Fixed Point Theory, 17(2016), No. 2, 495-508
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPROXIMATE CONTROLLABILITY OF FRACTIONAL NEUTRAL DIFFERENTIAL SYSTEMS WITH BOUNDED DELAY

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Abstract. In this paper, by using fractional power of operators and Schauder fixed point theorem, we study the approximate controllability of fractional neutral differential systems with bounded delay. The existence and uniqueness of mild solution of the system is also proved and an example is given to illustrate the theory.
Key Words and Phrases: Fractional neutral differential systems, Schauder fixed point theorem, compact semigroup, approximate controllability.
2010 Mathematics Subject Classification: 34H05, 47H10.

1. Introduction

Fractional neutral differential systems are abstract formulation for many problems arising in engineering and physics. The potential applications of fractional calculus are in diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory etc. (see [8], [2], [23], [6], [12]). There have been a great deal of interest in the solutions of fractional differential systems in analytic and numerical sense. One can see the monographs of Benchohra et al. [5], Podlubny [22], Kilbas et al. [13], Miller et al. [19], Tarasov [30] and the survey of Agarwal et al. [1] and the reference therein. In order to study the fractional systems in the infinite dimensional space, the first important step is how to introduce a new concept of mild solutions. A pioneering work has been reported by EI-Borai [9] and Zhou et al.[33].

In recent years, controllability problems for various types of nonlinear fractional dynamical systems in infinite dimensional spaces have been considered in many publications. Exact controllability for fractional order systems have been proved by many authors (see [7], [3], [23], [29], [31], [32]) and the boundary controllability problem is proved by Ahmed [4]. In these papers, the main tool used by the authors is to convert the controllability problem into a fixed point problem with the assumption that the controllability operator has an induced inverse on a quotient space. In [29],

[31], [32], [4] the authors made an assumption that the semigroup associated with the linear part is compact in order to prove the controllability results. Although the exact controllability of fractional differential systems in abstract space has been discussed, Sukavanam et al. [26], Hernańdez et al. [11] point out that some papers on controllability of abstract control systems contain a similar technical error when the compactness of semigroup and other hypotheses are satisfied, more precisely, in this case the application of controllability results are restricted to the finite dimensional space. Thus, the concept of exact controllability is too strong in infinite dimensional spaces and the approximate controllability is more appropriate.

The approximate controllability of the systems with integer order has been proved in [25]-[28] among others. However, there are only few papers which deal with the approximate controllability of fractional order system. In [24] Sakthivel et al. proved the approximate controllability by assumption that the C_0 semigroup $T(t)$ is compact and nonlinear function is continuous and uniformly bounded. Recently, Sukavanam et al. [26] have proved some sufficient conditions for the approximate controllability of a fractional order system which the nonlinear term depends on both state and control variables. Kumar et al. [14], [15] prove the approximate controllability for some semilinear delay control systems of fractional order under the natural assumption that the corresponding linear system is approximately controllable. In [16], Kumar et al. provided different sufficient conditions for the approximate controllability of fractional order semilinear system with fixed delay. In [17], Kumar et al. also give some sufficient conditions for the approximate controllability of fractional order neutral control systems with unbounded delay in the phase space. In this paper, we use the techniques similar to [16] with suitable modifications to prove the approximate controllability of the fractional neutral differential systems. The fractional neutral differential systems with bounded delay in the present paper generalize the fractional semilinear system with fixed delay appeared in [16]. Compared to [17], we use the different method and discuss the neutral fractional system in different space. So the conclusions in the present paper is the continuations of the conclusions in [16] and [17].

2. Preliminaries

Throughout this paper let V and $\hat{V}$ be Hilbert space and $Z = L_2([0,\tau];V)$, $Z_h = L_2([-h,\tau];V)$ be the function spaces corresponding to V and $Y = L_2([0,\tau];\hat{V})$ be the function space corresponding to $\hat{V}$. Consider the fractional order delay control system

$$\begin{cases} {}^cD_t^q(x(t) + F(t, x(t-h))) = Ax(t) + Bu(t) + G(t, x(t-h)), & t \in [0,\tau], \\ x(t) = \phi(t), & t \in [-h, 0], \end{cases} \tag{2.1}$$

where ${}^cD_t^q$ is the Caputo fractional of order $\frac{1}{2} < q < 1$. The state function $x(t)$ takes its value in the space V, the control function $u(t)$ takes its value in the space $\hat{V}$; $A : D(A) \subseteq V \to V$ is a closed linear operator with dense domain $D(A)$ and generates a C_0-semigroup $T(t)$; B is a bounded linear operator from Y to Z; the function $F, G : [0,\tau] \times V \to V$ is nonlinear and $\phi \in C([-h,0];V)$.

