg(f, U)$ is indeed well defined.

Step 2. Let us now extend the definition of degree to general admissible pairs.

Definition 5.2 (general definition of degree)**.** Let (f, U) be an admissible pair. Consider:

(1) a positively oriented smoothing map g of f;
(2) an open neighborhood V of the coincidence set $C(f, U)$ such that $\overline{V} \subseteq U$, g is proper on $\overline{V}$ and $(f - g)|_{\overline{V}}$ is compact;
(3) a CJ-multimap $\Xi : \overline{V} \multimap F$ having bounded finite dimensional image and such that
$$\|g(x) - f(x) - \Xi(x)\| < \rho, \quad \forall x \in \partial V,$$
where ρ is the distance in F between 0 and $f(\partial V)$ and the norm of a subset B of a normed space is defined as $\|B\| = \sup\{\|x\|,\ x \in B\}$.

Then, we define

$$\deg(f, U) = \deg(g - \Xi, V), \tag{5.3}$$

where the right hand side of the equality is the degree following the above step 1.

First of all observe that the right hand side of (5.3) is well defined since the pair $(g - \Xi, V)$ is admissible. Indeed, $g - \xi$ is proper on $\overline{V}$ and thus the coincidence set

$C(g-\Xi, U)$ is a compact subset of $\overline{V}$ which is actually contained in V by assumption (3).

We have to show that $\deg(f,U)$ is well-defined, in the sense that formula (5.3) does not depend on g, Ξ and V. Consider two positively oriented smoothing maps g_0 and g_1. For $i = 0, 1$, let V_i be an open neighborhood of the coincidence set $C(f,U)$ such that $\overline{V}_i \subseteq U$, g_i is proper on $\overline{V}_i$ and $(f-g_i)|_{\overline{V}_i}$ is compact. Moreover, consider a CJ-multimap $\Xi_i : \overline{V}_i \multimap F$ with bounded finite dimensional image and such that

$$\|g_i(x) - f(x) - \Xi_i(x)\| < \rho, \quad \forall x \in \partial V_i, \tag{5.4}$$

where ρ is the distance in F between 0 and the closed set $f((\overline{V}_0 \cup \overline{V}_1)\backslash(V_0 \cap V_1))$. For $i = 0, 1$, the map $f_i : \overline{V}_i \multimap F$, defined by

$$f_i(x) = g_i(x) - \xi_i(x),$$

is oriented having g_i as positively oriented smoothing map. In addition, since g_i is proper on $\overline{V}_i$, f_i turns out to be proper as well. By (5.4), $C(f_1, U)$ is a compact subset of $V_0 \cap V_1$. In particular, (f_0, V_0) and (f_1, V_1) are admissible. We need to show that

$$\deg(f_0, V_0) = \deg(f_1, V_1). \tag{5.5}$$

To see this, denoting $V = V_0 \cap V_1$, define $H : \overline{V} \times [0,1] \multimap F$ by

$$H(x, \lambda) = (1-\lambda) f_0(x) + \lambda f_1(x),$$

and $G : \overline{V} \times [0,1] \to F$ by

$$G(x, \lambda) = (1-\lambda) g_0(x) + \lambda g_1(x).$$

The map H is proper, being a compact perturbation of g_0. Hence, $H^{-1}(0)$ is compact and, by (5.4), contained in $V \times [0,1]$. Thus there exist an open subset W of $V \times [0,1]$ containing the coincidence set $C(H, V \times [0,1])$, and a subspace Z of F of finite dimension, say n, containing $\Xi_0(\overline{V})$ and $\Xi_1(\overline{V})$ such that every partial map G_λ is transverse to Z on

$$W_\lambda = \{x \in V : (x, \lambda) \in W\}.$$

Consequently, the set $M = G^{-1}(Z) \cap W$ is an $(n+1)$-manifold with boundary $(M_0 \times \{0\}) \cup (M_1 \times \{1\})$. In addition, the transversality of G_λ to Z implies that any section M_λ is a boundaryless n-manifold.

