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THE IMPLICIT MIDPOINT RULE FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The implicit midpoint rule (IMR) for nonexpansive mappings is established in Banach spaces. The IMR generates a sequence by an implicit algorithm. Weak convergence of this algorithm is proved in a uniformly convex Banach space which either satisfies Opial's property or has a Fréchet differentiable norm. Consequently, this algorithm applies in both ℓ_p and L^p for $1 < p < \infty$.

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1. Introduction

The implicit midpoint rule (IMR) for nonexpansive mappings in a Hilbert space H, inspired by the IMR for ordinary differential equations [2, 3, 4, 6, 21, 22, 23], is introduced in [1]. This rule generates a sequence $\{x_n\}$ via the semi-implicit procedure:

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in H$ is arbitrarily chosen, $t_n \in (0, 1)$ for all n, and T is a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$).

There are mainly in the literature two sorts of iteration methods for nonexpansive mappings, namely, the Halpern method [8] and the Krasnoselskii-Mann method [9, 17] which generate a sequence $\{x_n\}$ via the iteration procedures:

$$x_{n+1} = (1 - t_n)u + t_n T x_n, \quad n \geq 0, \tag{1.2}$$

and

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n, \quad n \geq 0. \tag{1.3}$$

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where T is a nonexpansive mapping.

Under certain conditions (see [11, 16, 13, 20, 24, 26, 15, 29]), Halpern's algorithm (1.2) can be strongly convergent, while Krasnoselskii-Mann's algorithm can have, in general, weak convergence [19, 14, 27]. It is however unclear how to compare the two algorithms (1.2) and (1.3) or how to identify the limit of the Krasnoselskii-Mann algorithm (1.3).

The IMR (1.1) is proved to converge weakly [1] in the Hilbert space setting provided the sequence $\{t_n\}$ satisfies the two conditions:

(C1) $t_{n+1}^2 \le a t_n$ for all $n \ge 0$ and some $a > 0$,

(C2) $\liminf_{n\to\infty} t_n > 0$.

It remains unclear if this algorithm can converge strongly.

The purpose of the present paper is to extend the IMR (1.1) to the setting of Banach spaces that are uniformly convex with either Opial's property or a Fréchet differentiable norm.

The paper is organized as follows. In the next section we introduce uniformly convex Banach spaces, Fréchet differentiability of a norm, and Opial's property. Included in this section is also the very powerful inequality tools [28] that characterize uniform convexity. The main results of this paper, Theorems 3.6 and 3.8, that is, the weak convergence of the algorithm (1.1) in a uniformly convex Banach space either with Opial's property or having a Fréchet differentiable norm, are proved in Section 3.

2. Preliminaries

Let X be a real Banach space. Recall that X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \le 2$, where δ_X is the modulus of convexity of X defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \ \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\right\} \tag{2.1}$$

for all $0 < \varepsilon \le 2$. Typical examples of uniformly convex spaces include ℓ^p and L^p spaces for all $1 < p < \infty$.

Uniform convexity can be characterized by inequalities. As a matter of fact, we have the following result which plays a key role in the proof to the main results, Theorems 3.6 and 3.8.

Lemma 2.1. *Suppose X is a uniformly convex Banach space and given $\rho > 0$. Then there exists a continuous, convex, and strictly increasing function γ depending only on ρ such that*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\gamma(\|x-y\|) \tag{2.2}$$

for all $x, y \in X$ such that $\|x\| \le \rho$, $\|y\| \le \rho$ and all $0 \le t \le 1$.

Recall that (the norm of) X is said to be Fréchet differentiable if, for each $x \in S_X := \{x \in X : \|x\| = 1\}$, the unit sphere of X, the limit:

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.3}$$

exists and is attained uniformly in $y \in S_X$. The (normalized) duality map $J : X \to X^*$, the dual space of X, is defined as

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}. \tag{2.4}$$

Furthermore, we say that X is uniformly smooth if the limit (2.3) exists and is attained uniformly in $x, y \in S_X$. It is known that X is Fréchet differentiable if and only if J is single-valued and is norm-to-norm continuous. It is also known that for $1 < p < \infty$, both ℓ^p and L^p are uniformly convex and uniformly smooth.

Recall also that a real Banach space X is said to satisfy Opial's property [18] if, for any sequence $\{x_n\}$ of X, there holds the implication:

$$x_n \to x \quad \text{weakly} \quad \Longrightarrow \quad \liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } y \in X,\ y \neq x.$$

It is known that for $1 < p < \infty$, the space ℓ^p satisfies Opial's property, while the space L^p fails to satisfy Opial's property unless $p = 2$. A profound result [5] is that each separable Banach space can be renormed to satisfy Opial's property.

