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COUPLED FIXED POINT THEOREMS FOR SYMMETRIC CONTRACTIONS IN b-METRIC SPACES WITH APPLICATIONS TO OPERATOR EQUATION SYSTEMS

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Abstract. In this paper, we will consider coupled fixed point problems in b-metric spaces for single-valued operators satisfying a symmetric contraction condition. On one hand, existence and uniqueness of the solution and, on the other hand, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing property of the coupled fixed point problem are discussed. The approach is based on the application of a Ran-Reurings type fixed point theorem for an appropriate operator on the Cartesian product space. Some applications to a system of integral equations and to a periodic boundary value problems are also given.

Key Words and Phrases: Single-valued operator, fixed point, ordered metric space, coupled fixed point, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing, integral equation, periodic boundary value problem.
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1. Introduction

Among many extensions of the Banach's contraction principle a very interesting one was given, in the framework of so-called b-metric spaces, by S. Czerwik in [7]. For the concept of b-metric space (also called quasimetric space) see I.A. Bakhtin [1], L.M. Blumenthal [5], Czerwik [7], Heinonen [11]. For several fixed point theorems in this context see [2], [6], Czerwik [7], [17],...

If (X, d) is a metric space and $T : X \times X \to X$ is an operator, then, by definition, a coupled fixed point for T is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*). \end{cases} \tag{1.1}$$

We will denote by $CFix(T)$ the coupled fixed point set for T.

Opoitsev in [15]-[16] considered, for the first time, the coupled fixed point problem, but the issue gets a fast development by the seminal works of D. Guo and V.

Lakshmikantham [10] and T. Gnana Bhaskar and V. Lakshmikantham in [8]. For other contributions, see [9], [14], [22], [23], ... A very interesting approach in this field was presented by V. Berinde (see [4]), by introducing a symmetric contraction type condition on the operator T. See also [3] for a consistent generalization.

The aim of this paper is to present, in the context of b-metric spaces with constant $s \geq 1$, some coupled fixed point theorems for symmetric contractions. On the other hand, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing property of the coupled fixed point problem are discussed. The approach is based on the application of a Ran-Reurings type theorem for an appropriate operator on the Cartesian product space. From this point of view, the results are new even for the case of metric spaces (i.e., $s = 1$). Some applications to a system of integral equations and to a periodic boundary value problems are also given.

2. Preliminaries

Throughout this paper $\mathbb{N}$ stands for the set of natural numbers, while $\mathbb{N}^*$ for the set of natural numbers except 0.

We will recall now the definition of a b-metric space.

Definition 2.1. (*Bakhtin* [1], *Czerwik* [7]) *Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric with constant s if the classical axioms of the metric functional are satisfied, with the following modification of the triangle inequality axiom:*

$(iii)'$ $d(x,z) \leq s[d(x,y) + d(y,z)]$*, for all* $x, y, z \in X$.

A pair (X, d) with the above properties is called a b-metric space.

Some examples of b-metric spaces are given in [2], [7], [6], [13], ...

It is worth to mention that the b-metric on a nonempty set X need not be continuous. Moreover, open balls in such spaces need not be open sets. For example, the definition of a closed set will be considered here in the sequential meaning, i.e., a set $Y \subset X$ is said to be closed if for any sequence (x_n) in Y which is convergent to some x, we have that $x \in Y$.

We also mention two continuity concepts. Let (X, d) be a b-metric space. Then f is called:

a) continuous on X if for every $x \in X$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to x in (X, d), it follows that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ in (X, d);

b) with closed graph if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to x in (X, d) and the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to y in (X, d) as $n \to \infty$, we have that $y = f(x)$.

If X is a nonempty set and $f : X \to X$ is a single-valued operator, then we denote $Fix(f) := \{x \in X \ : \ x = f(x)\}$, the fixed point set for f and by $Graph(f) := \{(x, f(x)) \mid x \in X\}$, the graph of the operator f.

3. Coupled fixed points for symmetric mixed monotone operators

We will present first a fixed point result, which will be used in the proof of our main result. Actually, this result is an extension to the case of b-metric spaces of the well known fixed point theorem given by Ran and Reurings in [21]. In particular, for $s = 1$ we get exactly Theorem 2.1 in [21].

Theorem 3.1. *Let X be a nonempty set endowed with a partial order "$\leq$" and $d : X \times X \to \mathbb{R}_+$ be a complete b-metric with constant $s \geq 1$. Let $f : X \to X$ be an operator which has closed graph (in particular, it is continuous) with respect to d and increasing with respect to "$\leq$". Suppose that there exist a constant $k \in (0, \frac{1}{s})$ and an element $x_0 \in X$ such that:*
(i) $d(f(x), f(y)) \leq kd(x, y)$, for all $x, y \in X$ with $x \leq y$.
(ii) $x_0 \leq f(x_0)$.
Then $Fix(f) \neq \emptyset$ and the sequence of successive approximations $(f^n(x))_{n\in\mathbb{N}}$ starting from any point $x \in X$ which is comparable to x_0 converges to a fixed point of f.

Proof. For $x_0 \in X$, we have that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is increasing. Hence, we can apply the contraction condition (i) and we get that f is asymptotically regular at x_0, i.e.

$$d(f^n(x_0), f^{n+1}(x_0)) \leq k^n d(x_0, f(x_0)) \to 0 \text{ as } n \to \infty.$$

Now, in order to show that $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence, we observe that, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$\begin{gathered} d(f^n(x_0), f^{n+p}(x_0)) \\ \leq sd\left(f^n(x_0), f^{n+1}(x_0)\right) + ... + s^{p-1}d\left(f^{n+p-2}(x_0), f^{n+p-1}(x_0)\right) \\ +s^{p-1}d\left(f^{n+p-1}(x_0), f^{n+p}(x_0)\right) \leq sk^n(1 + sk + \cdots + (sk)^{p-1})d(x_0, f(x_0)) \end{gathered}$$

Since $k < \frac{1}{s}$, we get that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq sk^n \frac{1-(sk)^p}{1-sk} d(x_0, f(x_0)) \to 0 \text{ as } n \to \infty.$$

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence and by the completeness of the metric d, there exists $x^* \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$. Because f has closed graph it follows that $x^* \in Fix(f)$. Moreover, if $x \leq x_0$ (or $x \geq x_0$), by the monotonicity condition on f, we have that $f^n(x) \leq f^n(x_0)$ (or reversely), for each $n \in \mathbb{N}$. By the contraction condition (i) we get that

$$d(f^n(x), f^n(x_0)) \leq k^n d(x, x_0) \to 0 \text{ as } n \to \infty.$$

Thus

$$\begin{aligned} d(f^n(x), x^*) &\leq s(d(f^n(x), f^n(x_0)) + d(f^n(x_0), x^*)) \\ &\leq s(k^n d(x, x_0) + d(f^n(x_0), x^*)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

which immediately implies that $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$. □

Remark 3.2. Notice here that uniqueness of the fixed point and global convergence of the successive approximations sequence can be obtained adding the hypothesis that every pair of elements of X has a lower bound or an upper bound. We may also remark that this condition is equivalent to the fact that for every $x, y \in X$, there exists $z \in X$ which is comparable to x and y. For some complementary results on this topic see also [18].

