Fixed Point Theory, 17(2016), No. 2, 449-456
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A COINCIDENCE AND FIXED POINT THEOREMS FOR SEMI-QUASI CONTRACTIONS

RAJENDRA PANT*, S.L. SINGH** AND S.N. MISHRA***

*Department of Mathematics, Visvesvaraya National Institute of Technology
Nagpur 440010, India
E-mail: pant.rajendra@gmail.com

** Vedic MRI, #21 Govind Nagar, Rishikesh 249201, India
E-mail: vedicmri@gmail.com

***Department of Mathematics, Walter Sisulu University
Mthatha 5117, South Africa
E-mail: swaminathmishra3@gmail.com

Abstract. In this paper, we introduce the notion of semi-quasi contraction and obtain coincidence and common fixed point theorems for such contractions without using any kind of continuity of mappings. Besides addressing an open question, our results extend and generalize some well known fixed point theorems including Ćirić's quasi contraction theorem.
Key Words and Phrases: Fixed point, quasi-contraction, semi-quasi contraction.
2010 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction and preliminaries

Throughout this paper Y denotes an arbitrary non-empty set, X a metric space (X, d) and $\mathbb{N}$ the set of all positive integers. In 1974, Ćirić [1] introduced the following notion of *quasi-contraction* and obtained a remarkable generalization of the classical Banach contraction principle (BCP):

Definition 1.1. A self-mapping T of a metric space X is a *quasi-contraction* if there exists a number $r \in [0, 1)$ such that

$$d(Tx, Ty) \le r\ M(x, y) \tag{1.1}$$

for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Theorem 1.2. [1]. *A quasi-contraction on a complete metric space has a unique fixed point.*

We remark that a quasi-contraction for a self-mapping on a metric space is considered as the most general among contractions listed by Rhoades [15].
Following Jungck [7] and Ćirić [1], Ranganathan [14] and Das and Naik [2] independently obtained the following extension of Theorem 1.2.

Theorem 1.3. *Let X be a complete metric space and $S, T : X \to X$ a pair of mappings satisfying the following:* **(a)** *S is continuous;* **(b)** *$T(X) \subseteq S(X)$;* **(c)** *S and T commute on X. Assume that there exists $r \in [0,1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq r\ M_{S,T}(x, y),$$

where $M_{S,T}(x,y) = \max\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\}$. Then S and T have a unique common fixed point.

Notice that Theorem 1.2 is Theorem 1.3 when S is the identity mapping on X. On the other hand, Suzuki [19] obtained the following forceful generalization of the BCP.

Theorem 1.4. *Let X be a complete metric space and T a mapping on X. Define a nondecreasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \leq r \leq 2^{-\frac{1}{2}}, \\ (1+r)^{-1} & \text{if } 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r) d(x, Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq r\ d(x,y) \text{ for all } x, y \in X.$$

Then T has a unique fixed point in X.

Using variants of $\theta(r)$, a number of extensions and generalizations of Theorem 1.4 have appeared in [3,4,9,13,17,18] and elsewhere. Popescu [11] obtained the following generalization of a fixed point theorem of Jungck [7], Theorem 1.4 and a result of Kikkawa and Suzuki [9, Th. 3].

Theorem 1.5. *Let X be a complete metric space and $S, T : X \to X$ a pair of mappings satisfying the conditions* **(a)**, **(b)** *and* **(c)**. *Assume that there exists $r \in [0,1)$ such that for all $x, y \in X$,*

$$\theta(r) d(Sx, Tx) \leq d(Sx, Sy) \text{ implies } d(Tx,Ty) \leq r\ N_{S,T}(x,y),$$

where $N_{S,T}(x,y) = \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \dfrac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}$.
Then S and T have a unique common fixed point.

We remark that a mapping $T : X \to X$ satisfying the condition

$$d(Tx,Ty) \leq r\ N_{S,T}(x,y),\ x, y \in X,$$

also satisfies the condition

$$d(Tx,Ty) \leq r\ M_{S,T}(x,y),\ x, y \in X,$$

but the reverse implication is not true.
Now a natural question arises whether it is possible to drop the continuity requirement on the mapping S and replace the condition $N_{S,T}(x,y)$ by $M_{S,T}(x,y)$ in Theorem 1.5. Indeed, this question has implicitly been open for about five years.
In this paper, using a different value of $\theta(r)$, we answer this question affirmatively and also relax the commutativity requirement considerably. Our results complement, extend and generalize a number of fixed point results including Theorems 1.2 - 1.5 and an important coincidence theorem due to Goebel [5].