Let Z be oriented and orient M in such a way that any M_λ is the oriented Fredholm G_λ-preimage of Z. By the definition of degree in the step 1 (formula (5.1)), one has

$$\deg(f_0, V_0) = \deg_B(f_0|_{M_0}, M_0, 0),$$

$$\deg(f_1, V_1) = \deg_B(f_1|_{M_1}, M_1, 0).$$

The homotopy invariance property of the Brouwer degree implies

$$\deg_B(f_0|_{M_0}, M_0, 0) = \deg_B(f_1|_{M_1}, M_1, 0).$$

Therefore,

$$\deg(f_0, V_0) = \deg(f_1, V_1),$$

and we can conclude that $\deg(f,U)$ is well-defined by (5.3).

Remark 5.3. Of course, a Fredholm map of index zero is also a quasi-Fredholm multimap, and Definition 5.2 can be applied to an admissible pair (f, U) with f of class C^1. In this case a definition of degree is given in [1] - for triples (f, U, y) - by a different approach. The reduction property proved in [1, section 3] shows that the two degrees coincide when both are defined (i.e., in the C^1 case).

We conclude by listing in the following theorem the main properties of the degree. The proof of the first two properties is a straightforward consequence of the definition of the degree and is omitted.

Theorem 5.4. *The following properties of the degree hold:*

1. *(Normalization) Let U be an open neighborhood of* 0 *in E and let the identity I of E be naturally oriented. Then,*
$$\deg(I, U) = 1.$$
2. *(Additivity) Let (f, U) be an admissible pair and U_1, U_2 two disjoint open subsets of U such that the coincidence set $C(f, U)$ is contained in $U_1 \cup U_2$. Then (f, U_1) and (f, U_2) are admissible and*
$$\deg(f, U) = \deg(f, U_1) + \deg(f, U_2).$$
3. *(Homotopy invariance) Let $G : U \times [0, 1] \to F$ be an oriented homotopy of Fredholm maps and $K : U \times [0, 1] \multimap F$ a locally compact CJ-multimap. Assume that the set $C := \{(x, \lambda) \in U \times [0, 1] : G(x, \lambda) \in K(x, \lambda)\}$ is compact. Then* $\deg(G_\lambda - K_\lambda, U)$ *is well defined and does not depend on* $\lambda \in [0, 1]$.

The proof of the homotopy invariance property is not a simple consequence of the definition. For a detailed proof the reader can see the text of Väth [27], em particular Theorems 12.23, 13.5 and 13.19. Let us give here some remarks concerning the difficulties in the proof. Since the set C is compact, its projection on E, say S, is a compact subset of U. Consequently, there exists an open neighborhood V of S, with $\overline{V} \subseteq U$, such that G is proper and K is compact on $\overline{V} \times [0, 1]$.

Recalling Definition 5.2, let $\Xi : \overline{V} \times [0, 1] \multimap F$ be a a CJ-multimap having bounded finite dimensional image and such that $\|K(x, \lambda) - \Xi(x, \lambda)\| < \rho$, for each $(x, \lambda) \in \partial V \times [0, 1]$, where ρ is the distance in F between 0 and $(G - K)(\partial V \times [0, 1])$. Then, we have
$$\deg(G_\lambda - K_\lambda, U) = \deg(G_\lambda - \Xi_\lambda, V), \quad \forall \lambda \in [0, 1].$$
By the compactness of C, there exist an open subset W of $V \times [0, 1]$ containing C, and a subspace Z of F of finite dimension, say n, containing $\Xi(\overline{V} \times [0, 1])$ such that every partial map G_λ is transverse to Z on W_λ. Proceeding analogously to the previous step 1 (of the construction of the degree), we observe that M contains the set $D := \{(x, \lambda) \in V \times [0, 1] : G(x, \lambda) \in \Xi(x, \lambda)\}$. In addition, still recalling the above step 1, there exists an open subset $\mathcal{V}$ of M, containing D and such that the restriction of Ξ to $\mathcal{V}$ is approximable.