We recall first the concept of mixed monotone mapping.

Definition 3.3. *Let $(X, \leq)$ be a partially ordered set and $T : X \times X \to X$. We say that T has the mixed monotone property if $T(\cdot, y)$ is monotone increasing for any $y \in X$ and $T(x, \cdot)$ is monotone decreasing for any $x \in X$.*

Let $(X, \leq)$ be a partially ordered set and d be a b-metric on X. Notice that we can endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, \text{ we write } (x, y) \leq_P (u, v) \Leftrightarrow x \leq u, y \geq v.$$

Our first main result is the following.

Theorem 3.4. *Let $(X, \leq)$ be a partially ordered set and let $d : X \times X \to \mathbb{R}_+$ be a complete b-metric on X with constant $s \geq 1$. Let $T : X \times X \to X$ be an operator with closed graph (or, in particular, continuous) which has the mixed monotone property on $X \times X$. Assume that the following conditions are satisfied:*

(i) *there exists $k \in (0, \frac{1}{s})$ such that*

$$d(T(x, y), T(u, v)) + d(T(y, x), T(v, u)) \leq k[d(x, u) + d(y, v)], \ \forall x \leq u, y \geq v;$$

(ii) *there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$;*

Then, the following conclusions hold:

(a) there exists $(x^*, y^*) \in X \times X$ a solution of the coupled fixed point problem (1.1), such that the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X defined, for $n \in \mathbb{N}$, by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n), \end{cases} \tag{3.1}$$

have the property that $(x_n)_{n\in\mathbb{N}} \to x^*$, $(y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$. Moreover, for every pair $(x, y) \in X \times X$ with $x \leq x_0$ and $y \geq y_0$ (or reversely), we have that $(T^n(x, y))_{n\in\mathbb{N}}$ converges to x^* and $(T^n(y, x))_{n\in\mathbb{N}}$ converges to y^*.

(b) In particular, if the b-metric d is continuous, then the following estimation holds

$$d(T^n(x_0, y_0), x^*) + d(T^n(y_0, x_0), y^*) \leq \frac{sk^n}{1 - sk} \cdot [d(x_0, T(x_0, y_0)) + d(y_0, T(y_0, x_0))],$$

for all $n \in \mathbb{N}^*$.

Proof. We denote $Z := X \times X$. By (ii), we have that $z_0 := (x_0, y_0) \leq_P (x_1, y_1) := z_1$. If we consider $x_2 := T(x_1, y_1)$ and $y_2 := T(y_1, x_1)$, then we get

$$x_2 := T(x_1, y_1) = T^2(x_0, y_0) \text{ and } y_2 := T(y_1, x_1) = T^2(y_0, x_0).$$

With these notations, due to the mixed monotone property of T, we have that

$$x_2 = T(x_1, y_1) \geq T(x_0, y_0) = x_1 \text{ and } y_2 = T(y_1, x_1) \leq T(y_0, x_0) = y_1.$$

Indeed, for example for the first relation $T(x_1, y_1) \geq T(x_0, y_0)$, notice that by $(x_0, y_0) \leq_P (x_1, y_1)$ and the mixed monotone property, we have that $T(x_0, y) \leq T(x_1, y)$, for any $y \in X$ and $T(x, y_0) \leq T(x, y_1)$, for any $x \in X$. Thus, for $y := y_0$ and $x := x_1$ and using the transitivity we obtain $T(x_0, y_0) \leq T(x_1, y_1)$. In a similar way one can prove the inequality $T(y_1, x_1) \leq T(y_0, x_0)$. Thus, we have

$$z_1 = (x_1, y_1) \leq_P (x_2, y_2) := z_2.$$

By this approach, we obtain the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X with

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n), \end{cases} \tag{3.2}$$

Then, by mathematical induction, we can easily verify that

$$z_n := (x_n, y_n) \leq_P (x_{n+1}, y_{n+1}) := z_{n+1}, \ \forall\, n \in \mathbb{N}.$$

Hence, $(z_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence in $(Z, \leq_P)$.

We introduce now the functional $\widetilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\widetilde{d}((x,y),(u,v)) := d(x,u) + d(y,v).$$

It is easy to see that $\widetilde{d}$ is a b-metric on Z with the same constant $s \geq 1$ and if the space (X,d) is complete, then $(Z,\widetilde{d})$ is complete too.

We consider now the operator $F : Z \to Z$ given by

$$F(x,y) := (T(x,y), T(y,x)).$$

Notice first that $z_{n+1} = F(z_n)$, for $n \in \mathbb{N}$, where $z_0 := (x_0, y_0)$. Secondly, let us observe that, by the mixed monotone property of T, we have that F is monotone increasing with respect to $\leq_P$, i.e.,

$$(x,y),(u,v) \in Z, \text{ with } (x,y) \leq_P (u,v) \ \Rightarrow \ F(x,y) \leq_P F(u,v).$$

Notice also that, since T has closed graph (or respectively is continuous on $X \times X$), then F has closed graph too (respectively is continuous on Z).

We will prove now that F is a contraction in $(Z,\widetilde{d})$ on all comparable (with respect to $\leq_P$) elements of Z. Indeed, for all $z, w \in Z$ with $z := (x,y) \leq_P (u,v) := w$, we have

$$\begin{aligned} \widetilde{d}(F(z),F(w)) &= \widetilde{d}(F(x,y),F(u,v)) = \widetilde{d}((T(x,y),T(y,x)),(T(u,v),T(v,u))) \\ &= d(T(x,y),T(u,v)) + d(T(y,x),T(v,u)) \leq k[d(x,u)+d(y,v)] = k\widetilde{d}(z,w). \end{aligned}$$

As a conclusion, by our hypotheses and the construction of F, we have the following properties for F:

1) $F : Z \to Z$ has closed graph (in particular it is continuous) on Z;
2) $F : Z \to Z$ is increasing on Z;
3) there exists $z_0 := (x_0, y_0) \in Z$ such that $z_0 \leq_P F(z_0)$;
4) there exists $k \in (0, \frac{1}{s})$ such that

$$\widetilde{d}(F(z),F(w)) \leq k\widetilde{d}(z,w), \text{ for all } z, w \in Z \text{ with } z \leq_P w.$$

Hence we can apply Theorem 3.1 and we get that F has at least one fixed point $z^* \in Z$ and, for any $z \in Z$ which is comparable with z_0, the sequence of successive approximations for F starting from z converges to a fixed point of F. In particular, the sequence $(z_n) = ((x_n,y_n))_{n\in\mathbb{N}}$ constructed below converges in $(Z,\widetilde{d})$ to $z^* := (x^*,y^*)$ as $n \to \infty$.

