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PARTIAL METRIC, FIXED POINTS, VARIATIONAL PRINCIPLES

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Abstract. The paper is mainly devoted to fixed point theorems. More general versions of important theorems of Matkowski, Romaguera, Takahashi, Brøndsted, and Banach are proved. Also, Ekeland's variational principle is extended. On the way to the main results, short proofs of crucial properties of partial metric spaces are proposed. In addition, the notion of kernel of partial metric is suggested together with a standard method of proving fixed point theorems. We also supply a useful criterion for Cauchy sequences.

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An Introduction

The paper is divided into three sections. The first one is devoted to the properties of partial metric (it **need not** be nonnegative). We offer short proofs of the crucial dependencies between partial metric and the related natural metric. All these facts are gathered in Lemma 1.7. Lemma 1.9 is a nice criterion for Cauchy sequences. It is also helpful in proving fixed point theorems.

The second section begins with the definition of a kernel $Ker\ p = \{x \in X : p(x, x) = 0\}$ of p, followed by Lemma 2.2, establishing some properties of it. We suggest a standard method of proving fixed point theorems involving the kernel. Two examples are given: the theorem of Romaguera [16, Corollary 3] (here Theorem 2.3) and the theorem of Aydi-Abbas-Vetro [1, Theorem 3.2] (here Theorem 2.4). Next, we present an extension of the Romaguera result (Theorem 2.6) with a good-looking condition (2.2), which is more general than the well known Matkowski's condition, also in the case of metric space. In it simplest form, Theorem 2.6 resolves the following well known problem: "Assume that $d(f^t(y), f^s(x)) \leq kd(y, x)$ for a $k < 1$. Does f have a fixed point?". The last theorem of this section concerns a common fixed point of a family of mappings.

The third section is devoted to variational type theorems. The variational principles, i.e. Theorems 3.5, 3.6, 3.9 (an extension of the Ekeland theorem), and fixed point theorems related to Theorem 3.6, i.e. Theorems 3.7, 3.8 (an extension of the

Takahashi theorem), follow from the respective general results of [14]. Still we require some lemmas to show that the theory developed in [14] applies to partial metric spaces. The section concludes with a theorem of Brøndsted type (Theorem 3.10) and a variational version of the Banach type fixed point theorem (Theorem 3.11).

1. Properties of partial metric spaces

Let us recall the notion of a partial metric space introduced by Matthews [10, Definition 3.1].

Definition 1.1. *A* **partial metric** *is a mapping* $p\colon X \times X \to R$ *such that,*

$$y = x \text{ iff } p(y, y) = p(y, x) = p(x, x),\ x, y \in X, \tag{1.1}$$

$$p(y, y) \leq p(y, x),\ x, y \in X, \tag{1.2}$$

$$p(y, x) = p(x, y),\ x, y \in X, \tag{1.3}$$

$$p(z, x) \leq p(z, y) + p(y, x) - p(y, y),\ x, y, z \in X. \tag{1.4}$$

This nonalphabetical order, is better suited to variational principles.

Neill [12] suggests that Matthews assumes the values of p to be nonnegative. Therefore, some authors use Neill's notion of dualistic partial metric space for the case of real valued p. We prefer the original definition given by Matthews, with $p\colon X \times X \to R$.

If p is a partial metric on X then $q\colon X \times X \to [0, \infty)$ defined by

$$q(y, x) = p(y, x) - p(y, y),\ x, y \in X \tag{1.5}$$

is a quasi-metric [10, Theorem 4.1] ($y = x$ iff $q(y, x) = q(x, y) = 0$, $q(z, x) \leq q(z, y) + q(y, x)$).

An open ball for $x \in X$, $\epsilon > 0$ is defined by:

$$B(x, \epsilon) = \{y \in X : q(x, y) < \epsilon\} = \{y \in X : p(x, y) < p(x, x) + \epsilon\}. \tag{1.6}$$

The family of open balls generates topology $\mathcal{T}_q$ on X. It is accepted that the **partial metric space** (X, p) is equipped with the topology $\mathcal{T}_q$.

It is known (see, e.g. [13]) that a metric d can be defined by a partial metric p as follows:

$$\begin{aligned} d(y, x) &= \max\{q(y, x), q(x, y)\} \\ &= \max\{p(y, x) - p(y, y), p(x, y) - p(x, x)\},\ x, y \in X. \end{aligned} \tag{1.7}$$

The norms $\max\{\mid x \mid, \mid y \mid\}$ and $[\mid x \mid + \mid y \mid]/2$ in R^2 are equivalent and therefore metric d as in (1.7) is equivalent to ρ:

$$\rho(y, x) = [q(y, x) + q(x, y)]/2 = p(y, x) - [p(y, y) + p(x, x)]/2,\ x, y \in X.$$

In this section it is understood that q, d are defined by (1.5), (1.7) respectively, for a partial metric p.

Remark 1.2. *Clearly, the metric topology of* (X, d) *is stronger than the topology* $\mathcal{T}_q$ *of* (X, p). *Therefore, every closed set in* (X, p) *is closed in* (X, d), *and* $x = \lim_{n\to\infty} x_n$ *in* (X, d) *implies* $x \in \lim_{n\to\infty} x_n$ *in* (X, p).

For partial spaces it is accepted (see, e.g. [13, p. 19]) that a sequence $(x_n)_{n\in N}$ is called a **Cauchy sequence** in (X,p) if $\lim_{m,n\to\infty} p(x_n,x_m)=\alpha\in R$, and (X,p) is **complete** if for every Cauchy sequence $(x_n)_{n\in N}$ in (X,p) there exists an $x\in \lim_{n\to\infty} x_n$ such that $\lim_{m,n\to\infty} p(x_n,x_m)=p(x,x)$.