Definition 2.1. *A function $x(t) \in Z_h$ is said to be the mild solution of* (2.1) *if it satisfies*

$$x(t) = \begin{cases} S_q(t)(\phi(0) + F(0, \phi(-h))) - F(t, x(t-h)) \\ + \int_0^t (t-s)^{q-1} T_q(t-s)(Bu(s) + G(s, x(s-h)))ds \\ - \int_0^t (t-s)^{q-1} T_q(t-s) AF(s, x(s-h))ds, & t \in [0, \tau], \\ \phi(t), & t \in [-h, 0], \end{cases} \tag{2.2}$$

where $S_q(t)$ and $T_q(t)$ are called characteristic solution operators and given by

$$S_q(t) = \int_0^\infty \xi_q(\theta) T(t^q\theta) d\theta, T_q(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q\theta) d\theta,$$

and for $\theta \in (0, \infty)$, $\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \bar{w}_q(\theta^{-\frac{1}{q}}) \geq 0$,

$$\bar{w}_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \vartheta^{-qn-1} \frac{\Gamma(nq+1)}{n!} sin(n\pi q).$$

Here, ξ_q is a probability density function defined on $(0, \infty)$, that is $\xi_q(\theta) \geq 0$, $\theta \in (0, \infty)$, and $\int_0^\infty \xi_q(\theta) d\theta = 1$.

Let $x(t)$ be the state value of system (2.1) at time t corresponding to the control u. The system (2.1) is said to be approximately controllable in time interval $[0, \tau]$, if for every desired final state ζ and $\epsilon > 0$, there exists a control function $u \in \hat{V}$ such that the solution of (2.1) satisfies $\|x(\tau) - \zeta\| < \epsilon$.

Lemma 2.1. ([34]) *For any fixed $t \geq 0$, $S_q(t)$ and $T_q(t)$ are bounded linear operators. Hence*

$$\|S_q(t)x\| \leq M\|x\|$$

and

$$\|T_q(t)x\| \leq \frac{Mq}{\Gamma(1+q)}\|x\|$$

for all $x \in V$, where M is a constant such that $\|T(t)\| \leq M$ for all $t \in [0, \tau]$.

3. Existence and uniqueness of mild solution

In this section we prove the existence and uniqueness of the mild solution of (2.1). To prove the result let us assume the following conditions:

(H_1) There exists a positive constant $\beta \in (0, 1)$ and $N, L > 0$ such that $F(t, x) \in D(A^\beta)$ and

$$\|A^\beta(F(t, x) - F(t, y))\| \leq L\|x - y\|_{Z_h},$$
$$\|A^\beta F(t, x)\| \leq N(1 + \|x\|_{Z_h}).$$

(H_2) For each $t \in [0, \tau]$, $G(t, \cdot)$ is continuous and $G(t, x)$ satisfy the generalized Lipschitz condition, i.e. there exists a function $H(t) \in L^{\frac{1}{l}}([0, \tau]; V), 0 < l < q$, such that

$$\|G(t, x) - G(t, y)\| \leq H(t)\|x - y\|_{Z_h}.$$

(H_3) $\gamma C + \frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l} \cdot \tau^{q-l} + \frac{\Gamma(1+\beta)C_{1-\beta}\gamma\tau^{q\beta}}{\beta\Gamma(1+q\beta)} < 1,$

where $\gamma = \max\{N, L\}$, $\|H\|_{\frac{1}{l}} = \left(\int_0^\tau (H(s))^{\frac{1}{l}} ds\right)^l$.