If additionally, the b-metric d is continuous, we have the following apriori estimation of the errors:

$$\widetilde{d}(z_n, z^*) \leq \frac{sk^n}{1-sk} \cdot \widetilde{d}(z_0, z_1), \text{ for all } n \in \mathbb{N}^*,$$

which means that

$$d(x_n, x^*) + d(y_n, y^*) \le \frac{sk^n}{1-sk} \cdot [d(x_0, x_1) + d(y_0, y_1)], \text{ for all } n \in \mathbb{N}^*.$$

As a consequence, we also have that $(x_n)_{n\in\mathbb{N}} \to x^*$, $(y_n)_{n\in\mathbb{N}} \to y^*$ in (X, d) as $n \to \infty$. Moreover, by Theorem 3.1, for every pair $(x, y) \in X \times X$ with $x \le x_0$ and $y \ge y_0$ (or reversely), we have that $(T^n(x, y))_{n\in\mathbb{N}}$ converges to x^* and $(T^n(y, x))_{n\in\mathbb{N}}$ converges to y^*. This completes the proof. □

Concerning the uniqueness of the coupled fixed point, by Remark 3.2, we get the following result.

Theorem 3.5. *If, in addition to the hypotheses of Theorem 3.4, we suppose that, at least one of the following assumptions takes place:*

(A) for every $(x, y), (u, v) \in X \times X$ there exists $(z, w) \in X \times X$ such that

$$\begin{cases} (x \le z, y \ge w) \text{ or } (z \le x, w \ge y) \\ (u \le z, v \ge w) \text{ or } (z \le u, w \ge v), \end{cases} \tag{3.3}$$

or

$$(B) \text{ every pair of elements } (x, y) \in X \times X \text{ there exists } z \in X \text{ such that } x \le z \le y, \tag{3.4}$$

then the coupled fixed point in Theorem 3.4 is unique and for every pair $(x, y) \in X \times X$ with $x \le x_0$ and $y \ge y_0$ (or reversely), we have that $(T^n(x, y))_{n\in\mathbb{N}}$ converges to x^* and $(T^n(y, x))_{n\in\mathbb{N}}$ converges to y^* as $n \to \infty$.

Concerning the existence of a fixed point for T (in the sense that $T(x, x) = x$), we can prove the following result.

Theorem 3.6. *If we assume that all the hypotheses of Theorem 3.5 take place, then for the unique coupled fixed point (x^*, y^*) of T we have that $x^* = y^*$, i.e., x^* is a fixed point for T.*

Proof. From Theorem 3.5, the coupled fixed point problem for T has a unique solution (x^*, y^*). We will consider two cases:

Case 1. If x^* and y^* are comparable, then, from the contraction condition on T, written for $x = v := x^*, y = u = y^*$, we obtain

$$d(T(x^*, y^*), T(y^*, x^*)) + d(T(y^*, x^*), T(x^*, y^*)) \le k[d(x^*, y^*) + d(y^*, x^*)],$$

which yields that $d(x^*, y^*) \le kd(x^*, y^*)$. Since $k < 1$, we get that $d(x^*, y^*) = 0$. Thus $x^* = y^*$ and $T(x^*, x^*) = x^*$.

Case 2. If x^* and y^* are not comparable, then there exists $z \in X$ which is comparable to x^* and y^*. Suppose, for example, that $x^* \le z$ and $y^* \le z$. In view of the definition of the partially order relation $\le_P$ on $X \times X$, we obtain that (x^*, y^*), (x^*, z), (z, x^*) and (y^*, x^*) are comparable with respect to $\le_P$. From the proof of Theorem 3.4, we know that $F : X \times X \to X \times X$, given by $F(x, y) = (T(x, y), T(y, x))$ is a k-contraction on all comparable (with respect to $\le_P$) elements of Z. Moreover

$$\widetilde{d}(F^n(u), F^n(v)) \le k^n \widetilde{d}(u, v), \forall u, v \in X \times X, \text{ with } u \le_P v.$$

Then, for $u := (x^*, z)$ and $v := (x^*, y^*)$, we get

$$\widetilde{d}(F^n(x^*, z), F^n(x^*, y^*)) \leq k^n \widetilde{d}((x^*, z), (x^*, y^*)) = k^n d(z, y^*).$$

Similarly, for $u := (x^*, z)$ and $v := (z, x^*)$, we obtain

$$\widetilde{d}(F^n(x^*, z), F^n(z, x^*)) \leq k^n \widetilde{d}((x^*, z), (z, x^*)) = 2k^n d(x^*, z),$$

while for $u := (y^*, x^*)$ and $v := (z, x^*)$, we can write that

$$\widetilde{d}(F^n(y^*, x^*), F^n(z, x^*)) \leq k^n \widetilde{d}((y^*, x^*), (z, x^*)) = k^n d(y^*, z).$$

As a consequence of the above three relations, we have

$$\begin{aligned}
d(x^*, y^*) &= \frac{1}{2}\widetilde{d}((x^*, y^*), (y^*, x^*)) = \frac{1}{2}\widetilde{d}(F^n(x^*, y^*), F^n(y^*, x^*)) \\
&\leq \frac{s}{2}(\widetilde{d}(F^n(x^*, y^*), F^n(x^*, z)) + \widetilde{d}(F^n(x^*, z), F^n(y^*, x^*))) \\
&\leq \frac{s}{2}\widetilde{d}(F^n(x^*, y^*), F^n(x^*, z)) + \frac{s^2}{2}(\widetilde{d}(F^n(x^*, z), F^n(z, x^*)) + \widetilde{d}(F^n(z, x^*), F^n(y^*, x^*))) \\
&\leq \frac{s}{2}k^n d(z, y^*) + \frac{s^2}{2}k^n(2d(x^*, z) + d(y^*, z)) \\
&= \frac{s}{2}k^n[(1+s)d(y^*, z) + 2sd(x^*, z)] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, we get that $x^* = y^*$.

In a similar way, if we suppose that $x^* \leq z$ and $y^* \geq z$, then we obtain that (x^*, y^*), (y^*, z), (z, y^*) and (y^*, x^*) are comparable with respect to $\leq_P$. From the contraction condition on F, we get

$$\begin{aligned}
\widetilde{d}(F^n(y^*, x^*), F^n(z, y^*)) &\leq k^n[d(y^*, z) + d(x^*, y^*)], \\
\widetilde{d}(F^n(z, y^*), F^n(y^*, z)) &\leq 2k^n d(y^*, z), \\
\widetilde{d}(F^n(y^*, z), F^n(x^*, y^*)) &\leq k^n[d(y^*, x^*) + d(z, y^*)].
\end{aligned}$$

As a consequence of the above three relations, we have

$$\begin{aligned}
d(x^*, y^*) &= \frac{1}{2}\widetilde{d}((y^*, x^*), (x^*, y^*)) = \frac{1}{2}\widetilde{d}(F^n(y^*, x^*), F^n(x^*, y^*)) \\
&\leq \frac{s}{2}(\widetilde{d}(F^n(y^*, x^*), F^n(z, y^*)) + \widetilde{d}(F^n(z, y^*), F^n(x^*, y^*))) \\
&\leq \frac{s}{2}\widetilde{d}(F^n(y^*, x^*), F^n(z, y^*)) + \frac{s^2}{2}(\widetilde{d}(F^n(z, y^*), F^n(y^*, z)) + \widetilde{d}(F^n(y^*, z), F^n(x^*, y^*))) \\
&\leq \frac{s}{2}k^n(d(y^*, z) + d(x^*, y^*)) + \frac{s^2}{2}k^n(2d(y^*, z) + d(y^*, x^*) + d(z, y^*)) \\
&= \frac{s}{2}k^n[(1+3s)d(y^*, z) + (1+s)d(x^*, y^*)] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, we get that $x^* = y^*$. □

It is worth to mention now that if we consider the coupled fixed point problem (1.1) in a complete b-metric space and we assume that the symmetric contraction condition on T holds on the whole space (and not only on comparable pairs of the space), then the following theorem can be deduced.