Proposition 1.3. *Let (X,p) be a partial metric space and $(x_n)_{n\in N}$ a sequence in X. If $x\in\lim_{n\to\infty}x_n$ then the following holds:*

$$\limsup_{m,n\to\infty} p(x_n,x_m)\le \lim_{n\to\infty} p(x,x_n)=p(x,x). \tag{1.8}$$

If $\lim_{n\to\infty} p(x,x_n)=p(x,x)$ is satisfied then $x\in\lim_{n\to\infty}x_n$.

Proof. The equivalence of $\lim_{n\to\infty} p(x,x_n)=p(x,x)$ and $x\in\lim_{n\to\infty}x_n$ follows directly from (1.6) and (1.2). Conditions (1.4), (1.3) yield

$$\begin{aligned} p(x_n,x_m)-p(x,x) &\le p(x_n,x)+p(x,x_m)-2p(x,x)\\ &= p(x,x_n)-p(x,x)+p(x,x_m)-p(x,x). \end{aligned}$$

Therefore, $\lim_{n\to\infty} p(x,x_n)=p(x,x)$ implies

$$\limsup_{m,n\to\infty} p(x_n,x_m)\le p(x,x).$$

□

The subsequent three propositions are included in the proof of Lemma 2.2 in [13]. We present shorter reasonings.

Proposition 1.4. *Let (X,p) be a partial metric space. A sequence $(x_n)_{n\in N}$ converges to an x in (X,d) iff the following is satisfied:*

$$\lim_{m,n\to\infty} p(x_n,x_m)=\lim_{n\to\infty} p(x,x_n)=p(x,x). \tag{1.9}$$

Proof. If (1.9) holds then

$$\begin{aligned} \lim_{n\to\infty} q(x,x_n) &= \lim_{n\to\infty}[p(x,x_n)-p(x,x)]\\ &= \lim_{n\to\infty}[p(x_n,x)-p(x_n,x_n)]\\ &= \lim_{n\to\infty} q(x_n,x)=0 \end{aligned}$$

and therefore $x=\lim_{n\to\infty}x_n$ in (X,d). Conversely, for $x=\lim_{n\to\infty}x_n$ in (X,d), i.e. $\lim_{n\to\infty} q(x_n,x)=\lim_{n\to\infty} q(x,x_n)=0$ from

$$\begin{aligned} 0 &\le p(x_n,x_m)-p(x_n,x_n)\le p(x_n,x)+p(x,x_m)-p(x,x)-p(x_n,x_n)\\ &= p(x_n,x)-p(x_n,x_n)+p(x,x_m)-p(x,x)=q(x_n,x)+q(x,x_m) \end{aligned}$$

we get (1.9). □

Proposition 1.5. *A sequence $(x_n)_{n\in N}$ is a Cauchy sequence in (X,p) iff it is a Cauchy sequence in (X,d).*

Proof. Assume that $(x_n)_{n\in N}$ is a Cauchy sequence in (X,p). From

$$\lim_{m,n\to\infty} p(x_n,x_m)=\alpha\in R$$

it follows that

$$\lim_{m,n\to\infty} q(x_n,x_m)=\lim_{m,n\to\infty}[p(x_n,x_m)-p(x_n,x_n)]=0$$

and $\lim_{m,n\to\infty} q(x_m, x_n) = 0$, i.e. $\lim_{m,n\to\infty} d(x_n, x_m) = 0$ and $(x_n)_{n\in N}$ is a Cauchy sequence in (X, d).

Assume that $(x_n)_{n\in N}$ is a Cauchy sequence in (X, d). We have

$$\begin{aligned} | p(x_n, x_n) - p(x_m, x_m) | &\leq | p(x_n, x_m) - p(x_n, x_n) | + | p(x_m, x_n) - p(x_m, x_m) | \\ &= q(x_n, x_m) + q(x_m, x_n) \end{aligned}$$

and consequently (see (1.7)), $(p(x_n, x_n))_{n\in N}$ is a Cauchy sequence in R, and so it converges, say to an $\alpha \in R$. Now $\lim_{m,n\to\infty}[p(x_n, x_m) - p(x_n, x_n)] = 0$ yields $\lim_{m,n\to\infty} p(x_n, x_m) = \alpha$ and $(x_n)_{n\in N}$ is a Cauchy sequence in (X, p). □

Proposition 1.6. *A partial metric space (X, p) is complete iff the metric space (X, d) is complete.*

Proof. In view of Proposition 1.5 any Cauchy sequence in (X, p) or (X, d) is a Cauchy sequence in both spaces. If (X, d) is complete and $x = \lim_{n\to\infty} x_n$ then $x \in \lim_{n\to\infty} x_n$ in (X, p) (Remark 1.2). If (X, p) is complete and $x \in \lim_{n\to\infty} x_n$ is such that $\lim_{m,n\to\infty} p(x_m, x_n) = p(x, x)$ then in view of Proposition 1.3 condition (1.9) is satisfied and $x = \lim_{n\to\infty} x_n$ in (X, d) (Proposition 1.4). □

Our propositions yield the following more precise version of [13, Lemma 2.2].

Lemma 1.7. *A partial metric space (X, p) is complete iff the metric space (X, d) is complete (see (1.7)). We have $x \in \lim_{n\to\infty} x_n$ in (X, p) iff (1.8) holds, and $x = \lim_{n\to\infty} x_n$ in (X, d) iff (1.9) is satisfied. A sequence $(x_n)_{n\in N}$ is a Cauchy sequence in (X, p) iff it is a Cauchy sequence in (X, d); in addition, if any of these spaces is complete then there exists an $x \in \lim_{n\to\infty} x_n$ in (X, p) such as to satisfy (1.9).*

The last three lemmas of this section have a role to play in Section 3. Lemma 1.9 is a nice criterion for Cauchy sequences.