2. Semi-quasi contractions

Definition 2.1. A pair of mappings $S, T : Y \to X$ with values in a metric space X will be called a *semi-quasi contraction* if there exists a number $r \in [0, 1)$ such that

$$(1-r)d(Sx, Tx) \leq d(Sx, Sy) \text{ implies } d(Tx, Ty) \leq r\ M_{S,T}(x, y) \qquad \text{(CS)}$$

for all $x, y \in Y$. It will be called a semi-quasi contraction for a self-mapping T of X when $Y = X$ and S is the identity mapping on X.

Following Itoh and Takahashi [6] (see also [16]), we have the following definition of (IT)-commuting for a pair of mappings.

Definition 2.2. A pair of mappings $S, T : X \to X$ is (IT)-commuting (also called weakly compatible by Jungck and Rhoades [8]) at $u \in X$ if $STu = TSu$ such that $Su = Tu$.

Now onwards, $O(y_k; n)$ denotes the set of points $\{y_k, y_{k+1}, ..., y_{k+n}\}$ and $\delta[O(y_k; n)]$ the diameter of $O(y_k; n)$. Notice that if $\delta[O(y_k; n)] > 0$ for $k, n \in \mathbb{N}$ then

$$\delta[O(y_k; n)] = d(y_k, y_j); \text{ where } k < j \leq k + n.$$

The main result of this paper is prefaced by the following lemmas. Lemmas 2.3 and 2.4 are modeled on the pattern of Ćirić [1].

Lemma 2.3. *Let X be a metric space and $S, T : Y \to X$ a semi-quasi contraction for a pair of mappings. Suppose that $T(Y) \subseteq S(Y)$ and $\delta[O(y_n; 1)] > 0$ for $n \in \mathbb{N} \cup \{0\}$. Then*

$$\delta[O(y_n; 1)] \leq r^n \delta[O(y_0; n + 1)],$$

where $y_n = Tx_n = Sx_{n+1}$ for $n \in \mathbb{N} \cup \{0\}$.

Proof. Let $x_0 \in Y$ be arbitrary. We define sequences $\{x_n\}$ in Y and $\{y_n\}$ in X by

$$y_n = Tx_n = Sx_{n+1} \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Since for $n \in \mathbb{N}$,

$$(1-r)d(Sx_n, Tx_n) \leq d(Sx_n, Tx_n) = d(Sx_n, Sx_{n+1}) = d(y_{n-1}, y_n),$$

by (CS), we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Tx_n, Tx_{n+1}) \\ &\leq r \max\{d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), \\ &\qquad d(Sx_n, Tx_{n+1}), d(Sx_{n+1}, Tx_n)\} \\ &= r \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1})\}. \end{aligned}$$

Thus for $n \in \mathbb{N}$,

$$d(y_n, y_{n+1}) \leq r\ \delta[O(y_{n-1}; 2)]$$

and evidently

$$\delta[O(y_n; 1)] \leq r\ \delta[O(y_{n-1}; 2)].$$

Inductively,

$$d(y_n, y_{n+1}) = \delta[O(y_n; 1)] \leq r\ \delta[O(y_{n-1}; 2)] ... \leq r^n \delta[O(y_0; n + 1)].$$ □

Lemma 2.4. *Under the hypotheses of Lemma* 2.3,

$$\delta[O(y_0; n+1)] \leq \frac{1}{(1-r)} d(y_0, y_1).$$

Proof. It is obvious that $\delta[O(y_0; n+1)] = d(y_0, y_k)$ for some positive integer $k \leq n+1$. Now

$$\begin{aligned}\delta[O(y_0; n+1)] &= d(y_0, y_k) \leq d(y_0, y_1) + d(y_1, y_k)\\ &= d(y_0, y_1) + d(Tx_1, Tx_k)\\ &= d(y_0, y_1) + \delta[O(y_1; n)]\\ &\leq d(y_0, y_1) + r\delta[O(y_0; n+1)].\end{aligned}$$

Therefore

$$\delta[O(y_0; n+1)] \leq \frac{1}{(1-r)} d(y_0, y_1).$$

□

Lemma 2.5. *Under the hypothesis of Lemma* 2.3,

$$d(Tx_n, Tx_{n+1}) \leq \frac{r}{1-r} d(Sx_n, Tx_n).$$

Proof. Pick $x_0 \in Y$. Define sequences $\{x_n\}$ and $\{y_n\}$ as in Lemma 2.3.
Since for $n \in \mathbb{N} \cup \{0\}$,

$$(1-r)d(Sx_n, Tx_n) \leq d(Sx_n, Tx_n) = d(Sx_n, Sx_{n+1}),$$

by (CS), we have

$$\begin{aligned}d(Tx_n, Tx_{n+1}) &\leq r \max\{d(Sx_n, Sx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}),\\ &\qquad d(Sx_n, Tx_{n+1}), d(Sx_{n+1}, Tx_n)\}\\ &= r \max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2}), d(Sx_n, Sx_{n+2})\}\\ &\leq r \max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2}), d(Sx_n, Sx_{n+1})\\ &\qquad + d(Sx_{n+1}, Sx_{n+2})\}\\ &= \frac{r}{1-r} d(Sx_n, Tx_n).\end{aligned}$$

