Fixed Point Theory, 17(2016), No. 2, 427-434
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

THE CONTINUOUS DEPENDENCE OF THE FIXED POINTS FOR NONEXPANSIVE AND QUASI-NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACE

M.O. OLATINWO

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
E-mail: memudu.olatinwo@gmail.com/molaposi@yahoo.com/polatinwo@oauife.edu.ng

Abstract. Recently, in the papers [14] and [15], the author established some results pertaining to the continuous dependence of the fixed points in normed space and Banach space settings for some iterative processes by using general contractive conditions. Both papers were devoted to the partial fulfillment of the open question in [3], that is, "apart from the Picard iteration, the continuous dependence of the fixed points has not been studied so far for other fixed point iteration procedures." In the present paper, our purpose is to further investigate the continuous dependence of the fixed points in uniformly convex Banach space for nonexpansive and quasi-nonexpansive mappings. Our results extend some recently announced ones in the current literature.
Key Words and Phrases: Normed space, Banach space, fixed point, continuous dependence, uniformly convex Banach space.
2010 Mathematics Subject Classification: 47H06, 47H10.

1. Introduction

Let (E, d) be a complete metric space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{p \in E \mid Tp = p\}$ is the set of fixed points of T.
There are several iterative processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \ n = 0, 1, \dots, \tag{1.1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality relation

$$d(Tx, Ty) \leq \alpha d(x, y), \ \forall \ x, \ y \in E \text{ and } \alpha \in [0, 1). \tag{1.2}$$

Condition (1.2) is called the *Banach's contraction condition.*

In the following, in a normed linear space or a Banach space setting, we state some of the iterative processes for which (1.1) is a special case:
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \ \lambda \in [0, 1], \ n = 0, 1, \dots, \tag{1.3}$$

is called the Schaefer's iteration process (see [21]).

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\ n = 0, 1, \dots, \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$, is called the Mann iteration process (see [11]).
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i\ T^i x_n,\ n = 0, 1, 2, \dots,\ \sum_{i=0}^{k} \alpha_i = 1, \tag{1.5}$$

$\alpha_i \geq 0,\ \alpha_0 \neq 0,\ \alpha_i \in [0, 1]$, where k is a fixed integer is called the Kirk iterative process (see [9]).
For $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,\ i}\ T^i x_n,\ n = 0, 1, 2, \dots,\ \sum_{i=0}^{k} \alpha_{n,\ i} = 1, \tag{1.6}$$

where $\alpha_{n,\ i} \geq 0,\ \alpha_{n,\ 0} \neq 0,\ \alpha_{n,\ i} \in [0, 1]$ and k being a fixed integer. This iterative algorithm was introduced in Olatinwo [12].

In many applications, the operator T in the Picard iteration of (1.1) depends on an additional parameter $\lambda \in Y$, where Y is a parameter space. Therefore, (1.1) is replaced by the equation

$$x_\lambda = S_\lambda\ x_\lambda,\ x_\lambda \in E,\ \lambda \in Y. \tag{1.7}$$

Condition (1.2) was employed in Zeidler [22] to prove a result on the stability of the fixed points (that is, continuous dependence of the fixed points on a parameter) for the Picard iteration. In Rus [16] and also Berinde [3], the continuous dependence of the fixed points on a parameter has been well formulated in the following general context in a metric space: Let $(E,\ d)$ be a complete metric space, (Y, τ) a topological space and $S_\lambda : E \times Y \to E$ a family of operators depending on the parameter $\lambda \in Y$. Suppose that $S_\lambda := S(.,\ \lambda),\ \lambda \in Y$, has a unique fixed point x_λ^*, for any $\lambda \in Y$.
Define the operator $U : Y \to E$ by

$$U(\lambda) = x_\lambda^*,\ \forall\ \lambda \in Y.$$

We are interested in finding sufficient conditions on S_λ that guarantee the continuity of U. In Olatinwo [13], the concept of the continuous dependence of the fixed points on a parameter has been studied for the Schaefer and Mann iterative processes in normed linear space. Also, in [14], the concept was studied for Kirk and Kirk-type iterative algorithms in Banach space.