Lemma 3.1. ([21]) *For all $x \in V, \beta \in (0,1)$ and $\eta \in (0,1)$, we have*

$$AT_q(t)x = A^{1-\beta}T_q(t)A^\beta x, 0 \le t \le \tau,$$

and

$$\|A^\eta T_q(t)\| \le \frac{qC_\eta}{t^{q\eta}} \cdot \frac{\Gamma(2-\eta)}{\Gamma(1+q(1-\eta))}, 0 < t \le \tau.$$

Lemma 3.2. ([21]) *For all $\beta \in (0,1)$, there exists a constant C such that*

$$\|A^{-\beta}\| \le C.$$

Theorem 3.1. *If the condition (H_1)-(H_3) hold, the system (2.1) admits a unique mild solution in Z_h for each control function $u(\cdot) \in Y$.*

Proof. Let

$$H_g = \max_{0 \le t \le \tau} \|G(t,0)\|$$

and $\|B\| \le M_B$. Define the mapping $\Phi : L_2([-h,\tau]; V) \to L_2([-h,\tau]; V)$ as

$$(\Phi x)(t) = \begin{cases} S_q(t)(\phi(0) + F(0,\phi(-h))) - F(t, x(t-h)) \\ + \int_0^t (t-s)^{q-1}T_q(t-s)(Bu(s) + G(s, x(s-h)))ds \\ - \int_0^t (t-s)^{q-1}T_q(t-s)AF(s, x(s-h))ds, & t \in [0,\tau], \\ \phi(t), & t \in [-h, 0], \end{cases} \tag{3.1}$$

Now, if we are able to show that Φ has a fixed point in the space $L_2([-h,\tau]; V)$, then (2.2) is the mild solution on $[-h,\tau]$.

Let

$$B_R = \{x(\cdot) \in L_2([-h,\tau]; V) : \|x\|_{Z_h} \le R, x(0) = \phi(0)\}$$

which is bounded and closed subset of $L_2([-h,\tau]; V)$.

For any $x(\cdot) \in B_R$, we have

$$\|S_q(t)(\phi(0) + F(0,\phi(-h)))\| \le M(\|\phi(0)\| + \|F(0,\phi(-h))\|),$$

$$\left\|\int_0^t (t-s)^{q-1}T_q(t-s)Bu(s)ds\| \le \frac{MqM_B}{\Gamma(1+q)}\sqrt{\frac{\tau^{2q-1}}{2q-1}}\|u\right\|_Y,$$

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{q-1}T_q(t-s)G(s,x(s-h))ds\right\| \\
&\leq \frac{Mq}{\Gamma(1+q)}\left\|\int_0^t (t-s)^{q-1}(G(s,x(s-h))-G(s,0)+G(s,0))ds\right\| \\
&\leq \frac{Mq}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}H(s)\|x\|_{Z_h}ds+\frac{\tau^q M}{\Gamma(1+q)}H_g \\
&\leq \frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\|x\|_{Z_h}\left(\int_0^t \left((t-s)^{q-1}\right)^{\frac{1}{1-l}}ds\right)^{1-l}+\frac{\tau^q M}{\Gamma(1+q)}H_g \\
&\leq \frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l}\cdot\tau^{q-l}R+\frac{\tau^q M}{\Gamma(1+q)}H_g, \\
&\left\|\int_0^t (t-s)^{q-1}T_q(t-s)AF(s,x(s-h))ds\right\| \\
&=\left\|\int_0^t (t-s)^{q-1}A^{1-\beta}T_q(t-s)A^{\beta}F(s,x(s-h))ds\right\| \\
&\leq \frac{\Gamma(1+\beta)qC_{1-\beta}}{\Gamma(1+q\beta)}\int_0^t (t-s)^{q\beta-1}N(1+\|x\|_{Z_h})ds \\
&\leq \frac{\Gamma(1+\beta)C_{1-\beta}N\tau^{q\beta}}{\beta\Gamma(1+q\beta)}+\frac{\Gamma(1+\beta)C_{1-\beta}N\tau^{q\beta}}{\beta\Gamma(1+q\beta)}R,
\end{aligned}$$

and

$$\|F(t,x(t-h))\|=\|A^{-\beta}A^{\beta}F(t,x(t-h))\|\leq CN(1+\|x\|_{Z_h})\leq NC(1+R).$$