□

Theorem 2.6. *Let $S, T : Y \to X$ be a semi-quasi contraction for a pair of mappings such that $T(Y) \subseteq S(Y)$. If $S(Y)$ or $T(Y)$ is a complete subspace of the metric space X then*

(A) *S and T have a coincidence point $u \in Y$.*

Further,

(B) *S and T have a unique fixed point provided that $Y = X$ and the mappings S and T are* (IT)*-commuting at u.*

Proof. Pick $x_0 \in Y$. Define sequences $\{x_n\}$ and $\{y_n\}$ as in Lemma 2.3.
If for some $n \in \mathbb{N} \cup \{0\}$, $\delta[O(y_n; 1)] = 0$, we have

$$y_n = y_{n+1}, \text{ i.e., } Sx_n = Tx_n \text{ and } x_n \text{ is a coincidence point of } S \text{ and } T.$$

If $\delta[O(x_n;1)] > 0$ then by Lemmas 2.3 and 2.4, we have

$$d(y_n, y_{n+1}) \leq \frac{r^n}{1-r} d(y_0, y_1).$$

Since $r < 1$, the sequence $\{y_n\}$ is Cauchy. Suppose $S(Y)$ is a complete subspace of X. Then $\{y_n\}$ being contained in $S(Y)$ has a limit in $S(Y)$. Call it z. Let $u \in S^{-1}z$. Then $Su = z$. We claim that

$$(1-r)d(Sx_n, Tx_n) \leq d(Sx_n, Su).$$

Suppose that for some $n \in \mathbb{N} \cup \{0\}$,

$$d(Sx_n, Su) < (1-r)d(Sx_n, Tx_n) \text{ and } d(Tx_n, Su) < (1-r)d(Tx_n, Tx_{n+1}).$$

Then

$$d(Sx_n, Tx_n) < d(Sx_n, Su) + d(Tx_n, Su) < (1-r)d(Sx_n, Tx_n) + (1-r)d(Tx_n, Tx_{n+1}).$$

Using Lemma 2.5,

$$\begin{aligned} d(Sx_n, Tx_n) &< (1-r)\left[d(Sx_n, Tx_n) + \frac{r}{1-r} d(Sx_n, Tx_n)\right] \\ &= (1-r)\left[1 + \frac{r}{1-r}\right] d(Sx_n, Tx_n) = d(Sx_n, Tx_n). \end{aligned}$$

a contradiction. Therefore $(1-r)d(Sx_n, Tx_n) \leq d(Sx_n, Su)$ holds for every $n \in \mathbb{N} \cup \{0\}$.
Now by (CS),

$$d(Tx_n, Tu) \leq r \ \max\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), d(Sx_n, Tu), d(Su, Tx_n)\}.$$

Making $n \to \infty$,

$$d(Su, Tu) \leq r \ d(Su, Tu),$$

and $Su = Tu$.
Further, if $Y = X$ and the mappings S and T are (IT)-commuting at u then $STu = TSu = SSu = TTu$. Again since

$$(1-r)d(STu, Tu) \leq d(STu, Tu) = d(STu, Su),$$

by (CS),

$$\begin{aligned} d(TTu, Tu) &\leq r \max\{d(STu, Su), d(STu, TTu), d(Su, Tu), d(STu, Tu), d(Su, TSu)\} \\ &= rd(TTu, Tu), \end{aligned}$$

a contradiction, and $Tu = z$ is a common fixed point of S and T.

In case $T(Y)$ is a complete subspace of X, the sequence $\{y_n\}$ converges in $S(Y)$ since $T(Y) \subseteq S(Y)$. So the previous argument works. The unicity of the common fixed point follows easily. □

Corollary 2.7. *Let X be a complete metric space and $T : X \to X$ a semi-quasi contraction. Then T has a unique fixed point in X.*

Proof. It comes from Theorem 2.6 when $Y = X$ and S is the identity mapping on X. □

3. Examples

In this section, we present a number of examples to illustrate the generality of our results.