Our purpose is to further investigate the continuous dependence of the fixed points in uniformly convex Banach space for nonexpansive and quasi-nonexpansive mappings by using Mann, Kirk and Kirk-Mann iterative processes. Our results extend some recently announced ones in the current literature. In particular, see [3, 2, 16, 17, 19, 20, 22] as well as the recent results of the author [13, 14]. For abstract view point concerning iterative approximation of fixed points, one can consult Rus [18].
We shall establish our results using the following contractive conditions:

Definition 2.1. For a continuous mapping $S : E \times Y \to E$, there exist a real number $a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and $\forall$ x, y $\in$ E, we have

$$\|S_\lambda x - S_\lambda y\| \leq \varphi(\|x - S_\lambda x\|) + a\|x - y\|, \ \lambda \in Y. \tag{1.8}$$

In addition to condition (1.8), we investigate the continuous dependence of fixed points for nonexpansive mappings:
Definition 2.2. A mapping $S : E \times Y \to E$ is said to be *nonexpansive* if

$$\|S_\lambda x - S_\lambda y\| \leq \|x - y\|, \ \forall\ x,\ y \in E,\ \lambda \in Y. \tag{1.9}$$

Lemma 2.1. (Olatinwo [14]) Let $(E, \ ||\cdot||)$ be a Banach space and let $S_\lambda : E\times Y \to E$ be a mapping satisfying (1.8), where $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a sublinear, monotone increasing function such that $\varphi(0) = 0$. Then, $\forall\ i \in \mathbb{N}$, and $\forall\ x,\ y \in E,\ \lambda \in Y$, we have

$$||S^i_\lambda x - S^i_\lambda y|| \leq \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j(||x - S_\lambda x||) + a^i||x - y||. \tag{1.10}$$

Lemma 2.2. (Groetsch [7]) Let X be a uniformly convex Banach space and let $x,\ y \in X$. If $||x|| \leq 1,\ ||y|| \leq 1$ and $||x - y|| \geq \epsilon > 0$, then

$$||\lambda x + (1 - \lambda)y|| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon) \text{ for } 0 \leq \lambda < 1.$$

2. Main results

Theorem 2.1. *Let $(E, ||.||)$ be a uniformly convex Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a continuous mapping satisfying (1.8). Suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear monotone increasing function such that $\varphi(0) = 0$. Let x^*_λ be the unique fixed point of S_λ (* where $S_\lambda x = S(x, \lambda)$, x $\in$ E, $\lambda \in$ Y). *Suppose $\{x_n\}_{n=0}^{\infty}$ is the Kirk-Mann iterative process defined by (1.6) with $\sum_{i=0}^{k} \alpha_{n,\ i} = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous.*
Proof. Let $\lambda_1,\ \lambda_2 \in Y$. Then, we shall apply Lemma 2.1 and the triangle inequality in the sequel. Let $S^0 = I$ (identity mapping). Then, $I(x_\lambda, \lambda) = I_\lambda x_\lambda = x_\lambda$. Let $\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \neq 0$,

$$u(\lambda_1, \lambda_2) = \frac{x^*_{\lambda_1} - x^*_{\lambda_2}}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|},\ v(\lambda_1, \lambda_2) = \frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i}(S^i(x^*_{\lambda_1}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_1))}{(1 - \alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}.$$

Then, we have

$$\|u(\lambda_1, \lambda_2)\| = \| \left(\frac{x^*_{\lambda_1} - x^*_{\lambda_2}}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \right) \| \leq \frac{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} = 1,$$

