Fixed Point Theory, 17(2016), No. 2, 413-426
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON GLOBAL EXISTENCE OF CERTAIN IMPULSIVE MULTI-ORDERS FRACTIONAL DIFFERENTIAL PROBLEM

HAMZA MEDJEKAL* AND SAÏD MAZOUZI**

*,**Laboratory of Applied Mathematics, Badji Mokhtar-Annaba University
P.O.Box 12, 23000 Annaba, Algeria
E-mails: h.medjekal@yahoo.com mazouzi_sa@yahoo.fr

Abstract. In this paper, we introduce a novel approach to tackle a class of fractional differential problems with impulses on the positive half-ray. So, we establish the existence of a bounded solution to certain impulsive multi-orders fractional initial value problem in a finite dimensional Banach space. The obtained result is based on the Schauder's fixed point theorem as well as certain continuation process. Finally, an illustrative example is provided.
Key Words and Phrases: Caputo's fractional derivative, multi-orders, impulsive conditions, Schauder's fixed point Theorem, PC-Ascoli-Arzela Theorem, continuation process.
2010 Mathematics Subject Classification: 26A33, 34A12, 34A37, 47H10.

1. Introduction

Due to their intensive applications in the modeling of many phenomena in various fields of science and engineering, Fractional Differential Equations (FDEs) have attracted the attention of a great deal of investigators in the last decade. Actually, the theory of FDEs has been rapidly developed, see the monographs of Kilbas *et al.* [14], Kilbas and Trujillo [15, 16], Lakshmikantham *et al.* [17], Miller and Ross [18], Srivastava and Saxena [23], Podlubny [22], Agarwal *et al.* [1], and so we can encounter in the literature several new results dealing with the FDEs such as in viscoelasticity, electrochemistry, control, porous media, etc. (see [9, 11, 13, 21] and the references therein).

Regarding the impulsive fractional differential equations (IFDE), we mention that they are nowadays an important tool for various mathematical models such as in physical and mathematical sciences. Due to their effectiveness, impulsive conditions have been used in specific models dealing with rapid changes which cause the discontinuity of the solution in a finite or infinite increasing temporal moments. For this reason several mathematicians are investigating the properties of solutions of such problems; unfortunately, most of the developed results are studied in the finite interval $[0, T]$, see [3, 4, 7, 19]. For instance in [10], the authors studied the existence and uniqueness of a class of impulsive fractional differential equations on $J = [0, T]$ and they introduced a new formula of solutions for an impulsive Cauchy problem with Caputo's fractional derivative by applying fixed point methods.

As far as we are involved in the study of impulsive fractional equations we must point out that Lemma 2.6 used in [10] to obtain the equivalence between an impulsive fractional problem and an integral equation is not correct as we see in the following counter-example:

Let us consider the function described on page 48 of the famous book of B. Nagy and F. Riesz [20] which is an example of a monotonic continuous function $F:[0,1]\to\mathbb{R}$, not constant in any subinterval of $[0,1]$ and satisfies $F'=0$, almost everywhere in $[0,1]$. So, in terms of Caputo's derivative we would have formally for any $\alpha\in(0,1)$

$$\begin{aligned} {}^{C}D_{0^+}^{\alpha}F(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}F'(s)\,ds=0,\ t\in[0,1] \\ F(a) &= F_0,\ (0<a<1),\ F_0 \text{ being the value of } F \text{ at } a. \end{aligned}$$

However, there is no apparent equivalence between this problem and the fractional integral representation of F defined in Lemma 2.6 [10], otherwise the function $F(t)$ would be constant and equal to F_0 throughout the interval $[0,1]$ which is a contradiction! Furthermore, since in the same work Lemma 2.7 is based on Lemma 2.6 then it is not correct and may lead to apparent contradictions... For further details see the recent paper of A. Bouzaroura and S. Mazouzi [8]. The reader may find in the recent comments about the concept of impulsive fractional differential equations of [24] that the proposed approach in [10] is incorrect.

Our main contribution in this paper is the study of new multi-orders fractional problems in a finite dimensional normed space $(X,\ \|.\|)$ such as either the Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$ subject to some impulsive conditions, namely

$$\begin{cases} {}^{C}D_{t_k^+}^{\alpha_k}y(t)=A(t)y(t)+f(t,y(t)),\ t\in J_k=(t_k,t_{k+1}],\ k=0,1,..., \\ y(0)=y_0\in X, \\ y(t_k^+)=y(t_k^-)+I_k(y(t_k^-)),\quad k\geq 1, \end{cases} \tag{1.1}$$

where ${}^{C}D_{t_k^+}^{\alpha_k}$ is the Caputo's fractional derivative of order $\alpha_k\in(0,1)$, $k=0,1,...$; $J_0=[0,t_1]$; $J_k=(t_k,t_{k+1}]$, for $k=1,...$; and y_0 is a given initial value in X. On the other hand, $A:J=[0,+\infty)\to\mathcal{B}(X)$, where $\mathcal{B}(X)$ is the Banach space of bounded linear operators on X into itself, $f:J\times X\to X$ and $I_k:X\to X$ are given continuous functions, $0=t_0<t_1<...<t_m<t_{m+1}<...$ Finally, $y(t_k^+)=\lim\limits_{h\to 0^+}y(t_k+h)$ and $y(t_k^-)=\lim\limits_{h\to 0^-}y(t_k+h)$ represent the right and left limits of $y(t)$ at $t=t_k$.