So

$$\begin{aligned}
\|(\Phi x)(t)\| &\leq M(\|\phi(0)\|+\|F(0,\phi(-h))\| \\
&+NC(1+R)+\frac{MqM_B}{\Gamma(1+q)}\sqrt{\frac{\tau^{2q-1}}{2q-1}}\|u\|_Y \\
&+\frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}R\left(\frac{1-l}{q-l}\right)^{1-l}\cdot\tau^{q-l}+\frac{\tau^q M}{\Gamma(1+q)}H_g \\
&+\frac{\Gamma(1+\beta)C_{1-\beta}N\tau^{q\beta}}{\beta\Gamma(1+q\beta)}+\frac{\Gamma(1+\beta)C_{1-\beta}N\tau^{q\beta}}{\beta\Gamma(1+q\beta)}R.
\end{aligned}$$

Now let $\|(\Phi x)(t)\|<R$, then

$$\begin{aligned}
&M(\|\phi(0)\|+\|F(0,\phi(-h))\| \\
&+NC(1+R)+\frac{MqM_B}{\Gamma(1+q)}\sqrt{\frac{\tau^{2q-1}}{2q-1}}\|u\|_Y \\
&+\frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}R\left(\frac{1-l}{q-l}\right)^{1-l}\cdot\tau^{q-l}+\frac{\tau^q M}{\Gamma(1+q)}H_g \\
&+\frac{\Gamma(1+\beta)C_{1-\beta}N\tau^{q\beta}}{\beta\Gamma(1+q\beta)}+\frac{\Gamma(1+\beta)C_{1-\beta}N\tau^{q\beta}}{\beta\Gamma(1+q\beta)}R<R.
\end{aligned}$$

Since the condition (H_3), we can obtain

$$NC + \frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l}\cdot\tau^{q-l} + \frac{\Gamma(1+\beta)C_{1-\beta}N\tau^{q\beta}}{\beta\Gamma(1+q\beta)} < 1,$$

then Φ maps the ball B_R of radius R into itself.

Next we show that Φ is contraction on B_R. For this, let us take $x_1, x_2 \in B_R$, then we get

$$\begin{aligned}
&\|\Phi x_1(t) - \Phi x_2(t)\| \le \|F(t, x_1(t-s)) - F(t, x_2(t-s))\| \\
&+ \int_0^t (t-s)^{q-1}T_q(t-s)\|G(s, x_1(s-h)) - G(s, x_2(s-h))\|ds \\
&+ \int_0^t (t-s)^{q-1}AT_q(t-s)\|F(s, x_1(s-h)) - F(s, x_2(s-h))\|ds \\
&\le \|A^{-\beta}A^{\beta}F(t, x_1(t-s) - F(s, x_2(t-s))\| \\
&+ \int_0^t (t-s)^{q-1}T_q(t-s)\|G(s, x_1(s-h)) - G(s, x_2(s-h))\|ds \\
&+ \int_0^t \|(t-s)^{q-1}A^{1-\beta}T_q(t-s)A^{\beta}F(s, x_1(s-h)) - F(s, x_2(s-h))\|ds \\
&\le LC\|x_1 - x_2\|_{Z_h} + \int_0^t (t-s)^{q-1}H(s)\|x_1 - x_2\|_{Z_h}ds \\
&+ \frac{\Gamma(1+\beta)C_{1-\beta}L\tau^{q\beta}}{\beta\Gamma(1+q\beta)}\|x_1 - x_2\|_{Z_h} \\
&\le LC\|x_1 - x_2\|_{Z_h} + \frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l}\cdot\tau^{q-l}\|x_1 - x_2\|_{Z_h} \\
&+ \frac{\Gamma(1+\beta)C_{1-\beta}L\tau^{q\beta}}{\beta\Gamma(1+q\beta)}\|x_1 - x_2\|_{Z_h} \\
&= \left(LC + \frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l}\cdot\tau^{q-l} + \frac{\Gamma(1+\beta)C_{1-\beta}L\tau^{q\beta}}{\beta\Gamma(1+q\beta)}\right)\|x_1 - x_2\|_{Z_h}.
\end{aligned}$$

Since the condition (H_3), we can obtain

$$\left(LC + \frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}R\left(\frac{1-l}{q-l}\right)^{1-l}\cdot\tau^{q-l} + \frac{\Gamma(1+\beta)C_{1-\beta}L\tau^{q\beta}}{\beta\Gamma(1+q\beta)}\right) < 1,$$

then Φ has a unique fixed point in B_R.