and by Lemma 2.1 we have that

$$\begin{aligned}
\|v(\lambda_1,\lambda_2)\| &= \| \left(\frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} (S^i(x^*_{\lambda_1},\lambda_1) - S^i(x^*_{\lambda_2},\lambda_1))}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \right) \| \\
&\leq \frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} \|S^i(x^*_{\lambda_1},\lambda_1) - S^i(x^*_{\lambda_2},\lambda_1)\|}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \\
&= \frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} \|S^i_{\lambda_1} x^*_{\lambda_1} - S^i_{\lambda_1} x^*_{\lambda_2}\|}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \\
&\leq \frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} \left\{ \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j (\|x^*_{\lambda_1} - S_{\lambda_1} x^*_{\lambda_1}\|) + a^i \|x^*_{\lambda_1} - x^*_{\lambda_2}\| \right\}}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \\
&= \frac{\left(\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} a^i\right)\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} = \frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} a^i}{1-\alpha_{\lambda_1,\ 0}} \leq \frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i}}{1-\alpha_{\lambda_1,\ 0}} = 1,
\end{aligned}$$

since

$$\begin{aligned}
\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} a^i &= \alpha_{\lambda_1,\ 1} a + \alpha_{\lambda_1,\ 2} a^2 + \ldots + \alpha_{\lambda_1,\ k} a^k \\
&\leq \alpha_{\lambda_1,\ 1} + \alpha_{\lambda_1,\ 2} + \ldots + \alpha_{\lambda_1,\ k} = \sum_{i=1}^{k} \alpha_{\lambda_1,\ i},
\end{aligned}$$

where $\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} = 1 - \alpha_{\lambda_1,\ 0},\ a \in [0,1)$.

$$\begin{aligned}
\|x^*_{\lambda_1} - x^*_{\lambda_2}\| &= \| \sum_{i=0}^{k} \alpha_{\lambda_1,\ i} S^i(x^*_{\lambda_1},\lambda_1) - \sum_{i=0}^{k} \alpha_{\lambda_2,\ i} S^i(x^*_{\lambda_2},\lambda_2) \| \\
&\leq \|\alpha_{\lambda_1,\ 0}(x^*_{\lambda_1} - x^*_{\lambda_2}) + \sum_{i=1}^{k} \alpha_{\lambda_1,\ i} \left(S^i(x^*_{\lambda_1},\lambda_1) - S^i(x^*_{\lambda_2},\lambda_1)\right) \| \\
&+ \| \sum_{i=0}^{k} \alpha_{\lambda_1,\ i} S^i(x^*_{\lambda_2},\lambda_1) - \sum_{i=0}^{k} \alpha_{\lambda_2,\ i} S^i(x^*_{\lambda_2},\lambda_2) \| \\
&\leq \|x^*_{\lambda_1} - x^*_{\lambda_2}\| \ \|\alpha_{\lambda_1,\ 0} \left(\frac{x^*_{\lambda_1} - x^*_{\lambda_2}}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \right) \\
&+ (1-\alpha_{\lambda_1,\ 0}) \frac{\sum_{i=1}^{k} \alpha_{\lambda_1,\ i} \left(S^i(x^*_{\lambda_1},\lambda_1) - S^i(x^*_{\lambda_2},\lambda_1)\right)}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \| \\
&+ \| \sum_{i=0}^{k} \alpha_{\lambda_1,\ i} S^i_{\lambda_1} x^*_{\lambda_2} - \sum_{i=0}^{k} \alpha_{\lambda_2,\ i} S^i_{\lambda_2} x^*_{\lambda_2} \| \\
&= \|x^*_{\lambda_1} - x^*_{\lambda_2}\| \ \|\alpha_{\lambda_1,\ 0} u(\lambda_1,\lambda_2) + (1-\alpha_{\lambda_1,\ 0}) v(\lambda_1,\lambda_2)\| \\
&+ \| \sum_{i=0}^{k} \alpha_{\lambda_1,\ i} S^i_{\lambda_1} x^*_{\lambda_2} - \sum_{i=0}^{k} \alpha_{\lambda_2,\ i} S^i_{\lambda_2} x^*_{\lambda_2} \|. \qquad (2.1)
\end{aligned}$$