Let us first anticipate our study by a concrete example of such a problem in $\mathbb{R}$ in a finite interval $[0,T]$, $T>1$, namely

$$\begin{cases} {}^{C}D_{0^+}^{1/2}y(t)=t^2-1,\ t\in J_0=[0,1], \\ {}^{C}D_{1^+}^{1/3}y(t)=t^2-2t+1,\ t\in J_1=(1,T], \\ y(0)=1, \\ y(1^+)=y(1)+2. \end{cases} \tag{1.2}$$

First, we look for a piecewise continuous function $y : [0,1] \to \mathbb{R}$ satisfying (1.2). Solving the subproblem

$$\begin{cases} {}^{C}D_{0^+}^{1/2} y(t) = t^2 - 1,\ t \in J_0, \\ y(0) = 1, \end{cases}$$

we get

$$\begin{aligned} y(t) &= 1 + \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} s^2 ds - \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} ds \\ &= 1 + \frac{16}{15\sqrt{\pi}} t^{5/2} - \frac{2}{\sqrt{\pi}} t^{1/2}, \end{aligned}$$

so that $y(1) = 1 - \frac{14}{15\sqrt{\pi}}$.

Next, solving the subproblem

$$\begin{cases} {}^{C}D_{1^+}^{1/3} y(t) = t^2 - 2t + 1,\ t \in J_1, \\ y(1^+) = y(1) + 2 = 3 - \frac{14}{15\sqrt{\pi}}, \end{cases}$$

we find

$$\begin{aligned} y(t) &= y(1^+) + \frac{1}{\Gamma(1/3)} \int_1^t (t-s)^{-2/3} (s-1)^2 ds \\ &= 3 - \frac{14}{15\sqrt{\pi}} + \frac{27}{14\Gamma(1/3)} (t-1)^{7/3}. \end{aligned}$$

Thus, the piecewise continuous function

$$y(t) = \begin{cases} 1 + \frac{16}{15\sqrt{\pi}} t^{5/2} - \frac{2}{\sqrt{\pi}} t^{1/2},\ t \in J_0, \\ 3 - \frac{14}{15\sqrt{\pi}} + \frac{27}{14\Gamma(1/3)} (t-1)^{7/3},\ t \in J_1, \end{cases}$$

is a solution to the impulsive fractional problem (1.2).

In view of these new ideas we intent to extend in this paper recent results on fractional differential equations on unbounded domains, for instance those of Benchohra *et al.* in [2, 6] established in the absence of impulses, as well as those of K. Balachandran *et al.* in [5] considered in a Banach space but only for the finite interval $[0, T]$. We point out that using the Schauder's fixed point theorem combined with the diagonalization process the authors in [2, 6] proved the existence of bounded real-valued solutions of some fractional order differential equation on the half-ray $J = [0, +\infty)$.

The paper is organized as follows, in section 2 we introduce some basic definitions and notations, and give some necessary lemmas that will be used throughout this paper. In section 3 we state and prove our main results. Finally, in the last section we give a concrete example that illustrates the existence result established in Theorem 3.1.

2. Preliminaries

Let $\mathbb{R}^+ = [0,\ \infty)$ and let I be an arbitrary interval of $\mathbb{R}$; we denote by $C(I, X)$ the linear space of all continuous functions $y : I \to X$.

First, we introduce the Caputo's fractional derivative as defined in the reference [14]. We have

Definition 2.1. *We define the left-sided fractional Riemann-Liouville integral of order $\alpha \in (0,1)$ of a function $f : [a,b] \to X$ as follows*

$$J_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a.$$

We define the left-sided fractional derivative of order $\alpha \in (0,1)$ of a function $f : [a,b] \to X$ in the sense of Caputo by

$$^{C}D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f^{'}(s)ds, \quad t > a.$$

Remark 2.1. *We point out that the above integrals are understood in the sense of Bochner and we assume that the function f satisfies the necessary conditions for which those integrals are well defined.*