Proposition 1.8. *Let $(\alpha_n)_{n\in N}$ be sequence in R satisfying:*

$$\begin{gathered} \text{for each } \epsilon > 0 \text{ there exists an } n_0 \in N \text{ such that} \\ \text{each } m, n \in N,\ n_0 < m < n \text{ yield } \alpha_m < \alpha_n + \epsilon. \end{gathered} \tag{1.10}$$

Then there exists $\lim_{n\to\infty} \alpha_n > -\infty$.

Proof. Suppose that $(\alpha_n)_{n\in N}$ has no finite limit. Then for a $\delta > 0$ there exist infinitely many $m < n$ such that $| \alpha_n - \alpha_m | > \delta$, i.e. $\alpha_n > \alpha_m + \delta$ or $\alpha_m > \alpha_n + \delta$. The last inequality contradicts (1.10). Let k_1 be such that $\alpha_n > \alpha_{k_1} - \delta/2$ holds for all $n > k_1$ (see (1.10)). There exist $k_2, n \in N$ such that $k_2 > n > k_1$ and $\alpha_{k_2} > \alpha_n + \delta$. Hence we obtain $\alpha_{k_2} > \alpha_n + \delta > \alpha_{k_1} + \delta/2$. By induction we define a subsequence $(\alpha_{k_n})_{n\in N}$ such that $\alpha_{k_{n+1}} > \alpha_{k_n} + \delta/2$, $n \in N$. This means that $\lim_{n\to\infty} \alpha_{k_n} = \infty$. Now for all $n > k_m > n_0$ we have (see (1.10)) $\alpha_n \geq \alpha_{k_m} - \delta/2$ and hence $\lim_{n\to\infty} \alpha_n = \infty$. □

Lemma 1.9. *Let (X, p) be a partial metric space. A sequence $(x_n)_{n\in N}$ is a Cauchy sequence in (X, p) (or (X, d)) iff $(p(x_n, x_n))_{n\in N}$ is upper bounded and the following condition is satisfied:*

$$\begin{gathered} \text{for each } \epsilon > 0 \text{ there exists an } n_0 \in N \text{ such that each } m, n \in N, \\ n_0 < m < n \text{ yield } q(x_n, x_m) = p(x_n, x_m) - p(x_n, x_n) < \epsilon. \end{gathered} \tag{1.11}$$

Proof. We have (see (1.4), (1.3))

$$p(x_m, x_m) - p(x_n, x_n) \le p(x_m, x_n) + p(x_n, x_m) - 2p(x_n, x_n)$$
$$= 2[p(x_n, x_m) - p(x_n, x_n)].$$

Therefore, (1.11) implies (1.10) for $\alpha_n = p(x_n, x_n)$, $n \in N$. Now we apply Proposition 1.8. From the upper boundedness of $(p(x_n, x_n))_{n \in N}$ it follows that $\lim_{n\to\infty} p(x_n, x_n) = \alpha \in R$. Now (1.2) and (1.11) yield

$$0 \le p(x_n, x_m) - p(x_n, x_n) < \epsilon,\ n_0 < m < n$$

for large $m < n$, which means (see (1.3)) that $\lim_{m,n\to\infty} p(x_n, x_m) = \alpha \in R$ and $(x_n)_{n\in N}$ is a Cauchy sequence in both spaces (Proposition 1.5).

If $(x_n)_{n\in N}$ is a Cauchy sequence in (X, p) then $\lim_{m,n\to\infty} p(x_n, x_m) = \alpha \in R$, $(p(x_n, x_n))_{n\in N}$ is bounded and clearly (see Proposition 1.5), (1.11) holds. □

Lemma 1.10. *Let (X, p) be a partial metric space. Then $q(\cdot, y)$ is a lower semicontinuous mapping on (X, p) and (X, d) for each $y \in X$.*

Proof. Let us assume $x \in \lim_{n\to\infty} x_n$ in (X, p). Then from

$$q(x, y) = p(x, y) - p(x, x) \le p(x, x_n) + p(x_n, y) - p(x_n, x_n) - p(x, x)$$
$$= p(x_n, y) - p(x_n, x_n) + p(x, x_n) - p(x, x) = q(x_n, y) + q(x, x_n)$$

and Proposition 1.3 ($\lim_{n\to\infty} q(x, x_n) = 0$) we get

$$q(x, y) \le \liminf_{n\to\infty} q(x_n, y).$$

The same reasoning applies to $x = \lim_{n\to\infty} x_n$ in (X, d) and Proposition 1.4. □

In fact, our Lemma 1.10 for (X, d) is equivalent to [15, Lemma 2.2], which states that $p(\cdot, x)$ is lower semicontinuous on (X, d), as by Proposition 1.4 $p(\cdot, \cdot)$ is continuous.

Lemma 1.11. *Let (X, p) be a partial metric space. If $f : (X, d) \to (X, d)$ is a continuous mapping then $p(f(\cdot), \cdot)$ and $q(f(\cdot), \cdot) = p(f(\cdot), \cdot) - p(f(\cdot), f(\cdot))$ are lower semicontinuous on (X, d).*

Proof. We have (see (1.4))

$$p(f(x), x) \le p(f(x), f(x_n)) + p(f(x_n), x) - p(f(x_n), f(x_n))$$

$$\le p(f(x), f(x_n)) - p(f(x_n), f(x_n)) + p(f(x_n), x_n) + p(x_n, x) - p(x_n, x_n).$$

Condition (1.9) applies for $x = \lim_{n\to\infty} x_n$ in (X, d) and therefore, for continuous f we obtain

$$p(f(x), x) \le \liminf_{n\to\infty} p(f(x_n), x_n).$$

Besides, $p(\cdot, \cdot)$ is continuous (see Proposition 1.4) and consequently, $q(f(\cdot), \cdot)$ is lower semicontinuous. □

2. Kernel and fixed points

Many fixed point theorems for partial metric spaces contain a conclusion that $f(x) = x$ for an x such that $p(x,x) = 0$. This suggests that the following notion should be useful.