Using Lemma 2.2 in (2.1) yields

$$\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \leq \left[1 - 2\alpha_{\lambda_1,\ 0}(1-\alpha_{\lambda_1,\ 0})\delta(\epsilon)\right] \|x^*_{\lambda_1} - x^*_{\lambda_2}\| + \|\sum_{i=0}^{k}\alpha_{\lambda_1,\ i}S^i_{\lambda_1}x^*_{\lambda_2} - \sum_{i=0}^{k}\alpha_{\lambda_2,\ i}S^i_{\lambda_2}x^*_{\lambda_2}\|, \tag{2.2}$$

from which it follows from (2.2) that

$$\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \leq \frac{\|\sum_{i=0}^{k}\alpha_{\lambda_1,\ i}S^i_{\lambda_1}x^*_{\lambda_2} - \sum_{i=0}^{k}\alpha_{\lambda_2,\ i}S^i_{\lambda_2}x^*_{\lambda_2}\|}{2\alpha_{\lambda_1,\ 0}(1-\alpha_{\lambda_1,\ 0})\delta(\epsilon)} \to 0 \text{ as } \lambda_2 \to \lambda_1,$$

which implies that, $||x^*_{\lambda_1} - x^*_{\lambda_2}|| \to 0$ as $\lambda_2 \to \lambda_1$.
That is, $||U(\lambda_1) - U(\lambda_2)|| \to 0$ as $\lambda_2 \to \lambda_1$.
Hence, the mapping $U : Y \to E$, defined by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous.

Theorem 2.2. *Let $(E, ||.||)$ be a uniformly convex Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a (continuous) nonexpansive mapping. Let x^*_λ be a fixed point of S_λ* (where $\mathrm{S_\lambda x = S(x,\lambda),\ x \in E,\ \lambda \in Y}$). *Suppose $\{x_n\}_{n=0}^{\infty}$ is the Kirk-Mann iterative process defined by (1.6) with $\sum_{i=0}^{k}\alpha_{n,\ i} = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous.*

Proof. Let $\lambda_1,\ \lambda_2 \in Y$. Let $S^0 = I$ (identity mapping). Then, $I(x_\lambda, \lambda) = I_\lambda x_\lambda = x_\lambda$. Let $\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \neq 0$,

$$u(\lambda_1, \lambda_2) = \frac{x^*_{\lambda_1} - x^*_{\lambda_2}}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|},\ v(\lambda_1, \lambda_2) = \frac{\sum_{i=1}^{k}\alpha_{\lambda_1,\ i}(S^i(x^*_{\lambda_1}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_1))}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}.$$

Then, we have

$$\|u(\lambda_1, \lambda_2)\| = \|\left(\frac{x^*_{\lambda_1} - x^*_{\lambda_2}}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}\right)\| \leq 1,$$

and by the nonexpansiveness of S, we have that

$$\begin{aligned}\|v(\lambda_1, \lambda_2)\| &\leq \tfrac{\sum_{i=1}^{k}\alpha_{\lambda_1,\ i}\|S^i(x^*_{\lambda_1},\lambda_1) - S^i(x^*_{\lambda_2},\lambda_1)\|}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} = \tfrac{\sum_{i=1}^{k}\alpha_{\lambda_1,\ i}\|S^i_{\lambda_1}x^*_{\lambda_1} - S^i_{\lambda_1}x^*_{\lambda_2}\|}{(1-\alpha_{\lambda_1,\ 0})\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} \\ &\leq \tfrac{\sum_{i=1}^{k}\alpha_{\lambda_1,\ i}}{1-\alpha_{\lambda_1,\ 0}} = 1.\end{aligned}$$

The second part of the proof of this theorem is the same as that of Theorem 2.1.