Now, an important point is: if G is C^1, the set $M := G^{-1}(Z) \cap W$ is an $(n + 1)$-manifold with boundary $(M_0 \times \{0\}) \cup (M_1 \times \{1\})$. Thus, orienting Z and M in such a way that any M_λ is the oriented Fredholm G_λ-preimage of Z, given $\varepsilon > 0$

sufficiently small and $\xi : \mathcal{V} \to Z$ an ε-aproximation of the restriction of Ξ to $\mathcal{V}$, we have, analogously to formula (5.1),

$$\deg(G_\lambda - \Xi_\lambda, \mathcal{V}_\lambda) = \deg_B((G_\lambda)|_{\mathcal{V}_\lambda} - \xi_\lambda, V_\lambda, 0).$$

Unfortunately, *M is not necessarily a C^1 manifold, since G is not assumed to be* C^1, even if any M_λ is an orientable C^1 manifold of the same dimension of Z. This problem is overcome in [27] analyzing the topological structure of M, see sections 7.3 e 8.8, and the homotopy invariance property is obtained.

6. Atypical bifurcation results

In this section we come back to problem (1.1) presenting some atypical bifurcation results (see the Introduction for the definition of bifurcation point). We assume that

$$(\lambda, x) \mapsto Lx + \lambda\,(h(\lambda, x) - \mathcal{H}(\lambda, x))$$

is an oriented multivalued quasi-Fredholm homotopy (recall Remark 4.9). Let $\pi : F \to F/\operatorname{Im} L$ denote the canonical projection, while $R : F \to \operatorname{Im} L$ stands for a bounded linear retraction, i.e. such that $Ry = y$ for every $y \in \operatorname{Im} L$. Problem (1.1) is clearly equivalent to the system

$$\begin{cases} Lx + \lambda Rh(\lambda, x) & \in \quad \lambda R\mathcal{H}(\lambda, x) \\ \lambda \pi h(\lambda, x) & \in \quad \lambda \pi \mathcal{H}(\lambda, x) \end{cases} \tag{6.1}$$

and, if $\lambda \neq 0$, to

$$\begin{cases} Lx + \lambda Rh(\lambda, x) & \in \quad \lambda R\mathcal{H}(\lambda, x) \\ \pi h(\lambda, x) & \in \quad \pi \mathcal{H}(\lambda, x). \end{cases} \tag{6.2}$$

Theorem 6.1 (necessary condition)**.** *Assume that p is a bifurcation point for the equation* (1.1)*. Then, $(h(0,p) - \mathcal{H}(0,p)) \cap \operatorname{Im} L \neq \emptyset$ or, equivalently, $\pi(h(0,p) - \mathcal{H}(0,p)) \ni 0$.*

Proof. Since p is a bifurcation point, there exists a sequence $\{(\lambda_n, p_n)\}$ of solutions of (6.2) converging to $(0, p)$. The result easily follows from the upper semicontinuity of $(\lambda, x) \mapsto \lambda\,(h(\lambda, x) - \mathcal{H}(\lambda, x))$. □

Theorem 6.3 below gives a sufficient condition for the existence of a bifurcation point. The proof is a straightforward extension of [3, Theorem 3.2] and it is given here for a sake of completeness. It uses the degree of a suitable finite-dimensional multimap between $\operatorname{Ker} L$ and $F/\operatorname{Im} L$. Even though these spaces must be oriented (and so we consider), the result is independent of the chosen orientations. Let us first recall the following result (see Lemma 1.4 of [12]) which plays a crucial role in the proof of Theorem 6.3. We point out that the degree in the statement of Theorem 6.3 is a finite-dimensional multivalued degree, which can be seen as a multivalued extension of the classical Brouwer degree or, equivalently, as the finite-dimensional particular case of the degree defined in section 5.