4. Controllability of system (2.1)

Define the linear operator L from Z to V by

$$Lp = \int_0^{\tau} (\tau - s)^{q-1}T_q(\tau - s)p(s)ds.$$

Let $N_0(L)$ be the null space of the operator L, which is a closed subspace in Z and its orthogonal space is $N_0^{\perp}(L)$. Denote the range of operator B by $R(B)$ and its closure by $\overline{R(B)}$.

Assumption. *We impose the following condition to prove the results:*

(H_4) *T(t) is compact for every* $t \geq 0$.

(H_5) *For each* $p \in Z$ *there exists a function* $q \in \overline{R(B)}$ *such that* $Lp = Lq$.

Clearly, assumption (H_5) implies that for any $p \in Z$ there exists a function $q \in \overline{R(B)}$ such that $L(p-q) = 0$. Hence $p - q = n \in N_0(L)$ which implies that $Z = N_0(L) \oplus \overline{R(B)}$. Therefore, we can define a linear and continuous mapping P from $N_0^{\perp}(L)$ into $\overline{R(B)}$ as $Pu^* = q^*$, where q^* is the unique minimum norm element in $\{u^* + N_0(L)\} \bigcap \overline{R(B)}$, that is

$$\|Pu^*\| = \|q^*\| = \min\{\|v\| : v \in \{u^* + N_0(L)\} \bigcap \overline{R(B)}\}.$$

From (H_5) it follows that for each $u^* \in N_0^{\perp}(L)$, the set $\{u^* + N_0(L)\} \bigcap \overline{R(B)}$ is not empty. Moreover, for each constant C_1 and $z \in Z$, there has a unique decomposition $z = n + q^*$. Thus for each $z \in Z$ and corresponding $n \in N_0(L)$, the following inequality holds $\|n\|_Z \leq (1 + C_1)\|z\|_Z$ for some constant C_1[20].

Define the operator $K : Z \to Z$ as

$$Kz(t) = \int_0^t (t-s)^{q-1} T_q(t-s) z(s) ds.$$

Let M_0 be the subspace of Z_h such that

$$M_0 = \left\{ \begin{array}{cc} m \in Z_h : m(t) = (Kn)(t), & n \in N_0(L), 0 \leq t \leq \tau; \\ m(t) = 0, & -h \leq t \leq 0, \end{array} \right\}. \tag{4.1}$$

Note that $m(\tau) = 0$, for all $m \in M_0$.

For each mild solution $x(\cdot)$ of system (2.1) with control u, we can define an operator $f_x : M_0 \to M_0$ as

$$f_x(m) = \left\{ \begin{array}{ll} Kn, & 0 \leq t \leq \tau; \\ 0, & -h \leq t \leq 0, \end{array} \right. \tag{4.2}$$

where n is given by the unique decomposition

$$z = n + q^*, z \in Z, n \in N_0(L), q \in \overline{R(B)}. \tag{4.3}$$

Theorem 4.1. *Under assumption* (H_5) *the fractional order system*

$$x(t) = S_q(t)(\phi(0) + F(0, \phi(-h))) - F(t, x(t-h)) + \int_0^t (t-s)^{q-1} T_q(t-s) Bu(s) ds \tag{4.4}$$

is approximately controllable.

Proof. Let $x(t)$ be the state value of system (4.4) at time t corresponding to the control u. The system (4.4) is said to be approximately controllable in time interval $[0, \tau]$, if for every desired final state ζ and $\epsilon > 0$ there exists a control function $u \in Y$ such that the solution of (4.4) satisfies $\|x(\tau) - \zeta\| < \epsilon$.

Since the domain $D(A)$ of the operator A is dense in V[21], to prove this, let us take $\zeta \in D(A)$, then $\zeta - S_q(\tau)(\phi(0) + F(0, \phi(-h))) + F(\tau, x(\tau - h)) \in D(A)$. It can be seen that there exists some $p \in C^1([0,\tau];V)$ such that

$$\eta = \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)p(s)ds,$$

where

$$\eta = \zeta - S_q(\tau)(\phi(0) + F(0, \phi(-h))) + F(\tau, x(\tau - h)).$$

The assumption (H_5) implies that there exists a function $q \in \overline{R(B)}$ such that the following equality holds

$$\eta = \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)p(s)ds = \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)q(s)ds.$$