We set

$J_0 = [0, t_1]$; $J_k = (t_k, t_{k+1}]$, $k = 1, ...$ and $m_n = \max\{k \in \mathbb{N}, t_k < n\}$, for $n \geq 1 + [t_1]$, and for each $n \geq 1 + [t_1]$, we introduce the Banach space

$$PC([0,n], X) = \left\{ \begin{array}{l} y : [0,n] \to X : y \in C(J_k \cap [0,n], X),\ k = 1, ..., m_n, \\ y(t_k^+) \text{ and } y(t_k^-) \text{ exist}, y(t_k^-) = y(t_k,),\ k = 1, ..., m_n \end{array} \right\},$$

equipped with the norm

$$\|y\|_n = \sup_{t \in [0,n]} \|y(t)\|.$$

We need the following hypotheses:
$(\mathcal{H}_0)$ $\{\alpha_k\}_{k \geq 0} \subset (0,1)$. We set $\alpha = \sup_{k \geq 0} \{\alpha_k\}$ and $\Gamma' = \inf_{k \geq 0} \{\Gamma(\alpha_k + 1)\}$.
$(\mathcal{H}_1)$ $f : J \times X \to X$ is a continuous function.
$(\mathcal{H}_2)$ There exist a continuous bounded function $P : J \to \mathbb{R}^+$ and a continuous nondecreasing function $\varphi : J \to \mathbb{R}^+$ such that

$$\|f(t,u)\| \leq P(t)\varphi(\|u\|), \quad t \in J,\ u \in X, \tag{2.1}$$

and

$$p^* = \sum_{i=1}^{\infty} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} P(s) ds < \infty.$$

We put $\|P\|_\infty = \sup_{t \geq 0} |P(t)|$.
$(\mathcal{H}_3)$ $A : J \to X$ is continuous and satisfies the estimate

$$\|A(t)\|_{\mathcal{B}(X)} \leq a(t),\ t \geq 0,$$

for some continuous bounded function $a : J \to \mathbb{R}^+$ such that

$$a^* = \sum_{i=1}^{\infty} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} a(s) ds < \infty.$$

We set $\|a\|_\infty = \sup_{t \geq 0} |a(t)|$.

$(\mathcal{H}_4)$ The functions $I_k : X \to X$ are continuous and there exists a sequence of positive numbers $\{\lambda_k\}_{k\geq 1}$ such that

$$\|I_k(u)\| \leq \lambda_k, \quad u \in X,$$

and

$$\Lambda = \sum_{k\geq 1} \lambda_k < \infty;$$

$(\mathcal{H}_5)$ There exists a constant $\rho > 0$ such that

$$\|y_0\| + \Lambda + a^*\rho + p^*\varphi(\rho) \leq \rho. \tag{2.2}$$

Next, consider the impulsive differential equation of fractional multi-orders

$$\begin{cases} {}^C D^{\alpha_k}_{t_k^+} y(t) = A(t)y(t) + f(t, y(t)), \ t \in J_k \cap [0,n], \ k = 0, 1, ..., m_n, \\ y(0) = y_0 \in X, \\ y(t_k^+) = y\left(t_k^-\right) + I_k(y(t_k^-)), \quad k = 1, 2, ..., m_n. \end{cases} \tag{2.3}$$

To be more rigorous we define a solution to the problem (2.3) as follows

Definition 2.2. *A function $y \in PC([0,n], X)$ is said to be a solution to the problem (2.3) if ${}^C D^{\alpha_k}_{t_{k+}} y\ (t)$ exists for $t \in J_k \cap [0,n]$, for each $k = 0, ..., m_n$ and satisfies the equation ${}^C D^{\alpha_k}_{t_{k+}} y(t) = A(t)y(t) + f(t, y(t))$ for $t \in J_k \cap [0,n]$, for each $k = 0, ..., m_n$ and the conditions*

$$\begin{cases} y(0) = y_0 \in X, \\ y(t_k^+) = y\left(t_k^-\right) + I_k(y(t_k^-)), \quad k = 1, 2, ..., m_n. \end{cases} \tag{2.4}$$

Next, we state and prove a useful equivalence between problem (2.3) and certain integral equation.
Let $h \in C(\mathbb{R}^+, X)$ and consider the fractional differential equation

$${}^C D^{\alpha_k}_{t_{k+}} y(t) = h(t), \quad t \in J_k \cap [0,n], k = 0, ..., m_n \tag{2.5}$$

We refer to (2.5)-(2.4) as *(pb)*, we have the following result:

Lemma 2.1. *A function $y \in PC([0,n], X)$ is a solution to the problem (pb) if and only if it satisfies the following integral equation*

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(\alpha_0)} \int\limits_0^t (t-s)^{\alpha_0 - 1} h(s) ds\ , \ t \in [0, t_1] \\ y_0 + \sum\limits_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} h(s) ds \\ + \frac{1}{\Gamma(\alpha_k)} \int\limits_{t_k}^{t} (t-s)^{\alpha_k - 1} h(s) ds + \sum\limits_{i=1}^{k} I_i(y(t_i^-)), \ t \in J_k \cap [0,n], \ k = 1, ..., m_n. \end{cases} \tag{2.6}$$

Proof. Let y be a solution to problem *(pb)* and let $t \in [0, t_1]$, then we have