Opial's property has many applications in fixed point theory of nonlinear operators, see, for instance, [10, 25].

3. The implicit midpoint rule

Let C be a closed convex subset of a real Banach space X. Recall that a a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \qquad x, y \in C.$$

A point $x \in C$ such that $Tx = x$ is said to be a fixed point of T. The set of all fixed points of T is denoted by $Fix(T)$, namely,

$$Fix(T) = \{x \in C : Tx = x\}.$$

Below we always assume $Fix(T) \neq \emptyset$.

For each fixed $u \in C$ and $t \in (0, 1)$, define a self-mapping of C, $T_t^u : C \to C$, by

$$T_t^u x := (1 - t)u + tT\left(\frac{u + x}{2}\right), \quad x \in C. \tag{3.1}$$

The following lemma is straightforward as T is nonexpansive.

Lemma 3.1. *The mapping T_t^u is a contraction with coefficient $t/2$, that is,*

$$\|T_t^u x - T_t^u y\| \leq \frac{t}{2}\|x - y\|, \quad x, y \in C. \tag{3.2}$$

Hence, by Banach's contraction mapping principle, T_t^u has a unique fixed point in C.

Lemma 3.1 guarantees that the following algorithm is well defined. Initializing with $x_0 \in C$, we define x_{n+1} by the iteration process which is referred to as the implicit midpoint rule (IMR) for nonexpansive mappings:

$$x_{n+1} = (1 - t_n)x_n + t_nT\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{3.3}$$

where $t_n \in (0, 1)$ for all n, and $T : C \to C$ is a nonexpansive mapping.

We first discuss useful properties of the IMR (3.3).

Lemma 3.2. *Assume X is uniformly convex and let $\{x_n\}$ be the sequence generated by the IMR (3.3). Then*

(i) $\|x_{n+1} - p\| \le \|x_n - p\|$ *for all* $n \ge 0$ *and* $p \in Fix(T)$. *In particular,* $\{x_n\}$ *is bounded and moreover, we have*

$$\lim_{n\to\infty} \|x_n - p\| \quad \textit{exists for every } p \in Fix(T). \tag{3.4}$$

Let $\rho > 0$ satisfy $\|x_n\| \le \rho$ for all n and let γ satisfy the inequality (2.2). Then we further have

(ii) $\sum_{n=1}^{\infty} t_n \gamma(\|x_n - x_{n+1}\|) < \infty$.

(iii) $\sum_{n=1}^{\infty} t_n(1-t_n)\gamma\left(\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\right) < \infty$.

Proof. To show (i), we take $p \in Fix(T)$ and deduce that

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|(1-t_n)(x_n - p) + t_n\left[T\left(\frac{x_n + x_{n+1}}{2}\right) - p\right]\right\| \\
&\le (1-t_n)\|x_n - p\| + t_n\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - p\right\| \\
&\le (1-t_n)\|x_n - p\| + t_n\left\|\frac{x_n + x_{n+1}}{2} - p\right\| \\
&\le (1-t_n)\|x_n - p\| + \frac{t_n}{2}(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

This straightforwardly implies that $\|x_{n+1} - p\| \le \|x_n - p\|$; consequently, $\{x_n\}$ is bounded and (3.4) holds. That is, (i) has been proved.

To prove (ii), we employ Lemma 2.1 with $\rho \ge \sup_{n\ge 0}\|x_n\|$. We then have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|(1-t_n)(x_n - p) + t_n\left[T\left(\frac{x_n + x_{n+1}}{2}\right) - p\right]\right\|^2 \\
&\le (1-t_n)\|x_n - p\|^2 + t_n\left\|T\left(\frac{x_n + x_{n+1}}{2}\right) - p\right\|^2 \\
&\quad - t_n(1-t_n)\gamma\left(\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|\right) \\
&\le (1-t_n)\|x_n - p\|^2 + t_n\left\|\frac{x_n + x_{n+1}}{2} - p\right\|^2 \\
&\quad - t_n(1-t_n)\gamma\left(\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|\right) \\
&\le (1-t_n)\|x_n - p\|^2 \\
&\quad + t_n\left(\frac{1}{2}\|x_n - p\|^2 + \frac{1}{2}\|x_{n+1} - p\|^2 - \frac{1}{4}\gamma(\|x_n - x_{n+1}\|)\right) \\
&\quad - t_n(1-t_n)\gamma\left(\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\|\right).
\end{aligned}$$