Theorem 2.3. *Let $(E, ||.||)$ be a uniformly convex Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a continuous mapping satisfying (1.8). Suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear monotone increasing function such that $\varphi(0) = 0$. Let x^*_λ be the unique fixed point of S_λ* (where $\mathrm{S_\lambda x = S(x,\lambda),\ x \in E,\ \lambda \in Y}$). *Suppose $\{x_n\}_{n=0}^{\infty}$ is the Kirk iterative process defined by (1.5) with $\sum_{i=0}^{k}\alpha_i = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous.*

Proof. Let $\lambda_1,\ \lambda_2 \in Y$. Let $S^0 = I$ (identity mapping). Then, $I(x_\lambda, \lambda) = I_\lambda x_\lambda = x_\lambda$. Let $\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \neq 0$,

$$u(\lambda_1, \lambda_2) = \frac{x^*_{\lambda_1} - x^*_{\lambda_2}}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|},\ v(\lambda_1, \lambda_2) = \frac{\sum_{i=1}^{k}\alpha_i(S^i(x^*_{\lambda_1}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_1))}{(1-\alpha_0)\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}.$$

Then, we have $\|u(\lambda_1, \lambda_2)\| \le 1$, and using Lemma 2.1 yields $\|v(\lambda_1, \lambda_2)\| \le 1$. Using the Kirk iterative process defined by (1.5) and applying Lemma 2.2 lead to

$$\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \le \frac{\| \sum_{i=1}^k \alpha_i S^i_{\lambda_1} x^*_{\lambda_2} - \sum_{i=1}^k \alpha_i S^i_{\lambda_2} x^*_{\lambda_2}\|}{2\alpha_0(1-\alpha_0)\delta(\epsilon)} \to 0 \text{ as } \lambda_2 \to \lambda_1,$$

which implies that, $||x^*_{\lambda_1} - x^*_{\lambda_2}|| \to 0$ as $\lambda_2 \to \lambda_1$.
That is, $||U(\lambda_1) - U(\lambda_2)|| \to 0$ as $\lambda_2 \to \lambda_1$.
Hence, the mapping $U : Y \to E$, defined by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous.

Theorem 2.4. *Let $(E, ||.||)$ be a uniformly convex Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a (continuous) nonexpansive mapping. Let x^*_λ be a fixed point of S_λ (where $\mathrm{S_\lambda x = S(x, \lambda),\ x \in E,\ \lambda \in Y}$). Suppose $\{x_n\}_{n=0}^\infty$ is the Kirk iterative process defined by (1.5) with $\sum_{i=0}^k \alpha_i = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous.*

Proof. This follows similarly as in that of Theorem 2.3.

Theorem 2.5. *Let $(E, ||.||)$ be a uniformly convex Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a continuous mapping satisfying (1.8). Suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear monotone increasing function such that $\varphi(0) = 0$. Let x^*_λ be the unique fixed point of S_λ (where $\mathrm{S_\lambda x = S(x, \lambda),\ x \in E,\ \lambda \in Y}$). Suppose $\{x_n\}_{n=0}^\infty$ is the Mann iterative process defined by (1.4) with $\sum_{i=0}^k \alpha_i = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous.*

Proof. Let $\lambda_1,\ \lambda_2 \in Y$. Let $\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \ne 0$,

$$u(\lambda_1, \lambda_2) = \frac{x^*_{\lambda_1} - x^*_{\lambda_2}}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|},\ v(\lambda_1, \lambda_2) = \frac{S(x^*_{\lambda_1}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_1)}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}.$$

Then, $\|u(\lambda_1, \lambda_2)\| \le 1$, and $\|v(\lambda_1, \lambda_2)\| \le 1$. For the Mann iterative process, let $k_{\lambda_1} = 1 - \alpha_{\lambda_1},\ k_{\lambda_2} = 1 - \alpha_{\lambda_2}$. Therefore,