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha_0)} \int\limits_0^t (t-s)^{\alpha_0 - 1} h(s) ds.$$

If $t \in (t_1,\ t_2] \cap [0,n]$, then

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 1} h(s) ds \\ &= y(t_1) + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 1} h(s) ds \\ &= y_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha_0)} \int_{0}^{t_1} (t_1-s)^{\alpha_0 - 1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t} (t-s)^{\alpha_1 - 1} h(s) ds. \end{aligned}$$

If $t \in (t_2, t_3] \cap [0,n]$, then we get once again

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha_2)} \int_{t_2}^{t} (t_2-s)^{\alpha_2 - 1} h(s) ds \\ &= y(t_2^-) + I_2(y(t_2^-)) + \frac{1}{\Gamma(\alpha_2)} \int_{t_2}^{t} (t-s)^{\alpha_2 - 1} h(s) ds \\ &= y_0 + I_1(y(t_1^-)) + I_2(y(t_2^-)) + \frac{1}{\Gamma(\alpha_0)} \int_{0}^{t_1} (t_1-s)^{\alpha_0 - 1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t_2} (t_2-s)^{\alpha_1 - 1} h(s) ds + \frac{1}{\Gamma(\alpha_2)} \int_{t_2}^{t} (t-s)^{\alpha_2 - 1} h(s) ds, \end{aligned}$$

and more generally, if $t \in J_k \cap [0,n]$, $k = 1, .., m_n$, then we get

$$\begin{aligned} y(t) &= y_0 + \sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha_{i-1} - 1} h(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} h(s) ds + \sum_{i=1}^{k} I_i(y(t_i^-)). \end{aligned}$$

Conversely, assume that y satisfies (2.6). If $t \in [0, t_1] \cap [0,n]$, then $y(0) = y_0$. Next, since the Caputo's derivative of a constant is zero, then we merely obtain ${}^C D_{t_{k+}}^{\alpha_k} y(t) = h(t)$, $t \in J_k \cap [0,n]$, $k = 0, 1, .., m_n$. On the other hand, one can easily verify that

$$y\left(t_k^+\right) = y\left(t_k\right) + I_k(y(t_k^-)), \qquad k = 1, ..., m_n,$$

which completes the proof. □

The next Theorem is a piecewise continuous version of the famous Ascoli-Arzela's theorem. It is essentially needed in the proof of our main result.

Theorem 2.1. [25] *Let X be a Banach space and $\mathcal{W}$ a subset of $PC([0,n],X)$. Then $\mathcal{W}$ is relatively compact if the following conditions are satisfied:*
(i) $\mathcal{W}$ is a uniformly bounded subset of $PC([0,n],X)$;
(ii) $\mathcal{W}$ is equicontinuous in (t_k,t_{k+1}), $k=0,1,...,m_n$, where $t_0=0$, $t_{m_{n+1}}=n$;
(iii) $\mathcal{W}(t)=\{u(t) \mid u\in\mathcal{W},\ t\in J_n^*\}$, $\mathcal{W}(t_k^+)=\{u(t_k^+) : u\in\mathcal{W}\}$ and $\mathcal{W}(t_k^-)=\{u(t_k^-) : u\in\mathcal{W}\}$ are relatively compact subsets of X, where $J_n^*=[0,n]\setminus\{t_k\}_{k=0}^{m_{n+1}}$.

3. Main results

We will establish in this section the global existence of at least one bounded solution to the problem (1.1) by using Schauder's fixed point theorem combined with certain continuation process based on Arzela-Ascoli's Theorem. Our main result is the following

Theorem 3.1. *If the hypotheses $(\mathcal{H}_1)-(\mathcal{H}_5)$ are satisfied, then problem (1.1) has at least one bounded solution $y\in PC(J,X)$.*

Proof. The proof will be given in two parts:
Part I. We begin by showing that problem (2.3) has a solution

$$y_n\in PC([0,n],X)$$

satisfying $\|y_n\|_n\le\rho$, for any $n\ge 1+[t_1]$. We set

$$\Phi(t,y(t))=A(t)y(t)+f(t,y(t)),$$

and for each $t\in J_k\cap[0,n]$, $k=0,1,...,m_n$, we define the mapping

$$\mathcal{F}:PC([0,n],X)\to PC([0,n],X)$$

by

$$(\mathcal{F}y)(t)=\begin{cases} y_0+\frac{1}{\Gamma(\alpha_0)}\int\limits_0^t(t-s)^{\alpha_0-1}\Phi(s,y(s))ds\ ,\ t\in[0,t_1],\\ y_0+\sum\limits_{i=1}^{k}\frac{1}{\Gamma(\alpha_{i-1})}\int\limits_{t_{i-1}}^{t_i}(t_i-s)^{\alpha_{i-1}-1}\Phi(s,y(s))ds\\ +\frac{1}{\Gamma(\alpha_k)}\int\limits_{t_k}^{t}(t-s)^{\alpha_k-1}\Phi(s,y(s))ds+\sum\limits_{i=1}^{k}I_i(y(t_i^-)),\\ \ t\in J_k\cap[0,n]\,,\ k=1,...,m_n.\end{cases}$$