It turns out that

$$\left(1-\frac{t_n}{2}\right)\|x_{n+1}-p\|^2 \leq \left(1-\frac{t_n}{2}\right)\|x_n-p\|^2-\frac{t_n}{4}\gamma(\|x_n-x_{n+1}\|) -t_n(1-t_n)\gamma\left(\left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\right)$$

and by dividing both sides by $(1-t_n/2)$ we get

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2-\frac{t_n}{2(2-t_n)}\gamma(\|x_n-x_{n+1}\|) -\frac{2t_n(1-t_n)}{2-t_n}\gamma\left(\left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\right). \tag{3.5}$$

This clearly implies that (noticing that $t_n\in(0,1)$)

$$\sum_{n=1}^{\infty} t_n\gamma(\|x_n-x_{n+1}\|)<\infty \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} t_n(1-t_n)\gamma\left(\left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\right)<\infty. \tag{3.7}$$

Namely, (ii) and (iii) are proved.

Lemma 3.3. *Let X be uniformly convex and let the sequence $\{x_n\}$ be generated by the IMR (3.3). Assume* $\liminf_{n\to\infty} t_n>0$*. Then we have*

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0 \tag{3.8}$$

and

$$\lim_{n\to\infty}\left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|=0. \tag{3.9}$$

Proof. By Lemma 3.2(ii) together with the assumption that $\liminf_{n\to\infty} t_n>0$, we immediately find that

$$\sum_{n=0}^{\infty}\gamma(\|x_{n+1}-x_n\|)<\infty.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.10}$$

Since the definition of IMR (3.3) yields that

$$\|x_{n+1}-x_n\|=t_n\left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|,$$

(3.9) follows from (3.10) and the assumption that $\liminf_{n\to\infty} t_n>0$.

Finally, (3.8) follows from (3.10) and (3.9). Indeed we have the following estimates:

$$\begin{aligned}\|x_n-Tx_n\| &\leq \left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\left\|Tx_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\\ &\leq \left\|x_n-T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\frac{1}{2}\|x_n-x_{n+1}\|\to 0.\end{aligned}$$

To prove the weak convergence of the IMR (3.3), we need the so-called demiclosedness principle for nonexpansive mappings.

Lemma 3.4. [7] *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a nonexpansive mapping with fixed points. Assume $\{u_n\}$ is a sequence in C such that $u_n \to u$ weakly and $(I - T)u_n \to 0$ strongly. Then $(I - T)x = 0$ (i.e., $Tx = x$).*

We use the notation: $\omega_w(u_n)$ to denote the set of all weak cluster points of the sequence $\{u_n\}$.

3.1. **Convergence in Banach Spaces with Opial's Property.**

We now prove in a uniformly convex Banach satisfying Opial's property, the IMR (3.3) generates a weakly convergent sequence.

Theorem 3.5. *Let X be a uniformly convex Banach space with Opial's property and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume $\{x_n\}$ is generated by the IMR (3.3) where the sequence $\{t_n\}$ of parameters satisfies the condition that $\liminf_{n\to\infty} t_n > 0$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. By Lemmas 3.3 and 3.4, we have $\omega_w(x_n) \subset Fix(T)$. Furthermore, by Lemma 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in Fix(T)$. Now assume $p_i \in \omega_w(x_n)$ and let $\{x_{n_k^{(i)}}\}$ be subsequences of $\{x_n\}$ weakly convergent to p_i, respectively, for $i = 1, 2$. Since $p_i \in Fix(T)$ and $\lim_{n\to\infty} \|x_n - p_i\|$ exists for $i = 1, 2$, if $p_1 \neq p_2$, we deduce by Opial's property that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p_1\| &= \lim_{k\to\infty} \|x_{n_k^{(1)}} - p_1\| \\
&< \lim_{k\to\infty} \|x_{n_k^{(1)}} - p_2\| = \lim_{k\to\infty} \|x_{n_k^{(2)}} - p_2\| \\
&< \lim_{k\to\infty} \|x_{n_k^{(2)}} - p_1\| = \lim_{n\to\infty} \|x_n - p_1\|.
\end{aligned}$$

This is an obvious contradiction. We therefore must have $p_1 = p_2$. This means that $\omega_w(x_n)$ consists of exactly one point which is equivalent to saying that $\{x_n\}$ is weakly convergent.