Definition 2.1. *Let $p : X \times X \to R$ be a mapping. The* **kernel** *of p is the set* **Ker p** $= \{x \in X : p(x,x) = 0\}$.

The next lemma presents some properties of the kernel in partial metric spaces.

Lemma 2.2. *Let (X,p) be a partial metric space with a nonempty kernel Z. Then $p = d$ on $Z \times Z$ (see (1.7)) and the metric topology of $(Z, p_{|Z\times Z})$ is induced by the topology of (X,p). The kernel is a closed set in (X,d). If (X,p) or (X,d) is complete then Z is complete in both spaces.*

Proof. For $x,y \in Z$ condition (1.1) means that $y = x$ iff $p(y,x) = 0$. Condition (1.4) yields

$$p(z,x) \le p(z,y) + p(y,x) - p(y,y) = p(z,y) + p(y,x),\ x,z \in X,\ y \in Z.$$

Thus $(Z, p_{|Z\times Z})$ is a metric space and, clearly $p = d$ on $Z \times Z$. For $z = x \in Z$ conditions (1.4), (1.3) yield $p(y,x) \ge 0$, $x,y \in Z$ (the same follows from (1.2)). The induced topology of a subspace of the partial metric space (X,p) is the partial metric topology for restricted p and therefore the metric topology of $(Z, p_{|Z\times Z})$ is induced by the topology of (X,p).

If $(x_n)_{n\in N}$ is a sequence in Z and $x = \lim_{n\to\infty} x_n$ in (X,d), then by (1.9) $0 = \lim_{n\to\infty} p(x_n,x_n) = \lim_{m,n\to\infty} p(x_n,x_m) = p(x,x)$, i.e. $x \in Z$ and consequently, Z is closed in (X,d).

For complete (X,p) and any Cauchy sequence $(x_n)_{n\in N}$ in Z there exists an $x \in \lim_{n\to\infty} x_n$ in (X,p) such that

$$0 = \lim_{n\to\infty} p(x_n,x_n) = \lim_{m,n\to\infty} p(x_n,x_m) = p(x,x).$$

Thus $x \in Z$ and Z is complete in (X,p). Clearly, the same concerns Z as a closed subset of the complete metric space (X,d). Consequently (see Proposition 1.6), Z is a complete set in (X,p) and (X,d), if any of these spaces is complete. □

Let us present two applications of Lemma 2.2.

Romaguera has proved the following extension of a theorem due to Matkowski [8, Theorem 1.2, p. 8].

Theorem 2.3 ([16, Corollary 3]). *Let (X,p) be a complete partial space with nonnegative p and let $f : X \to X$ be a mapping such that*

$$p(f(y), f(x)) \le \Phi(p(y,x)),\ \ x,y \in X, \tag{2.1}$$

where $\Phi : [0,\infty) \to [0,\infty)$ is a nondecreasing function such that

$$\lim_{n\to\infty} \Phi^n(t) = 0 \textit{ for all } t > 0.$$

Then f has a unique fixed point x and $p(x,x) = 0$.

It is sufficient to prove that $Z = Ker\, p \neq \emptyset$. Then from (2.1) it follows that for any $x \in Z$ we have

$$p(f(x), f(x)) \leq \Phi(p(x, x)) = 0,$$

i.e. $f(Z) \subset Z$ and Matkowski's theorem applies to $f_{|Z}$. In addition, each fixed point of f belongs to Z, as

$$p(x, x) = p(f(x), f(x)) \leq \Phi(p(x, x))$$

means that $p(x, x) = 0$.

Lemma 2.2 also applies in the proof of the following theorem of Aydi-Abbas-Vetro.

Theorem 2.4 ([1, Theorem 3.2])**.** *Let (X, p) be a complete partial metric space with nonnegative p. If $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_p(Ty, Tx) \leq kp(y, x)$$

where $k \in (0, 1)$. Then T has a fixed point.

The authors first prove that there exists an $x \in Z = Ker\, p$. For any $x \in Z$ we have $H_p(Tx, Tx) \leq kp(x, x) = 0$. Moreover, $H_p(Tx, Tx) = sup\{p(y, y) : y \in Tx\}$ (see [1, Proposition 2.2 (i)]), and consequently, $T(Z) \subset Z$. Now it is sufficient to apply the Nadler theorem [11, Theorem 5] to $T_{|Z}$.

The known notion of 0-completeness can be defined in the following way with the help of the concept of kernel.

Definition 2.5 (cp. [15, Definition 2.1])**.** *A partial metric space (X, p) is* **0-complete** *if every sequence $(x_n)_{n\in N}$ in X such that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ yields $\lim_{n\to\infty} x_n \cap Ker\, p \neq \emptyset$.*

It should be noted that nonnegative partial metric p defines metric δ in the following way: $\delta(x, y) = 0$ iff $x = y$, and $\delta(x, y) = p(x, y)$ for $x \neq y$. The topology of (X, δ) is clearly, larger than the topology of (X, d) (see (1.7)). Moreover, (X, p) is 0-complete iff (X, δ) is complete [7], [6, Proposition 2.1].

If the proof of a theorem is based on a sequence $(x_n)_{n\in N}$ such that $p(x_{n+1}, x_n) \neq 0$, $n \in N$, then it works for (X, δ) and the theorem itself looks like the respective result (if known) for metric spaces. Numerous examples can be found in [6]. The authors work on assumptions in order to replace p by δ, and then known "metric" theorems appear.

The method presented in [6] is very interesting and strong, nevertheless our results seem to be new also for metric spaces. As regards the kernel, it is a natural metric subspace of partial metric space and is better suited to the cases where topology of this subspace matters.

Now we are ready to present a considerable extension of the Romaguera theorem [16, Corollary 3]. Let us note that f^0 is the identity mapping.