Theorem 3.7. *Let (X,d) be a complete b-metric space with constant $s \geq 1$. Let $T : X \times X \to X$ be an operator. Assume that there exists $k \in (0,1)$ such that*

$$d(T(x,y),T(u,v)) + d(T(y,x),T(v,u)) \leq k[d(x,u) + d(y,v)],\ \forall (x,y),(u,v) \in X \times X$$

Then, the following conclusions hold:

(a) there exists a unique solution $(x^*, y^*) \in X\times X$ of the coupled fixed point problem (1.1), and, for any initial point $(x_0,y_0) \in X \times X$, the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ defined, for $n \in \mathbb{N}$, by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n), \end{cases} \tag{3.5}$$

converge to x^* and respectively to y^* as $n \to \infty$.

(b) In particular, if $k < \frac{1}{s}$ and the b-metric d is continuous, then the following estimation holds

$$d(T^n(x_0,y_0),x^*) + d(T^n(y_0,x_0),y^*) \leq \frac{sk^n}{1-sk} \cdot [d(x_0,T(x_0,y_0)) + d(y_0,T(y_0,x_0))],$$

for all $n \in \mathbb{N}^*$.

Proof. We introduce on $Z := X \times X$ the functional $\widetilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\widetilde{d}((x,y),(u,v)) := d(x,u) + d(y,v).$$

Notice that, as before, $\widetilde{d}$ is a b-metric on Z with the same constant $s \geq 1$ and, if the space (X,d) is complete, then $(Z,\widetilde{d})$ is complete too.

We consider now the operator $F : Z \to Z$ given by

$$F(x,y) := (T(x,y),T(y,x)).$$

It is easy to prove now that F is a contraction in $(Z,\widetilde{d})$ with constant $k \in (0,1)$, i.e.,

$$\widetilde{d}(F(z),F(w)) \leq k\widetilde{d}(z,w), \text{ for all } z,w \in Z.$$

Thus, we can apply for F the b-metric space version of the contraction principle (see [7] or Theorem 12.2 page 115 in [13]) and we get the first conclusion. For the estimation of the error, one can repeat the arguments from the proof of Theorem 3.4. □

4. Properties of the coupled fixed point set for symmetric contractions

In this section, we will study some qualitative properties of the coupled fixed point set related to a symmetric contraction condition, such as: data dependence, well-posedness, Ulam-Hyers stability and limit shadowing property.

For the data dependence problem, we have the following result.

Theorem 4.1. *Let $(X,\leq)$ be a partially ordered set and let $d : X \times X \to \mathbb{R}_+$ be a complete b-metric on X with constant $s \geq 1$. Let $T_i : X \times X \to X$ $(i \in \{1,2\})$ be two mappings such that T_1 has closed graph (or, in particular, is continuous) and satisfies the mixed monotone property on $X \times X$. Assume that the following conditions are satisfied:*

(i) *there exists* $k \in (0, \frac{1}{s})$ *such that*

$$d(T_1(x,y), T_1(u,v)) + d(T_1(y,x), T_1(v,u)) \le k[d(x,u) + d(y,v)],\ \forall x \le u, y \ge v;$$

(ii) *there exist* $x_0, y_0 \in X$ *such that* $x_0 \le T_1(x_0, y_0)$ *and* $y_0 \ge T_1(y_0, x_0)$;

(iii) *the operator* T_2 *has at least one coupled fixed point in* $X \times X$;

(iv) *for every* $(x,y), (u,v) \in X \times X$ *there exists* $(z,w) \in X \times X$ *such that*

$$\begin{cases} (x \le z, y \ge w) \text{ or } (z \le x, w \ge y) \\ (u \le z, v \ge w) \text{ or } (z \le u, w \ge v), \end{cases}$$

(v) *there exists* $\eta > 0$ *such that*

$$d(T_1(x,y), T_2(x,y)) \le \eta, \text{ for all } (x,y) \in X \times X.$$

In the above conditions, if (x^*, y^*) *denotes the unique coupled fixed point for* T_1, *then*

$$d(x^*, \bar{x}) + d(y^*, \bar{y}) \le \frac{2s\eta}{1-sk}, \forall (\bar{x}, \bar{y}) \in CFix(T_2)$$

with $(\bar{x} \le x^*, \bar{y} \ge y^*)$ *or* $(\bar{x} \ge x^*, \bar{y} \le y^*)$.

Proof. Since the operator T_1 satisfies the hypotheses of Theorem 3.5, there exists a unique coupled fixed point for T_1, say (x^*, y^*). Let $(\bar{x}, \bar{y}) \in CFix(T_2)$ such that $(\bar{x} \le x^*, \bar{y} \ge y^*)$ or $(\bar{x} \ge x^*, \bar{y} \le y^*)$. We consider again the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\widetilde{d}((x,y),(u,v)) := d(x,u) + d(y,v).$$

Then, we have:

$$\begin{aligned}
\widetilde{d}((x^*, y^*), (\bar{x}, \bar{y})) &= \widetilde{d}((T_1(x^*, y^*), T_1(y^*, x^*)), (T_2(\bar{x}, \bar{y}), T_2(\bar{y}, \bar{x}))) \\
&= d(T_1(x^*, y^*), T_2(\bar{x}, \bar{y})) + d(T_1(y^*, x^*), T_2(\bar{y}, \bar{x})) \\
&\le s[d(T_1(x^*, y^*), T_1(\bar{x}, \bar{y})) + d(T_1(\bar{x}, \bar{y}), T_2(\bar{x}, \bar{y}))] \\
&\quad + s[d(T_1(y^*, x^*), T_1(\bar{y}, \bar{x})) + d(T_1(\bar{y}, \bar{x}), T_2(\bar{y}, \bar{x}))] \\
&= s[d(T_1(x^*, y^*), T_1(\bar{x}, \bar{y})) + d(T_1(y^*, x^*), T_1(\bar{y}, \bar{x}))] \\
&\quad + s[d(T_1(\bar{x}, \bar{y}), T_2(\bar{x}, \bar{y})) + d(T_1(\bar{y}, \bar{x}), T_2(\bar{y}, \bar{x})) \\
&\le sk(d(x^*, \bar{x}) + d(y^*, \bar{y})) + 2s\eta.
\end{aligned}$$

Thus,

$$\widetilde{d}((x^*, y^*), (\bar{x}, \bar{y})) \le \frac{2s\eta}{1-sk},$$

and the proof is complete. □

We will study the well-posedness of the coupled fixed point problem (1.1).