Since $q \in \overline{R(B)}$, for a given $\epsilon > 0$ there exists a control function u_ϵ in Y such that

$$\|Bu_\epsilon - q\| < \left(\frac{M\tau^q}{\Gamma(1+q)}\right)^{-1} \epsilon.$$

Put

$$\eta_\epsilon = \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)Bu_\epsilon(s)ds,$$

$$\zeta_\epsilon = \eta_\epsilon - S_q(\tau)(\phi(0) + F(0, \phi(-h))) + F(\tau, x(\tau - h)).$$

Then

$$\begin{aligned}
&\|\zeta - \zeta_\epsilon\| = \|\eta - \eta_\epsilon\| \\
&= \|\int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)p(s)ds - \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)Bu_\epsilon(s)ds\| \\
&\le \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)\|Bu_\epsilon(s) - q(s)\|ds \\
&< \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we can obtain that for every desired final state ζ and $\epsilon > 0$ there exists a control function $u_\epsilon \in Y$ such that solution of (4.4) satisfies

$$\|x(\tau) - \zeta\| < \epsilon.$$

Lemma 4.1. *Under the assumption (H_1), (H_2) and (H_4), the operator f_x has a fixed point m_0 in the set M_0 if*

$$\frac{Mq(1 + C_1)\|H\|_{\frac{1}{l}}}{\Gamma(1+q)} \left(\frac{1-l}{q-l}\right)^{1-l} \tau^{q-l} < 1. \tag{4.5}$$

Proof. Let $B_r = \{z \in M_0 : \|z\|_{Z_h} \le r\}$ for some positive number r. First, we show that f_x maps B_r to B_r itself. If this is not true, then for each positive number r, there exists a function $m \in B_r$, such that $f_x(m)$ is not the element of B_r, i.e. $\|f_x(m)\| > r$.

On the other hand, from (H_1), (H_2), Lemma 3.1, Lemma 3.2 and (4.3), we have

$$
\begin{aligned}
r &< \|f_x(m)\| = \|Kn\| \leq \int_0^t (t-s)^{q-1}\|T_q(t-s)\|\|n(s)\|ds \\
&= \int_0^t (t-s)^{q-1}\|T_q(t-s)\|(1+C_1) \\
&\|G(s,(x+m)(s-h)) - AF(s,(x+m)(s-h))\|ds \\
&\leq (1+C_1)\int_0^t (t-s)^{q-1}\|T_q(t-s)\|(\|G(s,(x+m)(s-h)) \\
&- G(s,0) + G(s,0)\| + \|A^{1-\beta}A^{\beta}F(s,(x+m)(s-h))\|)ds \\
&\leq \frac{Mq(1+C_1)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}H(s)\|x+m\|_{Z_h}ds + \frac{Mq(1+C_1)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}H_g ds \\
&+ \frac{Mq(1+C_1)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\|A^{1-\beta}T_q(t-s)A^{\beta}F(s,(x+m)(s-h))\|ds \\
&\leq \frac{Mq(1+C_1)}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}H(s)(\|x\|_{Z_h} + \|m\|_{Z_h})ds + \frac{Mq(1+C_1)H_g}{\Gamma(1+q)}\frac{\tau^q}{q} \\
&+ \frac{\Gamma(1+\beta)qC_{1-\beta}(1+C_1)}{\Gamma(1+q\beta)}\int_0^t (t-s)^{-q(1-\beta)}(t-s)^{q-1}N(1+\|x\|_{Z_h})ds \\
&\leq \frac{Mq(1+C_1)\tau^{q-l}}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\|x\|_{Z_h}\left(\frac{1-l}{q-l}\right)^{1-l} \\
&+ \frac{Mq(1+C_1)\tau^{q-l}}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\|m\|_{Z_h}\left(\frac{1-l}{q-l}\right)^{1-l} \\
&+ \frac{Mq(1+C_1)H_g\tau^q}{\Gamma(1+q)} + \frac{C_{1-\beta}\Gamma(1+\beta)N\tau^{q\beta}(1+C_1)}{\beta\Gamma(1+q\beta)} \\
&+ \frac{C_{1-\beta}\Gamma(1+\beta)N\tau^{q\beta}(1+C_1)}{\beta\Gamma(1+q\beta)}\|x\|_{Z_h} \\
&\leq \frac{Mq(1+C_1)\tau^{q-l}}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l}\|x\|_{Z_h} + \frac{Mq(1+C_1)\tau^{q-l}}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l} r \\
&+ \frac{Mq(1+C_1)H_g\tau^q}{\Gamma(1+q)} + \frac{C_{1-\beta}\Gamma(1+\beta)N\tau^{q\beta}(1+C_1)}{\beta\Gamma(1+q\beta)} \\
&+ \frac{C_{1-\beta}\Gamma(1+\beta)N\tau^{q\beta}(1+C_1)}{\beta\Gamma(1+q\beta)}\|x\|_{Z_h}.
\end{aligned}
$$