Our main goal is to show that the mapping $\mathcal{F}$ has a fixed point which is a solution to the problem (2.3). Indeed, let

$$\mathcal{C}=\{y\in PC([0,n],X),\ \ \|y\|_n\le\rho\},$$

be the closed ball of $PC([0,n],X)$ centered at 0 and with radius ρ defined in $(\mathcal{H}_5)$. It is clear that $\mathcal{C}$ is a closed and convex subset of $PC([0,n],X)$.
We proceed in several steps:
Step 1. $\mathcal{F}$ maps $\mathcal{C}$ into itself.

Let $y \in \mathcal{C}$, then for $t \in [0, t_1]$, we have

$$\begin{aligned}
\|(\mathcal{F}y)(t)\| &\leq \|y_0\| + \frac{1}{\Gamma(\alpha_0)} \int\limits_0^t (t-s)^{\alpha_0 - 1} \|\Phi(s, y(s))\| ds \\
&\leq \|y_0\| + a^* \rho + p^* \varphi(\rho) \\
&\leq \rho.
\end{aligned}$$

Moreover, for each $t \in J_k \cap [0, n]$, $k = 0, 1, ..., m_n$, we have

$$\begin{aligned}
\|(\mathcal{F}y)(t)\| \leq \|y_0\| + & \sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} \|\Phi(s, y(s))\| ds \\
& + \frac{1}{\Gamma(\alpha_k)} \int\limits_{t_k}^{t} (t-s)^{\alpha_k - 1} \|\Phi(s, y(s))\| ds + \sum_{i=1}^{k} \|I_i(y(t_i^-))\| \\
& \leq \|y_0\| + \Lambda + \sum_{i=1}^{k+1} \frac{1}{\Gamma(\alpha_{i-1})} \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} \|\Phi(s, y(s))\| ds \\
& \leq \|y_0\| + \Lambda + \sum_{i=1}^{\infty} \frac{1}{\Gamma(\alpha_{i-1})} \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} \|\Phi(s, y(s))\| ds \\
& \leq \|y_0\| + \Lambda + \rho \sum_{i=1}^{\infty} \frac{1}{\Gamma(\alpha_{i-1})} \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} a(s) ds \\
& + \varphi(\rho) \sum_{i=1}^{\infty} \frac{1}{\Gamma(\alpha_{i-1})} \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1} - 1} P(s) ds \\
& \leq \|y_0\| + \Lambda + a^* \rho + p^* \varphi(\rho).
\end{aligned}$$

Thus

$$\|\mathcal{F}y\|_n \leq \|y_0\| + \Lambda + a^* \rho + p^* \varphi(\rho) \leq \rho, \tag{3.1}$$

showing that $\mathcal{F}\mathcal{C} \subset \mathcal{C}$.

Step 2. Let us prove that $\mathcal{F}$ is continuous. Indeed, consider a sequence $\{y_q\}_{q \geq 1}$ such that $y_q \to y$, in $PC([0, n], X)$, when $q \to \infty$. Then, for any $t \in [0, t_1]$, we have

$$\begin{aligned}
\|(\mathcal{F}y_q)(t) - (\mathcal{F}y)(t)\| &\leq \frac{1}{\Gamma(\alpha_0)} \int\limits_0^t (t-s)^{\alpha_0 - 1} \|\Phi(s, y_q(s)) - \Phi(s, y(s))\| ds \\
&\leq \frac{n^\alpha}{\Gamma'} \Big(\sup_{s \in [0, n]} \|A(s)(y_q(s)) - y(s))\| \\
&+ \sup_{s \in [0, n]} \|f(s, y_q(s)) - f(s, y(s)))\| \Big).
\end{aligned}$$

Since f and A are continuous, then

$$\|\mathcal{F}y_q - \mathcal{F}y\|_n \to 0,\ q \to \infty.$$

Next, for each $t \in J_k \cap [0,n]$, $k = 1, ..., m_n$, we have

$$\begin{aligned}
\|(\mathcal{F}y_q)(t) - (\mathcal{F}y)(t)\| \ &\leq \ \sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \|\Phi(s, y_q(s)) - \Phi(s, y(s))\| ds \\
&\quad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \|\Phi(s, y_q(s)) - \Phi(s, y(s))\| ds \\
&\quad + \sum_{i=1}^{k} \|I_k(y_q(t_k^-)) - I_k(y(t_k^-))\| \\
&\leq \ \frac{(m_n+1)n^{\alpha}}{\Gamma'} \Big(\sup_{s\in[0,n]} \|A(s)(y_q(s)) - y(s))\| \\
&\quad + \sup_{s\in[0,n]} \|f(s, y_q(s)) - f(s, y(s)))\| \Big) \\
&\quad + \sum_{i=1}^{k} \|I_i(y_q(t_i^-)) - I_i(y(t_i^-))\|.
\end{aligned}$$

The continuity of the functions f, A, I_i, for $i = 1, ..., k$, implies that

$$\|\mathcal{F}y_q - \mathcal{F}y\|_n \to 0,\ q \to \infty,$$

which establishes the continuity of $\mathcal{F}$.