Definition 4.2. *Let* (X,d) *be a* b*-metric space with constant* $s \ge 1$ *and* $T : X \times X \to X$ *be an operator. By definition, the coupled fixed point problem (1.1) is said to be well-posed if:*

(i) $CFix(T) = \{(x^*, y^*)\}$;

(ii) for any sequence $(x_n, y_n)_{n \in \mathbb{N}}$ *in* $X \times X$ *for which* $d(x_n, T(x_n, y_n)) \to 0$ *and respectively* $d(y_n, T(y_n, x_n)) \to 0$ *as* $n \to \infty$, *we have that* $(x_n) \to x^*$ *and* $(y_n) \to y^*$ *as* $n \to \infty$.

A well-posedness result is given in the following theorem.

Theorem 4.3. *Assume that all the hypotheses of Theorem 3.7 take place. Additionally, assume that the contraction constant k of T satisfies the condition $k < \frac{1}{s}$. Then the coupled fixed point problem (1.1) is well-posed.*

Proof. By Theorem 3.7 we get that $CFix(T) = \{(x^*, y^*)\}$. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in $X \times X$ for which $d(x_n, T(x_n, y_n)) \to 0$ and respectively $d(y_n, T(y_n, x_n)) \to 0$ as $n \to \infty$. As before, we will work on $Z := X \times X$ with the b-metric $\widetilde{d}((x, y), (u, v)) := d(x, u) + d(y, v)$. Then, we have:

$$\widetilde{d}((x_n, y_n), (x^*, y^*)) = d(x_n, x^*) + d(y_n, y^*)$$

$$\leq s[d(x_n, T(x_n, y_n)) + d(T(x_n, y_n), x^*)] + s[d(y_n, T(y_n, x_n)) + d(T(y_n, x_n), y^*)]$$

$$= s[d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))]$$

$$+s[d(T(x_n, y_n), T(x^*, y^*)) + d(T(y_n, x_n), T(y^*, x^*))]$$

$$\leq s[d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))] + sk[d(x_n, x^*) + d(y_n, y^*)]$$

$$= s[d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))] + sk\widetilde{d}((x_n, y_n), (x^*, y^*)).$$

Thus

$$\widetilde{d}((x_n, y_n), (x^*, y^*)) \leq \frac{s}{1 - sk}[d(x_n, T(x_n, y_n)) + d(y_n, T(y_n, x_n))] \to 0 \text{ as } n \to \infty.$$

Hence, we get that $(x_n) \to x^*$ and $(y_n) \to y^*$ as $n \to \infty$. □

We will consider the Ulam-Hyers stability of the coupled fixed point problem (1.1). For the case of metric spaces, see [20].

Definition 4.4. *Let (X, d) be a b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator. Let $\widetilde{d}$ be any b-metric on $X \times X$ generated by d. By definition, the coupled fixed point problem (1.1) is said to be Ulam-Hyers stable if there exists an increasing operator $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0 with $\psi(0) = 0$, such that for each $\varepsilon \in \mathbb{R}^*_+$ and for each solution $(\bar{x}, \bar{y}) \in X \times X$ of the inequality*

$$\widetilde{d}((x, y), (T(x, y), T(y, x)) \leq \varepsilon,$$

there exists a solution $(x^*, y^*) \in X \times X$ of the coupled fixed point problem (1.1) such that

$$\widetilde{d}((x^*, y^*), (\bar{x}, \bar{y})) \leq \psi(\varepsilon).$$

An Ulam-Hyers stability result for the coupled fixed point problem (1.1) is given in the following theorem.

Theorem 4.5. *Assume that all the hypotheses of Theorem 3.7 take place. Additionally, assume that the contraction constant k of T satisfies the condition $k < \frac{1}{s}$. Then the coupled fixed point problem (1.1) is Ulam-Hyers stable.*

Proof. By Theorem 3.7 we get that $CFix(T) = \{(x^*, y^*)\}$. Let any $\varepsilon > 0$ and let $(\bar{x}, \bar{y}) \in X \times X$ such that

$$d(\bar{x}, T(\bar{x}, \bar{y})) + d(\bar{y}, T(\bar{y}, \bar{x})) \leq \varepsilon.$$

As before, we consider on $Z := X \times X$ the b-metric $\widetilde{d}((x, y), (u, v)) := d(x, u) + d(y, v)$. Then, we have:

$$\begin{aligned}\widetilde{d}((\bar{x}, \bar{y}), (x^*, y^*)) &= d(\bar{x}, x^*) + d(\bar{y}, y^*) = d(\bar{x}, T(x^*, y^*)) + d(\bar{y}, T(y^*, x^*)) \\ &\leq s[d(\bar{x}, T(\bar{x}, \bar{y})) + d(T(\bar{x}, \bar{y}), T(x^*, y^*))] + s[d(\bar{y}, T(\bar{y}, \bar{x})) + d(T(\bar{y}, \bar{x}), T(y^*, x^*)) \\ &\leq s[d(\bar{x}, T(\bar{x}, \bar{y})) + d(\bar{y}, T(\bar{y}, \bar{x}))] + sk[d(\bar{x}, x^*) + d(\bar{y}, y^*)].\end{aligned}$$

Thus

$$\widetilde{d}((\bar{x}, \bar{y}), (x^*, y^*)) \leq \frac{s}{1 - sk}\varepsilon,$$

which leads to our conclusion. □

Finally, we will discuss the limit shadowing property of a coupled fixed point problem (1.1).

Definition 4.6. *Let (X, d) be a b-metric space with constant $s \geq 1$ and $T : X \times X \to X$ be an operator. By definition, the coupled fixed point problem (1.1) has the limit shadowing property if, for any sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ for which $d(x_{n+1}, T(x_n, y_n)) \to 0$ and respectively $d(y_{n+1}, T(y_n, x_n)) \to 0$ as $n \to \infty$, there exists a sequence $(T^n(x, y), T^n(y, x))_{n\in\mathbb{N}}$ such that $d(x_n, T^n(x, y)) \to 0$ and $d(y_n, T^n(y, x)) \to 0$ as $n \to \infty$.*

A shadowing type result for the coupled fixed point problem is given now.