Lemma 6.2. *Let K be a compact subset of a locally compact metric space X. Assume that any compact subset of X that contains K has nonempty boundary. Then $X \backslash K$ contains a not relatively compact component whose closure in X intersects K.*

Before stating Theorem 6.3, let us recall that, given any subset Ω of $\mathbb{R} \times E$, for any real λ we denote by Ω_λ the set $\{x \in E : (\lambda, x) \in \Omega\}$. The notation is analogous for any subset of $\mathbb{R} \times E$. See the presentation of problem (1.1) in the Introduction.

Theorem 6.3 (sufficient condition)**.** *Let* $v : \operatorname{Ker} L \cap \Omega_0 \multimap F/\operatorname{Im}L$ *be defined by* $v(p) = \pi(h(0,p) - \mathcal{H}(0,p))$. *Given an open subset* U *of* Ω, *assume* $\deg(v, U_0 \cap \operatorname{Ker} L)$ *is well defined and different from 0. Then there exists a connected set of nontrivial solutions of* (1.1) *whose closure in* U *is not compact and intersects* $\{0\} \times \operatorname{Ker} L$.

Proof. We denote by $\Sigma : \Omega \multimap \operatorname{Im} L \times F/\operatorname{Im} L$ the multimap

$$(\lambda, x) \mapsto (Lx + \lambda R(h(\lambda, x) - \mathcal{H}(\lambda, x)), \pi(h(\lambda, x) - \mathcal{H}(\lambda, x))),$$

which is clearly a quasi-Fredholm homotopy, and it is orientable since Ω is simply connected. Assume Σ oriented. This gives an orientation of any partial multimap $\Sigma_\lambda : \Omega_\lambda \multimap \operatorname{Im} L \times F/\operatorname{Im} L$. Consider the set

$$Y = \{(\lambda, x) \in U : \Sigma(\lambda, x) \ni 0\}$$

and observe that the zero section Y_0 is compact since $\deg(v, U_0 \cap \operatorname{Ker} L)$ is assumed to be defined. In addition it is possible to prove that Y is locally compact since

$(\lambda, x) \mapsto (Lx + \lambda Rh(\lambda, x), \pi h(\lambda, x))$ is locally proper

$(\lambda, x) \mapsto (\lambda R\mathcal{H}(\lambda, x), \pi\mathcal{H}(\lambda, x))$ is locally compact

(we omit the details). We apply Lemma 6.2 to the pair $(Y, \{0\} \times Y_0)$. Assume, by contradiction, there exists a compact set $C \subseteq Y$ containing $\{0\} \times Y_0$ that is open in Y. This implies the existence of an open subset W of $\mathbb{R} \times E$ such that $W \subseteq U$ and $W \cap Y = C$. Since C is compact, the homotopy invariance property implies that $\deg(\Sigma_\lambda, W_\lambda)$ does not depend on $\lambda \in \mathbb{R}$. Moreover, $W_\lambda \cap C_\lambda$ is empty for some λ. Hence, we obtain $\deg(\Sigma_0, W_0) = 0$. The inclusions $Y_0 \subseteq W_0 \subseteq U_0$ imply, using the excision property of the degree, $\deg(\Sigma_0, U_0) = 0$.

Now, observe that the subspace $F_0 = \{0\} \times F/\operatorname{Im} L$ is transverse to Σ_0 and $(\Sigma_0)_+^{-1}(F_0) = \operatorname{Ker} L \cap \Omega_0$. Thus, we have

$$\deg(\Sigma_0, U_0) = \deg(v, U_0 \cap \operatorname{Ker} L),$$

which can be seen as a reduction property for the degree for multivalued quasi-Fredholm maps and is a straightforward consequence of the definition. The above equality contradicts the assumption of the theorem and the assertion holds. □

The following corollary extends to the multivalued case an analogous result for single valued problem (see [3]).