Theorem 2.6. *Let (X, p) be a 0-complete partial metric space with nonnegative p. Assume that $f : X \to X$ is a mapping satisfying:*

$$p(f^t(y), f^s(x)) \leq \Phi(p(y, x)),\ x, y \in X \tag{2.2}$$

for fixed $s, t \in N \cup \{0\}$, *where* $\Phi\colon [0,\infty) \to [0,\infty)$ *is a nondecreasing mapping such that* $\lim_{n\to\infty}\Phi^n(\alpha) = 0$, $\alpha > 0$. *Then* f *has a unique fixed point. If* x *is a fixed point of* f *then* $x \in \lim_{n\to\infty} f^n(x_0) \cap \mathit{Ker}\, p$, $x_0 \in X$.

To make the proof of Theorem 2.6 more perspicuous we precede it with four propositions.

Proposition 2.7. *Assume that* $\Phi\colon [0,\infty) \to [0,\infty)$ *is a mapping such that* $\Phi(0) = 0$ *and* $\Phi(\alpha) < \alpha$, $\alpha > 0$. *Then* $\alpha \le \Phi^k(\alpha)$ *yields* $\alpha = 0$, $k \in N$.

Proof. Let us adopt $\Phi^0(\alpha) = \alpha$ and suppose $\Phi^{k-1}(\alpha) > 0$. Then by finite induction we get $\Phi^k(\alpha) = \Phi(\Phi^{k-1}(\alpha)) < \Phi^{k-1}(\alpha) < \cdots < \alpha$, which contradicts $\alpha \le \Phi^k(\alpha)$. Therefore, $0 \le \alpha \le \Phi(\Phi^{k-1}(\alpha)) = \Phi(0) = 0$ holds. □

Definition 2.8. *A family* $\mathcal{F}[X]$ **commutes on fixed points** *if it consists of mappings from* X *in itself such that* $h(x) = x$ *for a* $h \in \mathcal{F}[X]$ *and an* $x \in X$ *yields* $(h \circ g)(x) = g(x)$ *for all* $g \in \mathcal{F}[X]$.

Proposition 2.9. *Let* $p\colon X \times X \to R$ *be a mapping satisfying (1.1) on Ker p in place of* X *and let* $\mathcal{F}[X]$ *commute on fixed points and satisfy (2.3) for* $s = t = 1$, $|\, p \,|$ *in place of* p *and a* $\Phi\colon [0,\infty) \to [0,\infty)$ *such that* $\Phi(0) = 0$ *and* $\Phi(\alpha) < \alpha$, $\alpha > 0$. *Then each of* $f \in \mathcal{F}[X]$ *has at most one fixed point, the point is common for* $\mathcal{F}[X]$ *and belongs to Ker p.*

Proof. If $x, y \in X$ are fixed points of some $g, h \in \mathcal{F}[X]$, then (2.3) yields

$$|\, p(y,x) \,| = |\, p(h(y), g(x)) \,| \le \Phi(|\, p(y,x) \,|),$$

i.e. $p(y,x) = 0$ (Proposition 2.7). Similarly, we get

$$|\, p(x,x) \,| = |\, p(g(x), g(x)) \,| \le \Phi(|\, p(x,x) \,|),$$

i.e. $p(x,x) = 0$ and finally $p(x,x) = p(y,y) = 0$ ($x, y \in \mathit{Ker}\, p$). Now, from (1.1) it follows that $y = x$. □

Proposition 2.10. *Let* $p\colon X \times X \to R$ *be a mapping satisfying (1.1) for all* $x, y \in$ *Ker p, and (1.3). Assume that* $\mathcal{F}[X]$ *commutes on fixed points and satisfies (2.3) for* $s = t = 1$, $|\, p \,|$ *in place of* p *and a* $\Phi\colon [0,\infty) \to [0,\infty)$ *such that* $\Phi(\alpha) < \alpha$, $\alpha > 0$. *If* Φ *is nondecreasing, then* $h(x) = x$ *for an* $h \in \mathcal{F}[X]$ *and an* $x \in \mathit{Ker}\, p$ *yields* $g(x) = x$, $g \in \mathcal{F}[X]$; *if in addition,* $h = f^t$, $g = f^s$ *for some* $s + t > 0$ *then* $f(x) = x$ *holds.*

Proof. First let us prove that $g(x) = x$. We have

$$|\, p(g(x), x) \,| = |\, p(x, g(x)) \,| = |\, p(h(x), g(x)) \,| \le \Phi(|\, p(x,x) \,|) = 0.$$

Φ is nondecreasing and hence we get

$$\begin{aligned} |\, p(g(x), g(x)) \,| &= |\, p((h \circ g)(x), (g \circ h)(x)) \,| \le \Phi(|\, p(g(x), h(x)) \,|) \\ &= \Phi(|\, p(h(x), g(x)) \,|) \le \Phi^2(|\, p(x,x) \,|) = 0. \end{aligned}$$

Now $p(g(x), g(x)) = p(g(x), x) = p(x,x) = 0$ and (1.1) for $x \in \mathit{Ker}\, p$ yield $g(x) = x$. From (2.2) for $|\, p \,|$ in place of p we get

$$|\, p(f(x), x) \,| = |\, p(f^{t+1}(x), f^s(x)) \,| \le \Phi(|\, p(f(x), x) \,|),$$

and by Proposition 2.7 $p(f(x), x) = 0$. Now for nondecreasing Φ we obtain

$$| p(f(x), f(x)) |=| p(f^{t+s+1}(x), f^{s+t+1}(x)) |\leq \Phi^2(| p(f(x), f(x)) |),$$

and consequently, $p(f(x), f(x)) = p(f(x), x) = p(x, x) = 0$, i.e $f(x) = x$. □

Proposition 2.11. *Let $p : X \times X \to [0,\infty)$ be a mapping satisfying (1.1), (1.2), (1.3) for all $x \in Ker\, p$, the triangle inequality, and let $f : X \to X$ satisfy (2.2) for a $\Phi : [0,\infty) \to [0,\infty)$ such that $\Phi(\alpha) \leq \alpha$, $\alpha \geq 0$. Then for $x_n = f^n(x_0)$, $n \in N$ condition $\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0$ yields $f^t(x) = x$.*