Theorem 4.7. *Assume that all the hypotheses of Theorem 3.7 take place. Additionally, assume that the contraction constant k of T satisfies the condition $k < \frac{1}{s}$. Then the coupled fixed point problem (1.1) for T has the limit shadowing property.*

Proof. By Theorem 3.7 we know that $CFix(T) = \{(x^*, y^*)\}$ and, for any initial starting point $(x, y) \in X \times X$, we have that $(T^n(x, y)) \to x^*$ and $(T^n(y, x)) \to y^*$ as $n \to \infty$. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in $X \times X$ such that $d(x_{n+1}, T(x_n, y_n)) \to 0$ and respectively $d(y_{n+1}, T(y_n, x_n)) \to 0$ as $n \to \infty$. We consider on $Z := X \times X$ the b-metric $\widetilde{d}((x, y), (u, v)) := d(x, u) + d(y, v)$. Then, for every $(x, y) \in X \times X$, we have

$$\begin{aligned}&\widetilde{d}((x_{n+1}, y_{n+1}), (T^{n+1}(x, y), T^{n+1}(y, x))) \\ &\leq s\left[\widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) + \widetilde{d}((x^*, y^*), (T^{n+1}(x, y), T^{n+1}(y, x)))\right].\end{aligned}$$

For the first term of the above sum, we can write

$$\begin{aligned}\widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) &\leq s[\widetilde{d}((x_{n+1}, y_{n+1}), (T(x_n, y_n), T(y_n, x_n))) \\ &\quad + \widetilde{d}((T(x_n, y_n), T(y_n, x_n)), (T(x^*, y^*), T(y^*, x^*)))] \\ &\leq s\widetilde{d}((x_{n+1}, y_{n+1}), (T(x_n, y_n), T(y_n, x_n))) + sk\widetilde{d}((x_n, y_n), (x^*, y^*)) \\ &\leq s\widetilde{d}((x_{n+1}, y_{n+1}), (T(x_n, y_n), T(y_n, x_n))) \\ &\quad + sk[s\widetilde{d}((x_n, y_n), (T(x_{n-1}, y_{n-1}), T(y_{n-1}, x_{n-1}))) + sk\widetilde{d}((x_{n-1}, y_{n-1}), (x^*, y^*))]\end{aligned}$$

$$\leq \cdots \leq s\sum_{p=0}^{n}(sk)^{n-p}\widetilde{d}((x_{p+1},y_{p+1}),(T(x_p,y_p),T(y_p,x_p)))$$
$$+(sk)^{n+1}\widetilde{d}((x_0,y_0),(x^*,y^*)).$$

The first term of the above sum converges to zero by the Cauchy Lemma (see, for example [19]), while the second one goes to zero since $sk<1$.

Thus,

$$\widetilde{d}((x_{n+1},y_{n+1}),(T^{n+1}(x,y),T^{n+1}(y,x)))\to 0 \text{ as } n\to\infty,$$

and the conclusion follows. □

Remark 4.8. Notice that similar results (well-posedness property, Ulam-Hyers stability, limit shadowing property) can be established in the framework of a metric space endowed with a partial order relation and under a symmetric contraction condition on T with respect only on comparable elements, but then some additional assumptions (involving comparison properties of some elements) must be imposed.

5. Applications to integral and differential equations

We will discuss two applications of the previous results. Let us consider first the following system of Volterra type integral equations:

$$\begin{cases} x(t)=g(t)+\displaystyle\int_0^t G(s,t)f(s,x(s),y(s))ds \\ y(t)=g(t)+\displaystyle\int_0^t G(s,t)f(s,y(s),x(s))ds, \end{cases} \tag{5.1}$$

where $t\in[0,T]$.

A solution of the above system is a pair $(x,y)\in C[0,T]\times C[0,T]$ satisfying the above relations for all $t\in[0,T]$.

We consider $X:=C[0,T]$ endowed with the partial order relation:

$$x\leq_C y \Longleftrightarrow x(t)\leq y(t) \text{ for all } t\in[0,T].$$

Notice that the pair $(X,\leq)$ has the property (3.4).

We will also consider the following b-metric of Bielecki type on X

$$d(x,y):=\max_{t\in[0,T]}((x(t)-y(t))^2e^{-2\tau t}),$$

where $\tau>0$ could be arbitrarily chosen. Notice that d is a b-metric with constant $s=2$ and d can be represented using the classical Bielecki norm by $d(x,y)=\|(x-y)^2\|_B$.

Then, we have the following existence and uniqueness result.

Theorem 5.1. *Consider the integral equation system (5.1). We suppose:*

(i) $g:[0,T]\to\mathbb{R}$ and $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous and $G:[0,T]\times[0,T]\to\mathbb{R}_+$ is integrable with respect to the first variable.

(ii) $f(s,\cdot,\cdot)$ has the mixed monotone property with respect to the last two variables for all $s\in[0,T]$.

(iii) there exist $\alpha, \beta : [0,T] \to \mathbb{R}_+$ in $L^1[0,T]$ such that, for each $s \in [0,T]$ and every $x, y, u, v \in \mathbb{R}$, with $x \le u, y \ge v$, we have

$$|f(s,x,y) - f(s,u,v)| \le \alpha(s)|x-u| + \beta(s)|y-v|;$$

(iv) there exist $x_0, y_0 \in C[0,T]$ such that

$$\begin{cases} x_0(t) \le g(t) + \displaystyle\int_0^t G(s,t) f(s, x_0(s), y_0(s)) ds \\ y_0(t) \ge g(t) + \displaystyle\int_0^t G(s,t) f(s, y_0(s), x_0(s)) ds, \end{cases} \tag{5.2}$$

or

$$\begin{cases} x_0(t) \ge g(t) + \displaystyle\int_0^t G(s,t) f(s, x_0(s), y_0(s)) ds \\ y_0(t) \le g(t) + \displaystyle\int_0^t G(s,t) f(s, y_0(s), x_0(s)) ds, \end{cases} \tag{5.3}$$

for all $t \in [0,T]$.

Then, there exists a unique solution (x^*, y^*) of the system (5.1).

Proof. We will work in the space (X, d) with

$$d(x,y) := \max_{t \in [0,T]} ((x(t) - y(t))^2 e^{-2\tau t}) = \|(x-y)^2\|_B,$$

which is a b-metric space with $s = 2$, for any $\tau > 0$.

We can prove that all the assumptions of Theorem 3.5 are satisfied. We define $S : X \times X \to X$ by

$$S(x,y)(t) := g(t) + \int_0^t G(s,t) f(s, x(s), y(s)) ds, \text{ for each } t \in [0,T].$$