Dividing both side by r and taking limit as $r \to +\infty$, we get

$$
\frac{Mq(1+C_1)\|H\|_{\frac{1}{l}}}{\Gamma(1+q)}\left(\frac{1-l}{q-l}\right)^{1-l}\tau^{q-l} \geq 1, \tag{4.6}
$$

which is a contradiction to (4.5). Hence f_x maps B_r into itself.

Next we show that f_x is a compact operator. By assumption (H_4) the semigroup is compact. Hence $T_q(t)$ is also compact (see lemma 3.4 [20]). This implies that the integral operator K and hence f_x are compact.

Then by the Schauder fixed point theorem f_x has fixed point m_0 such that $f_x(m_0) = Kn = m_0$. This completes the proof of Lemma 4.1.

Theorem 4.2. *The fractional neutral differential control system* (2.1) *is approximately controllable under the conditions* (H_1)-(H_5) *and* (4.5).

Proof. Let $x(\cdot)$ be the mild solution of the following system given by

$$x(t) = \begin{cases} S_q(t)(\phi(0) + F(0, \phi(-h))) - F(t, x(t-h)) & \\ + \displaystyle\int_0^t (t-s)^{q-1} T_q(t-s) Bu(s) ds, & t \in [0, \tau]; \\ \phi(t), & t \in [-h, 0]. \end{cases} \tag{4.7}$$

Now, we have to prove that $y = x + m_0$ is the mild solution of the following system given by

$$\begin{cases} D_t^q (y(t) + F(t, y(t-h)) = Ay(t) + (Bu - q)(t) + G(t, y(t-h)) & t \in [0, \tau]; \\ y(t) = \phi(t), & t \in [-h, 0]. \end{cases} \tag{4.8}$$

From (4.3), we have

$$G(t, (x+m)(t-h)) - AF(t, (x+m)(t-h)) = n(t) + q(t).$$

Operating K on both sides at $m = m_0$ (a fixed point of f_x) and using the definition of M_0, we get

$$K(G(t, (x+m_0)(t-h)) - AF(t, (x+m_0)(t-h))) = K(n(t) + q(t)) = m_0(t) + Kq(t),$$

adding $x(\cdot)$ on both sides, we get

$$x(t) + K(G(t, (x+m_0)(t-h)) - AF(t, (x+m_0)(t-h))) = m_0(t) + Kq(t) + x(t),$$

then

$$\begin{aligned} & x(t) + \int_0^t (t-s)^{q-1} T_q(t-s)(G(s, (x+m_0)(s-h)) - AF(s, (x+m_0)(s-h))) ds \\ & = x(t) + m_0(t) + \int_0^t (t-s)^{q-1} T_q(t-s) q(s) ds. \end{aligned}$$

According to (4.7), so we have

$$\begin{aligned} x(t) + m_0(t) &= S_q(t)(\phi(0) + F(0, \phi(-h))) - F(t, x(t-h)) \\ &+ \int_0^t (t-s)^{q-1} T_q(t-s)(G(s, (x+m_0)(s-h)) - AF(s, (x+m_0)(s-h))) ds \\ &+ \int_0^t (t-s)^{q-1} T_q(t-s)(Bu(s) - q(s)) ds. \end{aligned}$$

This implies that $y(t) = x(t) + m_0(t)$ is the mild solution of (4.7) with control $Bu - q$. Moreover, since $m_0(0) = m_0(\tau) = 0$, we have

$$y(0) = x(0) + m_0(0) = x(0) = \phi(0),$$

$$y(\tau) = x(\tau) + m_0(\tau) = x(\tau).$$

Further, since $q \in \overline{R(B)}$ there exists a $v \in Y$ such that $\|Bv - q\| \leq \epsilon$ for any given $\epsilon > 0$.