Example 3.1. Let $Y = \{a, b, c, d\}$ and $X = \{1, 2, 3, 4\}$ be endowed with the metric d on X defined by $d(1, 2) = d(1, 3) = 1$, $d(1, 4) = \frac{3}{2}$, $d(2, 3) = d(2, 4) = d(3, 4) = 2$. Let $S, T : Y \to X$ be given by $Ta = Td = 1$, $Tb = Tc = 4$ and $Sa = 1$, $Sb = Sc = 3$, $Sd = 4$.
Notice that $d(Ta, Tb) = \frac{3}{2} > 1 = d(Sa, Sb)$ and Goebel's condition [5]:

$$d(Tx, Ty) \le rd(Sx, Sy) \text{ for all } x, y \in Y$$

is not satisfied. However, $T(Y) \subseteq S(Y)$ and

$$(1 - r)d(Sx, Tx) \le d(Sx, Sy) \le 2 \text{ implies}$$

$$d(Tx, Ty) \le \frac{3}{2} \le r \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$$

with $1 > r \ge \frac{3}{4}$. Therefore all the hypotheses of Theorem 2.6 hold and $Sa = Ta$.

Example 3.2. Let (X, d) be as in Example 3.1 and $S, T : X \to X$ such that $T1 = T4 = 1$, $T2 = T3 = 4$ and $S1 = 1$, $S2 = S3 = 3$, $S4 = 4$, then

1. $\theta(r)d(1, T1) = 0 \le d(T1, T2)$ and $d(T1, T2) = \frac{3}{2} > 1 = d(1, 2)$. So T does not satisfy the assumptions of Theorem 1.4. Further,
2. $\theta(r)d(S1, T1) = 0 \le d(S1, S2)$ and $d(T1, T2) = \frac{3}{2} > 1 = d(S1, S2)$. Therefore S and T do not satisfy the assumptions of Kikkawa and Suzuki [9, Th. 3].

Evidently, all the hypotheses of Theorem 2.6 hold and $S1 = T1 = 1$.

Example 3.3. Let (X, d) be as in Example 3.1. Define $T : X \to X$ by $T1 = T2 = T4 = 1$, $T3 = 4$.
Then for all $x, y \in X$,

$$(1 - r)d(x, Tx) \le d(x, y) \le 2 \text{ implies}$$

$$d(Tx, Ty) \le \frac{3}{2} \le r \ \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

with $r \ \ge \frac{3}{4}$. Therefore Corollary 2.7 applies.
On the other hand, $\theta(r)d(1, T1) = 0 \le 1 = d(1, 3)$ and $d(T1, T3) = \frac{3}{2} > rd(1, 3)$ for every $r \in [0, 1)$. So, the mapping T does not satisfy the assumptions of Theorem 1.4.

Example 3.4. In [19, Example 2], it is shown that T does not satisfy the assumption of Theorem 1.4. However, the same T satisfies all the conditions of Corollary 2.7.

The following example shows that Corollary 2.7 is more general than Theorem 1.2.

Example 3.5. [19, Example 1]. Let $X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$ be endowed with the metric d defined by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Let T be a mapping on X defined by

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & \text{if } x_1 \le x_2, \\ (0, x_2) & \text{if } x_1 > x_2. \end{cases}$$

We note that

$$d(Tx,Ty) \leq (4/5)M(x,y) \text{ if } (x,y) \neq ((4,5),(5,4)) \text{ and } (y,x) \neq ((4,5),(5,4)).$$

Since

$$(1-r)d((4,5),T(4,5)) > \frac{5}{2} > 2 = d((4,5),(5,4))$$

and

$$(1-r)d((5,4),T(5,4)) > d((5,4),(4,5)) \text{ for every } r \in [0,1).$$

Therefore Corollary 2.7 holds. But, since

$$d((T(5,4),T(4,5)) = 8 \geq M((5,4),(4,5)),$$

the mapping T does not satisfy the condition (1.1) of Theorem 1.2.

Finally, we present an example showing the superiority of Theorem 2.6 over Theorems 1.2-1.5.