$$\begin{aligned}\|x^*_{\lambda_1} - x^*_{\lambda_2}\| &\le \|k_{\lambda_1} x^*_{\lambda_1} + \alpha_{\lambda_1} S(x^*_{\lambda_1}, \lambda_1) - k_{\lambda_2} x^*_{\lambda_2} - \alpha_{\lambda_2} S(x^*_{\lambda_2}, \lambda_2) \\
&= \|k_{\lambda_1}(x^*_{\lambda_1} - x^*_{\lambda_2}) + (k_{\lambda_1} - k_{\lambda_2}) x^*_{\lambda_2} + \alpha_{\lambda_1}(S(x^*_{\lambda_1}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_1)) \\
&+ \alpha_{\lambda_1} S(x^*_{\lambda_2}, \lambda_1) - \alpha_{\lambda_2} S(x^*_{\lambda_2}, \lambda_2)\| \\
&\le \|(1 - \alpha_{\lambda_1})(x^*_{\lambda_1} - x^*_{\lambda_2}) + \alpha_{\lambda_1}(S(x^*_{\lambda_1}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_1))\| \\
&+ |\alpha_{\lambda_2} - \alpha_{\lambda_1}|\ \|x^*_{\lambda_2}\| + \|\alpha_{\lambda_1} S(x^*_{\lambda_2}, \lambda_1) - \alpha_{\lambda_2} S(x^*_{\lambda_2}, \lambda_2)\| \\
&\le \|x^*_{\lambda_1} - x^*_{\lambda_2}\|\ \|(1 - \alpha_{\lambda_1}) \frac{(x^*_{\lambda_1} - x^*_{\lambda_2})}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|} + \alpha_{\lambda_1} \frac{(S(x^*_{\lambda_1}, \lambda_1) - S(x^*_{\lambda_2}, \lambda_1))}{\|x^*_{\lambda_1} - x^*_{\lambda_2}\|}\| \\
&+ |\alpha_{\lambda_2} - \alpha_{\lambda_1}|\ \|x^*_{\lambda_2}\| + \|\alpha_{\lambda_1} S(x^*_{\lambda_2}, \lambda_1) - \alpha_{\lambda_2} S(x^*_{\lambda_2}, \lambda_2)\| \\
&= \|x^*_{\lambda_1} - x^*_{\lambda_2}\|\ \|(1 - \alpha_{\lambda_1}) u(\lambda_1, \lambda_2) + \alpha_{\lambda_1} v(\lambda_1, \lambda_2)\| \\
&+ |\alpha_{\lambda_2} - \alpha_{\lambda_1}|\ \|x^*_{\lambda_2}\| + \|\alpha_{\lambda_1} S_{\lambda_1} x^*_{\lambda_2} - \alpha_{\lambda_2} S_{\lambda_2} x^*_{\lambda_2}\|. \qquad (2.3)\end{aligned}$$

Applying Lemma 2.2 in (2.3) yields

$$\begin{aligned}\|x^*_{\lambda_1} - x^*_{\lambda_2}\| \quad &\le [1 - 2\alpha_{\lambda_1}(1 - \alpha_{\lambda_1})\delta(\epsilon)]\, \|x^*_{\lambda_1} - x^*_{\lambda_2}\| \\
&+ |\alpha_{\lambda_2} - \alpha_{\lambda_1}|\ \|x^*_{\lambda_2}\| + \|\alpha_{\lambda_1} S_{\lambda_1} x^*_{\lambda_2} - \alpha_{\lambda_2} S_{\lambda_2} x^*_{\lambda_2}\|,\end{aligned}$$

from which it follows that

$$\|x_{\lambda_1}^* - x_{\lambda_2}^*\| \leq \frac{|\alpha_{\lambda_2} - \alpha_{\lambda_1}|\ \|x_{\lambda_2}^*\| + \|\alpha_{\lambda_1} S_{\lambda_1} x_{\lambda_2}^* - \alpha_{\lambda_2} S_{\lambda_2} x_{\lambda_2}^*\|}{2\alpha_{\lambda_1}(1-\alpha_{\lambda_1})\delta(\epsilon)} \to 0 \text{ as } \lambda_2 \to \lambda_1,$$

which implies that $\|x_{\lambda_1}^* - x_{\lambda_2}^*\| \to 0$ as $\lambda_2 \to \lambda_1$.
That is, $||U(\lambda_1) - U(\lambda_2)|| \to 0$ as $\lambda_2 \to \lambda_1$.
Hence, the mapping $U : Y \to E$, defined by $U(\lambda) = x_\lambda^*,\ \lambda \in Y$, is continuous.