Proof. From

$$0 \leq p(f^t(x), x_n) \leq \Phi(p(x, x_{n-s})) \leq p(x, x_{n-s})$$

it follows that $\lim_{n\to\infty} p(f^t(x), x_n) = 0$. The triangle inequality and (1.3) yield

$$0 \leq p(f^t(x), x) \leq p(f^t(x), x_n) + p(x, x_n),$$

i.e. $p(f^t(x), x) = 0$. Now from

$$0 \leq p(f^t(x), f^t(x)) \leq p(f^t(x), x)$$

we get $p(f^t(x), f^t(x)) = p(f^t(x), x) = p(x, x)$ and $f^t(x) = x$. □

Proof of Theorem 2.6. It is well known that Φ as in Theorem 2.6 satisfies $\Phi(\alpha) < \alpha$ for each $\alpha > 0$ [9, Lemma]. Therefore, our propositions work. From Proposition 2.7 it follows that the case of $s = t = 0$ is trivial, as here X is a singleton (see (1.1)). Thus we assume that $s \leq t$ and $t \geq 1$. Let $x_0 \in X$ be arbitrary and let $x_n = f^n(x_0)$, $n \in N$, while $f^0(x_0) = x_0$. If $\lim_{n\to\infty} p(x_n, x_m) = 0$ then by the 0-completeness of X there exists an $x \in \lim_{n\to\infty} x_n \cap Ker\, p$. In view of Proposition 2.11 we have $f^t(x) = x$; by Proposition 2.10, $f(x) = x$; and x is the unique fixed point of f (Proposition 2.9). Therefore, the only step we need is to prove that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. We adapt the Romaguera reasoning from the proof of [16, Theorem 4].

For each $\epsilon > 0$ there exists a k_0 such that all $k > k_0$ yield

$$p(x_{(k+1)(s+t)+v}, x_{k(s+t)+u}) \leq \Phi^{2k}(p(x_{s+t+v}, x_u)) \leq \epsilon - \Phi(\epsilon),$$

for all $u, v < s + t$ (see the proof of Proposition 2.10).
Let us adopt $n = k(s + t) + v$, $m = k(s + t) + u$. Now we have

$$\begin{aligned} p(x_{n+2(s+t)}, x_m) &\leq p(x_{n+2(s+t)}, x_{n+(s+t)}) + p(x_{n+(s+t)}, x_m) \\ &\quad - p(x_{n+(s+t)}, x_{n+(s+t)}) \\ &\leq p(x_{n+2(s+t)}, x_{n+(s+t)}) + p(x_{n+(s+t)}, x_m) \\ &\leq \Phi^2(\epsilon - \Phi(\epsilon)) + \epsilon - \Phi(\epsilon) \leq \Phi(\epsilon) + \epsilon - \Phi(\epsilon) = \epsilon, \end{aligned}$$

and similarly,

$$\begin{aligned} p(x_{n+3(s+t)}, x_m) &\leq p(x_{n+3(s+t)}, x_{n+(s+t)}) + p(x_{n+(s+t)}, x_m) \\ &\leq \Phi^2(p(x_{n+2(s+t)}, x_n)) + p(x_{n+(s+t)}, x_m) \\ &\leq \Phi^2(\epsilon) + \epsilon - \Phi(\epsilon) \leq \Phi(\epsilon) + \epsilon - \Phi(\epsilon) = \epsilon. \end{aligned}$$

By induction we prove that $p(x_{n+r(s+t)}, x_m) \leq \epsilon$, $r \in N$.
Consequently, $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ holds for independent $m, n \in N$. □

For a complete metric space (X,p) Theorem 2.6 extends the Matkowski theorems [8, Theorem 1.2, p. 8] and [9, Theorem 2].

Theorem 2.12. *Let (X,p) be a 0-complete partial metric space with nonnegative p. Assume that $\mathcal{F}[X]$ is a family such that for fixed s,t in$N \cup \{0\}$ and all $g,h \in \mathcal{F}[X]$ mappings h^t, g^s commute on their fixed points and*

$$p(h^t(y), g^s(x)) \leq \Phi(p(y,x)),\ g,h \in \mathcal{F}[X],\ x,y \in X \tag{2.3}$$

holds for a nondecreasing $\Phi\colon [0,\infty) \to [0,\infty)$ such that $\lim_{n\to\infty} \Phi^n(\alpha) = 0$, $\alpha > 0$. Then each of $f \in \mathcal{F}[X]$ has the same unique fixed point; if $x \in Fix\, f$ then $x \in \lim_{n\to\infty} f^n(x_0) \cap Ker\, p$, $x_0 \in X$.

Proof. Any $f \in \mathcal{F}[X]$ has a unique $x \in Fix\, f$ (Theorem 2.6). Clearly, $x \in Fix\, f^t$ and hence, x is a common fixed point (Proposition 2.9). □

More sophisticated theorems on coincidences and fixed points (for continuous Φ) can be found in [5].

3. Variational conditions

In [14, Definition 14] the following notion was introduced.