Example 3.6. Let $Y = X = \{(0,0),(3,3),(4,0),(0,4),(4,5),(5,4)\}$ be a metric space endowed with the metric d defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1-y_1| + |x_2-y_2|.$$

Let $S,T : X \to X$ defined by

$$T(x_1,x_2) = \begin{cases} (0,4) & \text{if } x_1 = x_2 = 3, \\ (x_1,0) & \text{if } x_1 \leq x_2, \\ (0,x_2) & \text{if } x_1 > x_2. \end{cases} \quad \text{and } S(x_1,x_2) = (x_1,x_2).$$

We note that $d(Tx,Ty) = (4/5)M_{S,T}(x,y)$ if

$$(x,y) \neq ((4,5),(5,4)),((4,5),(3,3))$$

and

$$(y,x) \neq ((4,5),(5,4)),((4,5),(3,3)).$$

Since, in these cases

$$(1-r)d(Sx,Tx) > d(Sx,Sy) \text{ for every } r \in [0,1),$$

T satisfies the assumption of Theorem 2.6. Notice that

1. $d((T(5,4),T(4,5)) = 8 \geq M((5,4),(4,5))$, and Theorem 1.2 does not apply.
2. $d((T(5,4),T(4,5)) = 8 \geq M_{S,T}((5,4),(4,5))$, and Theorem 1.3 does not apply.
3. For every $r \in [0,1)$,

$$\theta(r)d((4,0),T(4,0)) = 4 \leq d((4,0),(3,3))$$

and

$$d(T(4,0),T(3,3)) = 4 > r\ d((4,0),(3,3)).$$

So, the mapping T does not satisfy the assumption of Theorem 1.4.

4. For every $r \in [0,1)$,

$$\theta(r)d((3,3),T(0,4)) = 4 \leq d((3,3),(0,4))$$

and

$$d(T(3,3),T(0,4)) = 4 > r\ N_{S,T}((3,3),(0,4))).$$

So the mapping T does not satisfy the assumption of Theorem 1.5.

Question. Whether the condition (CS) of Theorem 1.4 can be replaced by the following:

$$\theta(r)d(Sx,Tx) \leq d(Sx,Sy) \text{ implies } d(Tx,Ty) \leq rM_{S,T}(x,y).$$

References

[1] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45**(1974), 267-273.
[2] K.M. Das, K.V. Naik, *Common fixed point theorems for commuting maps on a metric space*, Proc. Amer. Math. Soc. **77**(1979), 369-373.
[3] S. Dhompongsa, H. Yingtaweesittikul, *Fixed points for multivalued mappings and metric completeness*, Fixed Point Theory and Appl., (2009), Art. ID 972395, 15 pp.
[4] D. Doric, R. Lazovic, *Some Suzuki-type fixed point theorems for generalized multivalued mappings and applications*, Fixed Point Theory Appl. 2011, 2011:40, 8 pp.
[5] K. Goebel, *A coincidence theorem*, Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys., **16**(1968), 733-735.
[6] S. Itoh, W. Takahashi, *Single valued mappings, mutivalued mappings and fixed point theorems,* J. Math. Anal. Appl., **59**(1977), 514-521.
[7] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly., **83**(1976), 261-263.
[8] G. Jungck, B.E. Rhoades, *Fixed points for set-valued functions without continuity*, Indian J. Pure Appl. Math., **29**(3)(1988), 227-238.
[9] M. Kikkawa, T. Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*, Nonlinear Anal., **69**(2008), 2942-2949.
[10] M. Kikkawa, T. Suzuki, *Some similarity between contractions and Kanannan mappings*, Fixed Point Theory and Appl., (2008), Art. ID 649749, 8 pp.
[11] O. Popescu, *Two fixed point theorems for generalized contractions with contants in complete metric space*, Cent. Eur. J. Math., **7**(3)(2009), 529-538.
[12] O. Popescu, *Two generalizations of some fixed point theorems*, Comput. Math. Appl., **62**(2011), no. 10, 3912–3919.
[13] G. Moţ, A. Petruşel, *Fixed point theory for a new type of contrative multivalued operators*, Nonlinear Anal., **70**(2009), 3371-3377.
[14] S. Ranganathan, *A fixed point theorem for commuting mappings*, Math. Sem. Notes Kobe Univ., **6**(1978), 351-357.
[15] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257–290.
[16] S.L. Singh, S.N. Mishra, *Coincidence and fixed points of nonself hybrid contractions*, J. Math. Anal. Appl., **256**(2001), 486-497.
[17] S.L. Singh, H.K. Pathak, S.N. Mishra, *On a Suzuki type general fixed point theorem with applications*, Fixed Point Theory and Appl., (2010), Art. ID 234717, 15 pp.
[18] S.L. Singh, S.N. Mishra, Renu Chugh, Raj Kamal, *General Common Fixed Point Theorems and Applications*, J. Appl. Math., (2012), Art. ID 902312, 14 pp.
[19] T. Suzuki, *A generalized Banach contration principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **136**(2007), no. 5, 1861-1869.

Received: November 25, 2013; Accepted: March 2, 2014.