Theorem 2.6. *Let $(E, ||.||)$ be a uniformly convex Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a (continuous) nonexpansive mapping. Let x_λ^* be a fixed point of S_λ (where $\mathrm{S_\lambda x = S(x, \lambda),\ x \in E,\ \lambda \in Y}$). Suppose $\{x_n\}_{n=0}^\infty$ is the Mann iterative process defined by (1.4) with $\sum_{i=0}^k \alpha_i = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x_\lambda^*,\ \lambda \in Y$, is continuous.*

Proof. As in Theorem 2.5.

Remark 2.7. To the best of our knowledge, our results are original, extend and improve the recent results of Olatinwo [13, 14] and Proposition 1.2 of Zeidler [22]. See also the recent results of [3, 2, 17, 19, 20].

References

[1] V. Berinde, *A priori and a posteriori error estimates for a class of φ-contractions,* Bull. Applied & Comput. Math., (1999), 183-192.

[2] V. Berinde, M. Pacurar, *Fixed points and continuity of almost contractions,* Fixed Point Theory, **9**(1)(2008), 23-34.

[3] V. Berinde, *Iterative Approximation of Fixed Points,* Springer-Verlag, Berlin, Heidelberg, 2007.

[4] Lj. B. Ciric, *Generalized contractions and fixed point theorems,* Publ. Inst. Math. (Beograd) (N.S.), **12**(26)(1971), 19-26.

[5] Lj. B. Ciric, *A generalization of Banach's contraction principle,* Proc. Amer. Math. Soc., **45**(1974), 267-273.

[6] Lj. B. Ciric, *Some Recent Results in Metrical Fixed Point Theory,* University of Belgrad, 2003.

[7] C.W. Groetsch, *A Note on Segmenting Mann Iterates*, J. Math. Anal. Appl., **40**(1972), 369-372.

[8] R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc., **10**(1968), 71-76.

[9] W.A. Kirk, *On successive approximations for nonexpansive mappings in Banach spaces*, Glasgow Math. J., **12**(1971), 6-9.

[10] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory,* John Wiley & Sons, Inc., 2001.

[11] W.R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc., **44**(1953), 506-510.

[12] M.O. Olatinwo, *Some stability results for two hybrid fixed point iterative algorithms in normed linear space,* Matematicki Vesnik, **61**(4)(2009), 247-256.

[13] M.O. Olatinwo, *Some results on the continuous dependence of the fixed points in normed linear space,* Fixed Point Theory, **10**(1)(2009), 151-157.

[14] M.O. Olatinwo, *Some results on the continuous dependence of the fixed points for Kirk-type iterative processes in Banach space,* Ann. Univ. Ferrara, **56**(1)(2010), 53-63.

[15] M.O. Olatinwo, *Some stability results for nonexpansive and quasi-nonexpansive operators in uniformly convex Banach space using two new iterative processes of Kirk-type,* Fasciculi Math., **43**(2010), 101-114.

[16] I.A. Rus, *Generalized Contractions and Applications,* Cluj Univ. Press, Cluj Napoca, 2001.

[17] I.A. Rus, *The theory of metrical fixed point theorem: Theoretical and applicative relevances,* Fixed Point Theory, **9**(2)(2008), 541-559.

[18] I.A. Rus, *An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations,* Fixed Point Theory, **13**(2012), 179-192.

[19] I.A. Rus, S. Mureşan, *Data dependence of the fixed points set of weakly Picard operators,* Studia Univ. Babeş-Bolyai Math., **43**(1998), 79-83.

[20] I.A. Rus, S. Mureşan, *Data dependence of the fixed points set of some weakly Picard operators,* Tiberiu Popoviciu Sem. Cluj-Napoca, (2000), 201-208.

[21] H. Schaefer, *Uber die Methode Sukzessiver Approximationen,* Jahresber. Deutsch. Math. Verein., **59**(1957), 131-140.

[22] E. Zeidler *Nonlinear Functional Analysis and its Applications-Fixed Point Theorems,* Springer-Verlag, New York, Inc., 1986.

Received: December 28, 2013; Accepted: February 10, 2014.