Step 3. $\mathcal{F}$ maps $\mathcal{C}$ into an equicontinuous family of $PC([0,n], X)$.
Let $0 < \tau_1 < \tau_2 < t_1$ and $y \in \mathcal{C}$, then

$$\begin{aligned}
\|(\mathcal{F}y)(\tau_2) - (\mathcal{F}y)(\tau_1)\| &\leq \frac{1}{\Gamma(\alpha_0)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha_0 - 1} - (\tau_1 - s)^{\alpha_0-1}| \|\Phi(s, y(s))\| ds \\
&\quad + \frac{1}{\Gamma(\alpha_0)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha_0 - 1} \|\Phi(s, y(s))\| ds \\
&\leq \frac{\|a\|_\infty \|y\| + \|P\|_\infty \varphi(\|y\|)}{\Gamma(\alpha_0)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha_0 - 1} - (\tau_1 - s)^{\alpha_0 - 1}| ds \\
&\quad + \frac{\|a\|_\infty \|y\| + \|P\|_\infty \varphi(\|y\|)}{\Gamma(\alpha_0)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha_0 - 1} ds.
\end{aligned}$$

Hence

$$\|(\mathcal{F}y)(\tau_2) - (\mathcal{F}y)(\tau_1)\| \leq \frac{\|a\|_\infty \rho + \|P\|_\infty \varphi(\rho)}{\Gamma'} (\tau_1^{\alpha_0} - \tau_2^{\alpha_0} + 2(\tau_2 - \tau_1)^{\alpha_0}).$$

Obviously, the right-hand side of the above inequality tends to zero as $\tau_1 \longrightarrow \tau_2$, therefore $\mathcal{FC}$ is equicontinuous in the interval $[0, t_1]$.
More generally, regarding the interval $J_k \cap [0, n]$, $k = 1, ..., m_n$, if $y \in \mathcal{C}$, then for every τ_1, τ_2 satisfying $t_k < \tau_1 < \tau_2 < t_{k+1}$, we obtain the following estimate

$$\begin{aligned}\|(\mathcal{F}y)(\tau_2) - (\mathcal{F}y)(\tau_1)\| \quad &\leq \quad \frac{\|a\|_\infty \rho + \|P\|_\infty \varphi(\rho)}{\Gamma'} (\tau_1^{\alpha_k} - \tau_2^{\alpha_k} + 2(\tau_2 - \tau_1)^{\alpha_k}) \\ &+ \sum_{\tau_1 \leq t_i \leq \tau_2} \|I_i(y(t_i^-))\|,\end{aligned}$$

which shows once again that $\mathcal{FC}$ is equicontinuous in the interval $J_k \cap [0, n]$, for $k = 1, ..., m_n$.
Thanks to the finite dimension assumption of X condition (iii) of Theorem 2.1 is obviously satisfied. As a consequence of the steps 1-3 together with the fact that X is of finite dimension we conclude by the PC-type Arzela-Ascoli theorem that $\mathcal{F}$ is completely continuous.
Accordingly, in virtue of Schauder's fixed point theorem $\mathcal{F}$ has a fixed point y_n in $\mathcal{C}$ which is a bounded solution to the problem (2.3) satisfying

$$\|y_n\|_n \leq \rho.$$

Next, in order to extend the foregoing existence result from the finite interval J_n, $n \geq 1 + [t_1]$ to the positive half ray we use certain continuation process which can be described as follows:
Part II. Continuation process. Let $\{n_j\}_{j \geq 0}$ be an increasing sequence of integer numbers satisfying

$$n_0 = 0 < n_1 < n_2 < ... < n_j < ... \uparrow \infty.$$

For each $j \in \mathbb{N}^*$, we define the function $u_j(t)$ as follows:

$$u_j(t) = \begin{cases} y_j(t), & t \in [0,\ n_j] \\ y_j(n_j), & t \in [n_j,\ \infty), \end{cases} \tag{3.2}$$

where $y_j(t)$ is the piecewise continuous solution on the interval $[0,\ n_j]$ obtained in the above steps. It is easy to see that

$$\|u_j\|_{n_1} \leq \rho, \quad j \in \mathbb{N}^*,$$

and for each $j \in \mathbb{N}^*$, u_j satisfies the following integral equation for $t \in J_k \cap [0,\ n_1]$, $k = 1,\ 2, ..., m_{n_1}$

$$\begin{aligned} u_j(t) \quad &= \quad y_0 + \sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \Phi(s, u_j) ds \\ &+ \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t - s)^{\alpha_k - 1} \Phi(s, u_j) ds \ + \sum_{i=1}^{k} I_i(u_j(t_i^-)). \end{aligned}$$