Corollary 6.4. *Let the assumptions of Theorem 6.3 be satisfied. In addition, assume that*

(1)	$(\lambda, x) \mapsto Lx + \lambda h(\lambda, x)$	*is proper*
(2)	$(\lambda, x) \mapsto \mathcal{H}(\lambda, x)$	*is compact*

on bounded and closed subsets of U. *Then* (1.1) *admits a connected set* S *of nontrivial solutions such that its closure in* $\mathbb{R} \times E$ *intersects* $\{0\} \times \operatorname{Ker} L$ *and is either unbounded or reaches the boundary of* U. *In particular, if* $U = \mathbb{R} \times E$, S *is unbounded.*

Proof. Call $\overline{S}$ the closure in $\mathbb{R} \times E$ of a connected branch S as in Theorem 6.3. Suppose $\overline{S} \cap \partial U = \emptyset$. Thus, the closure of S in U coincides with $\overline{S}$ and cannot be bounded according to (1) and (2) above. □

We conclude by studying the following boundary value problem depending on a parameter $\lambda \geq 0$:

$$\begin{cases} x'(t) + \lambda\phi(t, x(t), x'(t)) + \lambda\psi(t, x(t)) \in \lambda \int_0^t \gamma(s, x(s))\, ds \\ x(0) = x(T), \end{cases} \tag{6.3}$$

where $\phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ are as in Example 4.12. We assume that ϕ and ψ are T-periodic with respect to the first variable, while $\gamma : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an u.s.c. CJ-multimap, T-periodic in the first variable and such that

$$\|\gamma(t, c)\| \leq \nu(t)$$

for any $c \in \mathbb{R}^n$ and a.e. $t \in [0, T]$, with $\nu \in L^1([0, T])$.

Given E and F as in the Example 4.12, for technical reasons define

$$\begin{aligned} &L : E \to F, && Lx(t) = x'(t), \\ &h : E \to F, && h(x)(t) = \phi(t, x(t), x'(t)), \\ &k : E \to F, && k(x)(t) = \psi(t, x(t)), \\ &\mathbf{G} : E \multimap F, && \mathbf{G}(x)(t) = \textstyle\int_0^t \gamma(s, x(s))\, ds. \end{aligned}$$

The Banach spaces E and F are as in Example 4.12. Thus, problem (6.3) is equivalent to the semilinear operator inclusion

$$Lx \in \lambda(h(x) + k(x) + \mathbf{G}(x)) \tag{6.4}$$

in $E \times [0, +\infty)$. We assume that, for any $(t, a, b) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, the jacobian matrix $\partial_3\phi(t, a, b)$ has no negative eigenvalues; so that, as in Example 4.12,

$$H(x, \lambda) = Lx + \lambda(h(x) + k(x) + \mathbf{G}(x)),$$

is an orientable multivalued quasi-Fredholm homotopy.

By a solution of (6.4) we mean a pair (x, λ) such that $H(x, \lambda) \ni 0$ and we regard the distinguished subset $\operatorname{Ker} L \times \{0\}$ of the set of solutions as the set of *trivial solutions* of (6.4).

The following is a particular case of Theorem 6.3. To avoid cumbersome notation, any point $p \in \mathbb{R}^n$ is identified with the constant function $t \mapsto p$, so that $\mathbb{R}^n$ can be regarded as the set of trivial solutions of (6.4).

Theorem 6.5. *Let $v : \mathbb{R}^n \multimap \mathbb{R}^n$ be the vector field defined by*

$$v(p) = \frac{1}{T} \int_0^T \left(\phi(t, p, 0) + \psi(t, p) + \int_0^t \gamma(s, p)\, ds \right) dt.$$

Let U be an open subset of $E \times [0, +\infty)$ and let $U_0 = \{p \in \mathbb{R}^n : (p, 0) \in U\}$. Assume that the degree $\deg(v, U_0)$ is defined and nonzero. Then U contains a connected set of nontrivial solutions of problem (6.3) whose closure in U is not compact and intersects $\operatorname{Ker} L \times \{0\} \cong \mathbb{R}^n$ in the compact set $\{p \in U_0 : v(p) \ni 0\}$. In particular U_0 contains at least one bifurcation point.

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