3.2. **Convergence in Banach Spaces with Fréchet Differentiable Norm.**

One of the key ingredients of the weak convergence of the IMR (3.3) is that it can be equivalently rewritten as an explicit scheme via the resolvent [20] of the accretive operator $I - T$.

Lemma 3.6. *The IMR (3.3) can equivalently be rewritten as*

$$x_{n+1} = T_n x_n, \quad T_n := 2J_{s_n}^{I-T} - I, \tag{3.11}$$

where $s_n = \frac{t_n}{2-t_n}$ and J_s^{I-T} denotes the resolvent of $I - T$ of index $s > 0$, that is, $J_s^{I-T} = (I + s(I - T))^{-1}$.

We also have $Fix(T_n) = Fix(T)$ for all n.

Proof. Set $U = I - T$. Observe that U is accretive for T being nonexpansive; thus the resolvent $J_s^U := (I + sU)^{-1}$ is well defined and moreover, $2J_s^U - I$ is nonexpansive.

Now upon some manipulations, it is not hard to reformulate the IMR (3.3) equivalently as (3.11).

The next lemma points out that the weak ω-limit of the sequence $\{x_n\}$ and the fixed point set of T are of certain 'orthogonality' in some sense via the duality map J.

Lemma 3.7. *Assume X is uniformly convex and has a Fréchet differentiable norm. Let $\{x_n\}$ be the sequence generated by the IMR (3.3). Then there holds the relation*

$$\langle w_1 - w_2, J(p_1 - p_2)\rangle = 0, \qquad w_1, w_2 \in \omega_w(x_n),\ p_1, p_2 \in Fix(T). \tag{3.12}$$

Proof. Set

$$\begin{aligned} S_{n,m} &= T_{n+m-1} \cdots T_{n+1} T_n, \\ a_n &= \|\lambda x_n + (1-\lambda)p_1 - p_2\|, \\ d_{n,m} &= \|S_{n,m}(\lambda x_n + (1-\lambda)p_1) - (\lambda x_{n+m} + (1-\lambda)p_1)\|. \end{aligned}$$

Then $S_{n,m} x_n = x_{n+m}$. (Recall $x_{n+1} = T_n x_n$.)

By Lemma 2.1, we have a continuous, convex, strictly increasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma(0) = 0$, such that

$$d_{n,m} \le \gamma^{-1}\left(\|x_n - p_1\| - \|x_{n+m} - p_1\|\right)$$

for all m, n. It follows immediately that $\lim\limits_{n,m\to\infty} d_{n,m} = 0$. Now since $a_{n+m} \le d_{n,m} + a_n$, we get that $\lim\limits_{n\to\infty} a_n$ exists, which implies that $\lim\limits_{n\to\infty} \langle x_n - p_1, J(p_1 - p_2)\rangle$ exists (see [19] for more details) and (3.12) follows.

Theorem 3.8. *Let X be a uniformly convex Banach space with a Frechet differentiable norm and let $T : X \to X$ be a nonexpansive mapping with $Fix(T) \ne \emptyset$. Assume $\{x_n\}$ is generated by the IMR (3.3) where the sequence $\{t_n\}$ of parameters satisfies the condition that $\liminf_{n\to\infty} t_n > 0$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It suffices to show that the weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, consists of exactly one point. To see this, we take $w_1, w_2 \in \omega_w(x_n)$. By Lemmas 3.3 and 3.4, we get $w_1, w_2 \in Fix(T)$. Consequently, upon setting $p_1 = w_1$ and $p_2 = w_2$ in Lemma 3.7 immediately yields $\|w_1 - w_2\|^2 = \langle w_1 - w_2, J(w_1 - w_2)\rangle = 0$ and $w_1 = w_2$.

Remark 3.9. In the setting of a Hilbert space H, the operator $2J_\lambda^A - I$, where A is a maximal monotone in H with a zero and J_λ^A is the resolvent of index $\lambda > 0$, is a reflection. Hence, the sequence $\{v_n\}$ defined by the iteration process:

$$v_{n+1} = (2J_\lambda^A - I)v_n, \quad n \ge 0 \tag{3.13}$$

may fail to converge even if H is finite-dimensional, see a counterexample in [12]. As a consequence of our Theorems 3.6 and 3.8, we can, however, confirm the convergence of the algorithm (3.13) for the class of those maximal monotone operators A such that $A = I - T$ with T being nonexpansive and with a fixed point.

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