Let $x_w(\cdot)$ be the mild solution of the control system (2.1) corresponding to the control $w = u - v$. Then we can easily prove that

$$\begin{aligned}
&\|y(\tau) - x_w(\tau)\| = \|x(\tau) - x_w(\tau)\| = \\
&\|S_q(\tau)(\phi(0) + F(0, \phi(-h))) - F(\tau, x(\tau - h)) + \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)(Bu(s) - q(s))ds \\
&+ \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)(G(s, (x + m_0)(s - h)) - AF(s, (x + m_0)(s - h)))ds \\
&- S_q(\tau)(\phi(0) + F(0, \phi(-h)) + F(\tau, x(\tau - h)) - \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)(Bu(s) - Bv(s))ds \\
&- \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)(G(s, (x + m_0)(s - h)) - AF(s, (x + m_0)(s - h)))ds\| \\
&= \| \int_0^\tau (\tau - s)^{q-1} T_q(\tau - s)(Bv(s) - q(s))ds\| < \frac{M\tau^q}{\Gamma(1 + q)}\epsilon.
\end{aligned}$$

This implies that for every desired final state ζ and $\epsilon > 0$ there exists a control function $u - v \in Y$ such that the solution of (2.1) $x_w(\cdot)$ satisfies

$$\begin{aligned}
\|x_w(\tau) - \zeta\| &= \|x_w(\tau) - x(\tau) - \zeta + x(\tau)\| < \|x_w(\tau) - x(\tau)\| + \|x(\tau) - \zeta\| \\
&\leq \frac{q\tau^q}{\Gamma(1 + q)}\epsilon + \epsilon < \left(\frac{q\tau^q}{\Gamma(1 + q)} + 1\right)\epsilon.
\end{aligned}$$

Hence, the control system (2.1) is approximately controllable.

Remark 4.1. *If the system is without $F(t, x(t - h))$, then the main results of* [16] *are obtained under the condition $\frac{Ml\tau^q(1+C)}{\Gamma(1+q)} < 1$ as a corollary to Theorem 4.2.*

5. Example

As an application of Theorem 4.2, we consider the following system:

$$\begin{cases}
\dfrac{\partial^{\frac{2}{3}}}{\partial t^{\frac{2}{3}}}[z(t, x) + F(t, z(t - h, x))] = \dfrac{\partial^2}{\partial x^2} z(t, x) \\
+Bu(t) + G(t, z(t - h, x)), \\
0 \leq t \leq \tau, 0 \leq x \leq \pi, \\
z(t, 0) = z(t, \pi) = 0, \\
z(t, x) = \phi(t, x), -h \leq t \leq 0.
\end{cases} \tag{5.1}$$

To write system (5.1) to the form of (2.1), let $V = L_2(0, \pi)$ and A defined by $Af = f^{''}$ with domain

$$D(A) = \{f(\cdot) \in V : f, f^{'} \text{ absolutely continuous, } f^{''} \in V, f(0) = f(\pi) = 0\}.$$

Then A generates a uniformly bounded analytic semigroup which satisfies the condition (H_4). Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with the corresponding normalized eigenvectors $z_n(x) = (2/\pi)^{1/2} sin(nx)$. Then the following properties hold.

(i) If $f \in D(A)$, then $Af = \sum_{n=1}^{\infty} n^2 \langle f, z_n \rangle z_n$.

(ii) For each $f \in X$, $A^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \frac{1}{n} \langle f, z_n \rangle z_n$.

In particular, $\|A^{-\frac{1}{2}}\| = 1$.

(iii) The operator $A^{\frac{1}{2}}$ is given by $A^{\frac{1}{2}} f = \sum_{n=1}^{\infty} n \langle f, z_n \rangle z_n$ on the space

$$D(A^{\frac{1}{2}}) = \{f(\cdot) \in X, A^{\frac{1}{2}} f \in X\}.$$

If the conditions $(H_1) - (H_5)$ and (4.5) are satisfied, then the approximate controllability of the system (5.1) follows from Theorem 4.2.

Acknowledgments. The author is highly grateful for the referees careful reading and comments on this paper. The first author is supported by the NSFC Granted 11526038, 11301039, 11301040. The second author is supported by the NSFC Granted 11431015, 11271381.

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Received: January 15, 2014; Accepted: May 19, 2014.