Definition 3.1. *Let $q\colon X \times X \to [0,\infty)$ be a mapping. Then $(x_n)_{n\in N}$ is a Cauchy sequence in (X,q) if for each $\epsilon > 0$ there exists an $n_0 \in N$ such that each $m,n \in N$, $n_0 < m < n$ yield $q(x_n,x_m) < \epsilon$.*

Now Lemma 1.9 can be given the following form:

Lemma 3.2. *Let (X,p) be a partial metric space with upper bounded $p(\cdot,\cdot)$ and let q be defined by (1.5). Then $(x_n)_{n\in N}$ is a Cauchy sequence in (X,q) in the sense of Definition 3.1 iff it is a Cauchy sequence in (X,p) (or in the metric space (X,d) (see (1.7)).*

Another notion from [14] is presented in the following:

Definition 3.3 ([14, Definition 15])**.** *Let (X,d) be a metric space. A mapping $q\colon X \times X \to [0,\infty)$ is a* d**-istance** *in X if the following conditions are satisfied:*

(i) $q(z,x) \leq q(z,y) + q(y,x)$, $x,y,z \in X$,
(ii) $q(\cdot,x)$, is lower semicontinuous, $x \in X$,
(iii) each Def. 3.1 Cauchy sequence in (X,q) is a Cauchy sequence in (X,d).

From condition (1.4), [15, Lemma 2.2] and Proposition 1.5 (see Definition 3.1) it follows that each nonnegative partial metric p is a d-istance for d as in (1.7).

Now, this last statement, Lemma 1.10, and Lemma 3.2 (see also Proposition 1.6) yield:

Corollary 3.4. *For a partial metric space (X,p) let q, d be as in (1.5), (1.7) respectively. If $p(\cdot,\cdot)$ is upper bounded then q is a d-istance. If p is nonnegative then it is a d-istance. Metric space (X,d) is complete iff (X,p) is complete.*

Corollary 3.4 and the main results of [14] yield the subsequent five theorems.

For a mapping $\psi\colon X \to R$ let us adopt

$$\beta = \inf\{\psi(z) \colon z \in X\},\ B = \{z \in X \colon \psi(z) = \beta\}. \tag{3.1}$$

Theorem 3.5 (cp. [14, Theorem 21])**.** *Let (X,p) be a complete partial metric space and let d be as in (1.7). Assume that $\psi : (X,d) \to R$ is a lower semicontinuous mapping bounded below, and $q = p$ for nonnegative p or q is defined by (1.5) for upper bounded $p(\cdot,\cdot)$. Let us adopt $y \preccurlyeq x$ iff $\psi(y) + q(y,x) - \psi(x) \leq 0$, $x, y \in X$, and assume that the following holds (see (3.1)):*

$$\text{for each } x \in X \setminus B \text{ there exists a } y \in X \setminus \{x\} \text{ such that } y \preccurlyeq x. \tag{3.2}$$

Then for any $x_0 \in X \setminus B$, each maximal chain $A \subset X$ containing x_0 has a unique smallest element x, in addition satisfying:

(i) $\psi(x) = \inf\{\psi(z) \colon z \in X\}$ *(i.e. $x \in B$),*
(ii) $\psi(x) + q(x,x_0) - \psi(x_0) = \inf\{\psi(z) + q(z,x_0) - \psi(x_0) \colon z \in A\} \leq 0,$
(iii) $0 < \psi(y) + q(y,x) - \psi(x)$*, for each* $y \in X \setminus \{x\}$.

Theorem 3.6 (cp. [14, Theorem 22])**.** *Let (X,p) be a complete partial metric space and let d be as in (1.7). Assume that $\psi : (X,d) \to R$ is a lower semicontinuous mapping bounded below, and $q = p$ for nonnegative p or q is defined by (1.5) for upper bounded $p(\cdot,\cdot)$. Let us adopt $y \preccurlyeq x$ iff $\psi(y) + q(y,x) - \psi(x) \leq 0$, $x, y \in X$, and assume that the following condition holds:*

$$\text{for each } x \in X \text{ there exists a } y \text{ such that } y \preccurlyeq x. \tag{3.3}$$

Then for any $xo \in X$, each maximal chain $A \subset X$ containing x_0 has a unique smallest element x, in addition satisfying:

(i) $\psi(x) = \inf\{\psi(z) \colon z \in A\},$
(ii) $\psi(x) + q(x,x_0) - \psi(x_0) = \inf\{\psi(z) + q(z,x_0) - \psi(x_0) \colon z \in A\} \leq 0,$
(iii) $0 < \psi(y) + q(y,x) - \psi(x)$*, for each* $y \in X \setminus \{x\}$*,*
(iv) $q(x,x) = 0.$

Let 2^Y be the family of all subsets of Y. We say that $F \colon X \to 2^Y$ is a (multivalued) mapping if $F(x) \neq \emptyset$, for all $x \in X \neq \emptyset$.

Theorem 3.7 (cp. [14, Theorem 23])**.** *Let (X,p) be a complete partial metric space and let d be as in (1.7). Assume that $\psi : (X,d) \to R$ is a lower semicontinuous mapping bounded below, and $q = p$ for nonnegative p or q is defined by (1.5) for upper bounded $p(\cdot,\cdot)$. Assume $X \subset Y$ and $F \colon X \to 2^Y$ is a mapping satisfying:*

$$\begin{aligned} &\text{for each } x \in X \setminus F(x) \text{ there exists a } y \in X \setminus \{x\} \\ &\qquad\qquad \text{such that } \psi(y) + q(y,x) - \psi(x) \leq 0. \end{aligned} \tag{3.4}$$

Then F has a fixed point.

The next theorem extends the theorems of Caristi [2, Theorem (2.1)'] and Takahashi [4, Theorem 5].

Theorem 3.8 (cp. [14, Theorem 24])**.** *Let (X,p) be a complete partial metric space and let d be as in (1.7). Assume that $\psi : (X,d) \to R$ is a lower semicontinuous mapping bounded below, and $q = p$ for nonnegative p or q is defined by (1.5) for upper bounded $p(\cdot,\cdot)$. Let us adopt $y \preccurlyeq x$ iff $\psi(y) + q(y,x) - \psi(x) \leq 0$, $x, y \in X$, and assume that $X \subset Y$ and $F \colon X \to 2^Y$ is a mapping satisfying:*

$$\text{for each } x \in X \text{ there exists a } y \in F(x) \text{ such that } y \preccurlyeq x. \tag{3.5}$$

Then for any $x_0 \in X$, each maximal chain $A \subset X$ containing x_0 has a unique smallest element x, in addition satisfying conditions (i),…,(iv) of Theorem 3.6 and such that $x \in F(x)$.