Following the above steps we can prove that the sequence $\{u_j\}_{j \geq 1}$ is relatively compact in $PC([0,\ n_1], X)$, so there is an infinite proper subset N_1 of $\mathbb{N}^*$ and a function

$v_1 \in PC([0,\ n_1], X)$ such that

$$u_j \to v_1 \text{ in } PC([0,\ n_1], X), \text{ as } \ j \to \infty, \ \text{(through } \ N_1).$$

We notice that

$$\|u_j\|_{n_2} \le \rho, \quad j \in N_1,$$

and since the sequence $\{u_j(t),\, j \in N_1\}$ is relatively compact in $PC([0,\ n_2], X)$, then there exists an infinite proper subset N_2 of N_1 and a function $v_2 \in PC([0,\ n_2], X)$ such that

$$u_j \to v_2 \text{ in } PC([0,\ n_2], X), \text{ as } j \to \infty, \ \text{(through } \ N_2).$$

Continuing in this way we obtain a sequence of decreasing subsets $\{N_l\}_{l\ge 1}$ satisfying

$N_{l+1} \subset N_l$, for every $l \ge 1$, and a sequence of piecewise continuous functions $\{v_l\}_{l\ge 1}$ such that, for each $l = 1, 2, ...$, we have
(1) $v_l \in PC([0,\ n_l], X)$,
(2) $\{u_j,\, j \in N_l\} \to v_l$ in $PC([0,\ n_l], X)$, as $j \to \infty$, (through N_l),
(3) $v_{l+1} = v_l$ on the interval $[0,\ n_l]$ since $N_{l+1} \subset N_l$,
(4) For each $l \in \mathbb{N}^*$, the subsequence $\{u_j,\ j \in N_l\}$ satisfies the following integral equation

$$\begin{aligned} u_j(t) \quad &= \quad y_0 + \sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \Phi(s, u_j) ds \qquad (3.3) \\ &\quad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \Phi(s, u_j) ds + \sum_{i=1}^{k} I_i(u_j(t_i^-)), \end{aligned}$$

for every $t \in J_k \cap [0, n_l]$.

Next, define a function $y : J = [0, \infty) \to X$ as follows

$$\begin{cases} y(0) = y_0 \\ y(t) = v_l(t), \quad t \in (n_{l-1}, n_l], \ l = 1, 2, ... \end{cases} \qquad (3.4)$$

It follows that $y \in PC(J, X)$ and $\|y(t)\| \le \rho, \ \ t \in J$. Furthermore, letting $j \to \infty$ in (3.3), we obtain the following integral equation

$$\begin{aligned} v_l(t) \quad &= \quad y_0 + \sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \Phi(s, v_l(s)) ds \qquad (3.5) \\ &\quad + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \Phi(s, v_l(s)) ds \ + \sum_{i=1}^{k} I_i(v_l(t_i^-)), \end{aligned}$$

for every $t \in J_k \cap [0, n_l]$. Now since we have $v_l(s) = y(s)$, for every $s \in (n_{l-1}, n_l]$, $l = 1, 2, ...$, then equation (3.5) merely becomes

$$y(t) = y_0 + \sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} \Phi(s, y(s)) ds + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^{t} (t-s)^{\alpha_k - 1} \Phi(s, y(s)) ds + \sum_{i=1}^{k} I_i(y(t_i^-)),\ t \in J_k \cap [0, n_l]$$

from which we infer that y satisfies problem (1.1). Accordingly, y is a piecewise continuous bounded solution to the given problem which completes the proof of the main theorem. □

Here is a concrete example illustrating the above global existence theorem.

4. Example

Let $\alpha_k = \frac{1}{2} - \frac{1}{3+k}$, $k = 0, 1, ...$ and consider the problem

$$\begin{cases} {}^C D_{t_0^+}^{\alpha_0} y(t) = \frac{e^{-t/2}}{5} y(t) + \frac{e^{-t}}{10} |y(t)|^{1/2},\ t \in J_0 = [0, \pi] \\ y(0) = 1/6, \\ {}^C D_{t_k^+}^{\alpha_k} y(t) = \frac{e^{-t/2}}{5} y(t) + \frac{e^{-t}}{10} |y(t)|^{1/2},\ t \in J_k = (k\pi, (k+1)\pi] \\ y((k\pi)^+) = y(k\pi) - \frac{|y(k\pi)|}{k^2(4 + |y(k\pi)|)},\ k = 1, 2, ... \end{cases} \tag{4.1}$$

Here

$$A(t)x = \frac{e^{-t/2}}{5} x,\ f(t, x) = \frac{e^{-t}}{10} |x|^{1/2},\ \text{for every } (t, x) \in J \times \mathbb{R},$$
$$I_k(t) = -\frac{t}{k^2(4+t)},\ \lambda_k = \frac{1}{k^2},\ k = 1, 2, ...$$