Then, the system (5.1) can be re-written as a coupled fixed point problem for S:

$$\begin{cases} x = S(x,y) \\ y = S(y,x). \end{cases} \tag{5.4}$$

Then we have

$$|S(x,y)(t) - S(u,v)(t)|^2 \le \left[\int_0^t G(s,t) |f(s,x(s),y(s)) - f(s,u(s),v(s))| ds \right]^2$$

$$\le \left[\int_0^t G(s,t)(\alpha(s)|x(s) - u(s)| + \beta(s)|y(s) - v(s)|) ds \right]^2$$

$$= \left[\int_0^t G(s,t)(\alpha(s)\sqrt{(x(s)-u(s))^2 e^{-2\tau s}} e^{\tau s} + \beta(s)\sqrt{(y(s)-v(s))^2 e^{-2\tau s}} e^{\tau s}) ds \right]^2$$

$$= \left[\int_0^t G(s,t)(\alpha(s)\sqrt{|x(s)-u(s)|^2 e^{-2\tau s}}\ e^{\tau s} + \beta(s)\sqrt{|y(s)-v(s)|^2 e^{-2\tau s}}\ e^{\tau s}) ds \right]^2$$

$$\le \left[\int_0^t G(s,t)\alpha(s)\sqrt{\|(x-u)^2\|_B}\ e^{\tau s} ds + \int_0^T G(s,t)\beta(s)\sqrt{\|(y-v)^2\|_B}\ e^{\tau s} ds \right]^2$$

$$\leq 2\left[\left(\int_0^t G(s,t)\alpha(s)\sqrt{\|(x-u)^2\|_B}\; e^{\tau s}ds\right)^2 + \left(\int_0^T G(s,t)\beta(s)\sqrt{\|(y-v)^2\|_B}\; e^{\tau s}ds\right)^2\right]$$

$$= 2\left\{\left[\int_0^t G(s,t)\alpha(s)e^{\tau s}ds\right]^2 \cdot \|(x-u)^2\|_B + \left[\int_0^t G(s,t)\beta(s)e^{\tau s}ds\right]^2 \cdot \|(y-v)^2\|_B\right\}$$

$$\leq 2\left\{\int_0^t (G(s,t)\alpha(s))^2\, ds \cdot \int_0^t e^{2\tau s}ds \cdot \|(x-u)^2\|_B\right.$$
$$\left. + \int_0^t (G(s,t)\beta(s))^2\, ds \cdot \int_0^t e^{2\tau s}ds \cdot \|(y-v)^2\|_B\right\}$$

$$\leq \left\{\int_0^t (G(s,t)\alpha(s))^2\, ds \cdot \|(x-u)^2\|_B + \int_0^t (G(s,t)\beta(s))^2\, ds \cdot \|(y-v)^2\|_B\right\} \cdot \frac{e^{2\tau t}}{\tau}$$

$$\leq \left[\max_{t\in[0,T]}\left(\int_0^t (G(s,t)\alpha(s))^2\, ds\right) \cdot \|(x-u)^2\|_B\right.$$
$$\left. + \max_{t\in[0,T]}\left(\int_0^t (G(s,t)\beta(s))^2\, ds\right) \cdot \|(y-v)^2\|_B\right] \cdot \frac{e^{2\tau t}}{\tau}.$$

Thus, after a multiplication with $e^{-2\tau t}$ and taking the maximum over $t \in [0,T]$, we get that

$$\|(S(x,y) - S(u,v))^2\|_B \leq k_1\|(x-u)^2\|_B + k_2\|(y-v)^2\|_B,$$

where

$$k_1 := \frac{1}{\tau}\max_{t\in[0,T]}\left(\int_0^t (G(s,t)\alpha(s))^2\, ds\right)$$

and

$$k_2 := \frac{1}{\tau}\max_{t\in[0,T]}\left(\int_0^t (G(s,t)\beta(s))^2\, ds\right).$$

Hence, we get that

$$d(S(x,y),S(u,v)) \leq k_1 d(x,u) + k_2 d(y,v).$$

By a similar approach, we get that

$$d(S(y,x),S(v,u)) \leq k_1 d(y,v) + k_2 d(x,u).$$

Thus,

$$d(S(x,y),S(u,v)) + d(S(y,x),S(v,u)) \leq (k_1+k_2)(d(x,u)+d(y,v)), \text{ for all } x \leq u, y \geq v.$$

Since $\tau > 0$ is arbitrary, we can chose $\tau > 0$ such that

$$k_1 + k_2 = \frac{1}{\tau}\max_{t\in[0,T]}\left(\int_0^t (G(s,t))^2(\alpha^2(s) + (\beta(s))^2)ds\right) < \frac{1}{s} = \frac{1}{2}.$$

As a conclusion, all the assumptions of Theorem 3.4 are satisfied and the result follows by Theorem 3.5. □

Another application to a periodic boundary value problem will be presented now. More precisely, we consider the following periodic boundary value problem of the following type:

$$\begin{cases} x'(t) = f(t, x(t)) + g(t, x(t)), t \in (0, T) \\ x(0) = x(T). \end{cases} \tag{5.5}$$

A solution of the above problem is a function $x \in C^1[0, T]$ satisfying the above relations.

The above problem was considered for the first time in D. Guo, V. Lakshmikantham [10] and then in some related papers (see, for example, [3], [12], [14], ...). Notice also that a nice extension was given in V. Berinde [4].

We will follow [10], in order to present the statement and the terms of our problem. Let $\lambda_1, \lambda_2 > 0$ be such that:

$$\begin{cases} \ln \frac{2e-1}{e} \le (\lambda_2 - \lambda_1)T, \\ (\lambda_1 + \lambda_2)T \le 1. \end{cases} \tag{5.6}$$

In order to study the existence and uniqueness problem for (5.5), we will study first the existence and uniqueness of a solution for the following periodic system of differential equations:

$$\begin{cases} x' + \lambda_1 x - \lambda_2 y = f(t, x) + g(t, y) + \lambda_1 x - \lambda_2 y, t \in (0, T) \\ y' + \lambda_1 y - \lambda_2 x = f(t, y) + g(t, x) + \lambda_1 y - \lambda_2 x \\ x(0) = x(T) \\ y(0) = y(T). \end{cases} \tag{5.7}$$

Under some usual continuity assumptions (which will be imposed in the main theorem) the above system is equivalent to the following system of integral equations

$$\begin{cases} x(t) = \displaystyle\int_0^T (G_1(t, s)[f(s, x) + g(s, y) + \lambda_1 x - \lambda_2 y] \\ \qquad\qquad +G_2(t, s)[f(s, y) + g(s, x) + \lambda_1 y - \lambda_2 x])ds \\ y(t) = \displaystyle\int_0^T (G_1(t, s)[f(s, y) + g(s, x) + \lambda_1 y - \lambda_2 x] \\ \qquad\qquad +G_2(t, s)[f(s, x) + g(s, y) + \lambda_1 x - \lambda_2 y])ds, \end{cases}$$

where $t \in [0, T]$ and G_1, G_2 have the same expressions as in [10]. By the assumptions (5.6), we obtain (see [10]) that $G_1(t, s) \ge 0$ and $G_2(t, s) \le 0$, for all $t, s \in [0, T]$.

Then, we have the following existence and uniqueness result for the periodic system (5.7).