The subsequent theorem extends Ekeland's variational principle [3, Theorem 1].

Theorem 3.9 (cp. [14, Theorem 25])**.** *Let (X, p) be a complete partial metric space and let d be as in (1.7). Assume that $\psi: (X, d) \to R$ is a lower semicontinuous mapping bounded below, and $q = p$ for nonnegative p or q is defined by (1.5) for upper bounded $p(\cdot,\cdot)$. Then the following are satisfied:*

(i) for each $x_0 \in X$ there exists an $x \in X$ such that $\psi(x) \le \psi(x_0)$ and $\psi(x) - q(y, x) < \psi(y)$, for each $y \in X \setminus \{x\}$,

(ii) for any $\epsilon > 0$ and each $x_0 \in X$ with $q(x_0, x_0) = 0$ and $\psi(x_0) \le \epsilon + \inf\{\psi(z) : z \in X\}$ there exists an $x \in X$ such that $\psi(x) \le \psi(x_0)$, $q(x, x_0) \le 1$ and $\psi(x) - \epsilon q(y, x) < \psi(y)$, for each $y \in X \setminus \{x\}$.

The assumption of the lower semicontinuity of ψ can be replaced by the assumption of "continuity" of F as shown in the following extension of the Brøndsted theorem to the case of multivalued mappings in partial metric spaces.

Theorem 3.10 (cp. [4, (B.6), p. 33])**.** *Let (X, p) be a complete partial metric space and let d be as in (1.7). Assume that $\psi: X \to R$ is a mapping bounded below, and $q = p$ for nonnegative p or q is defined by (1.5) for upper bounded $p(\cdot,\cdot)$. Let us adopt $y \preccurlyeq x$ iff $\psi(y) + q(y, x) - \psi(x) \le 0$, $x, y \in X$, and assume that mapping $F: X \to 2^X$ has a closed graph (for (X, d) or (X, p)) and satisfies (3.5). Then F has a fixed point.*

Proof. Let $x_0 \in X$ be arbitrary and let $x_1 \in F(x_0)$ satisfy $x_1 \preccurlyeq x_0$. If x_n is defined, then $x_{n+1} \in F(x_n)$ is such that $x_{n+1} \preccurlyeq x_n$. From

$$\psi(x_n) + q(x_n, x_m) - \psi(x_m) \le \psi(x_n) + q(x_n, x_{n-1}) - \psi(x_{n-1})$$
$$+\psi(x_{n-1}) + q(x_{n-1}, x_{n-2}) - \psi(x_{n-2}) + \cdots + +\psi(x_{m+1}) + q(x_{m+1}, x_m) - \psi(x_m) \le 0$$

for $m < n$ it follows that

$$0 \le q(x_n, x_m) \le \psi(x_m) - \psi(x_n).$$

Therefore, $(\psi(x_n))_{n\in N}$ is nonincreasing, i.e. convergent, as ψ is bounded below. Now it is clear that $(x_n)_{n\in N}$ is a Cauchy sequence in (X, q) (see Definition 3.1). In view of Lemma 3.2 $(x_n)_{n\in N}$ is a Cauchy sequence in (X, d), which is complete (Proposition 1.6). The graph of F is closed for (X, d) (see Remark 1.2) and $\lim_{n\to\infty} d(x_n, F(x_n)) \le \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ means that $x = \lim_{n\to\infty} x_n \in F(x)$. □

Clearly, in the present paper one can use λq for a $\lambda > 0$ in place of q, and a metric equivalent to d.

Now we are ready to present a Banach type fixed point theorem in a variational version.

Theorem 3.11. *Let (X, p) be a complete partial metric space and let d be as in (1.7). Assume that p is bounded below with upper bounded $p(\cdot,\cdot)$ and $f: (X, d) \to (X, d)$ is a continuous mapping which for a $c \in [0, 1)$ satisfies:*

$$q(f^2(x), f(x)) \le cq(f(x), x),\ x \in X, \tag{3.6}$$

where $q(y,x) = p(y,x) - p(y,y)$, $x, y \in X$. *For* $\psi(\cdot) = q(f(\cdot),\cdot)$ *let us adopt* $y \preccurlyeq x$ *iff* $\psi(y) + (1-c)q(y,x) - \psi(x) \leq 0$, $x, y \in X$. *Then for any* $x_0 \in X$, *each maximal chain* $A \subset X$ *containing* x_0 *has a unique smallest element* x *such that* $f(x) = x$ *and the following conditions are satisfied:*

(i) $(1-c)q(x,x_0) - q(f(x_0),x_0) = \inf\{q(f(z),z) + (1-c)q(z,x_0)$
$-q(f(x_0),x_0)\colon z \in A\} \leq 0$,

(ii) $0 < q(f(y),y) + (1-c)q(y,x)$, *for each* $y \in X \setminus \{x\}$.

Proof. Condition (3.6) can be rewritten in the form

$$\begin{aligned} &q(f^2(x),f(x)) + (1-c)q(f(x),x) - q(f(x),x) \\ &= \psi(f(x)) + (1-c)q(f(x),x) - \psi(x) \leq 0,\ x \in X. \end{aligned}$$

Now we can see (Lemma 1.11) that Theorem 3.8 applies. Clearly, for $f(x) = x$ we have $\psi(x) = q(f(x),x) = q(x,x) = 0$ and conditions (i),(iv) of Theorem 3.6 can be disregarded (see (1.2)). □

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