Theorem 5.2. *Consider the periodic system of differential equations (5.7). We suppose:*

(i) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *and* $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ *are continuous;*

(ii) there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ *such that, for every* $x, y \in \mathbb{R}$ *with* $y \le x$*, we have*

$$0 \le (f(t, x) + \lambda_1 x) - (f(t, y) + \lambda_1 y) \le \mu_1(x - y)$$

$$-\mu_2(x - y) \le (g(t, x) - \lambda_2 x) - (g(t, y) - \lambda_2 y) \le 0,$$

where

$$\max\left\{\left(\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2}\right)^2, \left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^2\right\} < \frac{1}{4}.$$

(iii) the relations (5.6) are satisfied;

(iv) there exist lower and upper solution $x_0, y_0 \in C[0,T]$ *for the system (5.7) such that*

$$\begin{cases} \lambda_1(x_0(T)-x_0(0)) + \lambda_2(y_0(0)-y_0(T)) \le \dfrac{x_0(T)-x_0(0)}{T} \\ \lambda_1(y_0(T)-y_0(0)) + \lambda_2(x_0(0)-x_0(T)) \le \dfrac{y_0(T)-y_0(0)}{T}; \end{cases} \tag{5.8}$$

Then, there exists a unique solution (x^*, y^*) *of the system (5.7).*

Proof. We consider the space $X := C[0,T]$ endowed with the partial order relation:

$$x \le_C y \Longleftrightarrow x(t) \le y(t) \text{ for all } t \in [0,T].$$

Notice again that the pair $(X, \le)$ has the property (3.4).

On X we will also consider the functional

$$d(x,y) := \max_{t\in[0,T]}((x(t)-y(t))^2).$$

Notice that d is a b-metric with constant $s = 2$ and d can be represented using the classical Chebyshev norm by $d(x,y) = \|(x-y)^2\|_C$.

We can prove that all the assumptions of Theorem 3.5 are satisfied. We define $S : X \times X \to X$ by

$$\begin{aligned} S(x,y)(t) := \int_0^T &(G_1(t,s)[f(s,x) + g(s,y) + \lambda_1 x - \lambda_2 y] \\ &+ G_2(t,s)[f(s,y) + g(s,x) + \lambda_1 y - \lambda_2 x])ds. \end{aligned}$$

Then, the system (5.3) can be written as a coupled fixed point problem for S:

$$\begin{cases} x = S(x,y) \\ y = S(y,x). \end{cases} \tag{5.9}$$

By (ii), it follows in a similar way to [10], that S satisfies the mixed monotone condition.

By (iv), we have that $x_0(t) \le S(x_0,y_0)(t)$ and $y_0(t) \ge S(y_0,x_0)(t)$, for all $t \in [0,T]$.

Moreover, for $x \le u, y \ge v$, we can write

$$\begin{aligned} &(S(x,y)(t) - S(u,v)(t))^2 \\ &\le \Bigg(\int_0^T (G_1(t,s)[(f(s,x)+g(s,y)+\lambda_1 x-\lambda_2 y) - (f(s,u)+g(s,v)+\lambda_1 u-\lambda_2 v)] \\ &\quad + G_2(t,s)[(f(s,y)+g(s,x)+\lambda_1 y-\lambda_2 x) - (f(s,v)+g(s,u)+\lambda_1 v-\lambda_2 u)])ds\Bigg)^2 \\ &= \Bigg(\int_0^T (G_1(t,s)[(f(s,x)+g(s,y)+\lambda_1 x-\lambda_2 y) - (f(s,u)+g(s,v)+\lambda_1 u-\lambda_2 v)] \end{aligned}$$

$$
\begin{aligned}
&-G_2(t,s)[(f(s,v)+g(s,u)+\lambda_1 v-\lambda_2 u)-(f(s,y)+g(s,x)+\lambda_1 y-\lambda_2 x)])ds\Big)^2\\
&\leq \left(\int_0^T (G_1(t,s)[\mu_1(u-x)+\mu_2(y-v)]-G_2(t,s)[\mu_1(y-v)+\mu_2(u-x)]ds)\right)^2\\
&\qquad= \Bigg(\int_0^T G_1(t,s)[\mu_1\sqrt{(u-x)^2}+\mu_2\sqrt{(y-v)^2}]\\
&\qquad\quad -G_2(t,s)[\mu_1\sqrt{(y-v)^2}+\mu_2\sqrt{(u-x)^2}]ds\Bigg)^2\\
&\qquad= \Bigg(\int_0^T (\mu_1 G_1(t,s)-\mu_2 G_2(t,s))\sqrt{(u-x)^2}ds\\
&\qquad\quad +\int_0^T (\mu_2 G_1(t,s)-\mu_1 G_2(t,s))\sqrt{(y-v)^2}ds\Bigg)^2\\
&\qquad\leq 2\Bigg(\left[\int_0^T (\mu_1 G_1(t,s)-\mu_2 G_2(t,s))\sqrt{(u-x)^2}ds\right]^2\\
&\qquad\quad +\left[\int_0^T (\mu_2 G_1(t,s)-\mu_1 G_2(t,s))\sqrt{(y-v)^2}ds\right]^2\Bigg)\\
&\qquad= 2\Bigg(\|(x-u)^2\|_C\left[\int_0^T (\mu_1 G_1(t,s)-\mu_2 G_2(t,s))ds\right]^2\\
&\qquad\quad +\|(y-v)^2\|_C\left[\int_0^T (\mu_2 G_1(t,s)-\mu_1 G_2(t,s))ds\right]^2\Bigg)\\
&\qquad\leq \left(\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2}\right)^2\cdot\|(x-u)^2\|_C+\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^2\cdot\|(y-v)^2\|_C\\
&\leq \max\left\{\left(\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2}\right)^2,\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^2\right\}\cdot(\|(x-u)^2\|_C+\|(y-v)^2\|_C),\forall t\in[0,T].
\end{aligned}
$$

Thus, by taking the maximum over $[0,T]$, we obtain

$$d(S(x,y),S(u,v))\leq k(d(x,u)+d(y,v)),$$

where $k:=\max\left\{\left(\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2}\right)^2,\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^2\right\}$.

By a similar approach, we get that

$$d(S(y,x),S(v,u))\leq k(d(x,u)+d(y,v)).$$

Hence, by adding the above two relations, we get

$$d(S(x,y),S(u,v))+d(S(y,x),S(v,u))\}\leq 2k(d(x,u)+d(y,v)),\ \text{for all } x\leq u, y\geq v.$$

Since $2k<\frac{1}{2}$, all the assumptions of Theorem 3.5 are satisfied and the result follows by Theorem 3.5. □

We will discuss now the existence of the solution for the periodic boundary value problem (5.5).

Theorem 5.3. *Consider the periodic boundary value problem (5.5). Suppose that all the assumptions of Theorem 5.2 are satisfied. Then, the periodic boundary value problem (5.5) has a unique solution x^*.*

Proof. The conclusion follows by Theorem 3.6, since we get that, for the unique solution (x^*, y^*) of the coupled system, we also have $x^* = y^*$. Hence x^* is the unique solution of the periodic boundary value problem (5.5). □

Remark 5.4. It is worth to mention, that different qualitative properties (data dependence, well-posedness, Ulam-Hyers stability, limit shadowing) of the above studied problems can be discussed, using the coupled fixed point approach given in Section 4.

We also notice that the above examples illustrate how fixed point and coupled fixed point results in b-metric spaces can be applied to different operator equations and systems. It is an open question to find applications of the abstract results in some relevant complete b-metric spaces, for example, in b-metric spaces which cannot be organized as metric spaces.

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