It is clear that hypotheses $(\mathcal{H}_0)$– $(\mathcal{H}_3)$ are satisfied with

$$a(t) = \frac{e^{-t/2}}{5},\ P(t) = \frac{e^{-t}}{10},\quad \varphi(|u|) = |u(t)|^{1/2},\quad \|P\|_\infty = \frac{1}{10},\ \|a\|_\infty = \frac{1}{5}.$$

On the other hand, we have

$$p^* = \frac{1}{10} \sum_{i=1}^{\infty} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} e^{-s} ds \leq \frac{1}{10} \sum_{i=1}^{\infty} \frac{e^{-(i-1)\pi}}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} ds \leq 0.231\,56 < \infty.$$

Proceeding in the same manner we find that

$$a^* = \frac{1}{5} \sum_{i=1}^{\infty} \frac{1}{\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha_{i-1}-1} e^{-\frac{s}{2}} ds \leq 0.349\,63 < \infty.$$

In addition, we have

$$\Lambda = \sum_{k\geq 1} \lambda_k = \sum_{k\geq 1} \frac{1}{k^2} = \frac{\pi^2}{6} < \infty.$$

Regarding condition $(\mathcal{H}_5)$ we notice that we have

$$\|y_0\| + \Lambda + a^*\rho + p^*\varphi(\rho) \leq \frac{1}{6} + \frac{\pi^2}{6} + 0.349\,63\rho + 0.231\,56\sqrt{\rho} \leq \rho,$$

which is satisfied for any $\rho \geq 3.45$. Since all the assumptions of Theorem 3.1 are satisfied, then problem (4.1) has a bounded global solution on J.

References

[1] R.P. Agarwal, M. Benchohra, B.A. Slimani, *Existence results for differential equations with fractional order and impulses*, Mem. Diff. Eq. Math. Phys., **44**(2008), 1–21.

[2] A. Arara, M. Benchohra, N. Hamidi, J.J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal., **72**(2010), 580-586.

[3] K. Balachandran, S. Kiruthika, J.J. Trujillo, *Existence results for fractional impulsive integro-differential equations in Banach spaces*, Commun. Nonlinear Sci. Numer. Simul., **16**(2011), 1970–1977.

[4] K. Balachandran, S. Kiruthika, J.J. Trujillo, *On fractional impulsive equations of Sobolev type with nonlocal condition in Banach spaces*, Comput. Math. Appl., **62**(2011), 157–1165.

[5] K. Balachandran, S. Kiruthika, *Existence of solutions of abstract fractional impulsive semilinear evolution equations*, Electron. J. Qual. Theory Differ. Equ., (2010), no. 4, 1–12.

[6] M. Benchohra, F. Berhoun, G. N'Guérékata, *Bounded solutions for fractional order differential equations on the half-line,* Bull. Math. Anal. Appl., **4**(2012), no. 1, 62-71.

[7] M. Benchohra, B.A. Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Electron. J. Diff. Eq., **2009**(2009), no. 10, 1-11.

[8] A. Bouzaroura, Mazouzi, *An alternative method for the study of impulsive differential equations of fractional orders in a Banach space*, Int. J. Differ. Equ., **2013**(2013), Article ID 191060.

[9] K. Diethelm, A.D. Freed, *On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity*, in: Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties (F. Keil, W. Mackens, H. Voss, J. Werther - Eds.), Springer-Verlag, Heidelberg, 1999, 217–307.

[10] M. Fečkan, Y. Zhou, J. Wang, *On the concept and existence of solution for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **17**(2012), 3050–3060.

[11] L. Gaul, P. Klein, S. Kempfle, *Damping description involving fractional operators*, Mech. Syst. Signal Process, **5**(1991), 81–88.

[12] E. Hernandez, D. O'Regan, K. Balachandran, *On recent developments in the theory of abstract differential equations with fractional derivatives*, Nonlinear Anal., **73**(2010), 3462–3471.

[13] R. Hilfer, *Applications of Fractional Calculus in Physics,* World Scientific, 2000.

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies 204, Ed. Van Mill, Amsterdam, 2006.

[15] A.A. Kilbas, J.J. Trujillo, *Differential equations of fractional order: Methods, results and problems I*, Appl. Anal., **78**(2001), 153–192.

[16] A.A. Kilbas, J.J. Trujillo, *Differential equations of fractional order: methods, results and problems II*, Appl. Anal., **81**(2002), 435–493.

[17] V. Lakshmikantham, A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., **69**(2008), 2677–2682.

[18] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[19] G.M. Mophou, G.M. N'Guérékata, *Existence of mild solution for some fractional differential equations with a nonlocal condition*, Semigroup Forum, **79**(2009), 315-322.

[20] B. Nagy, F. Riesz, *Functional Analysis*, Blackie and Son Limited, London 1956.

[21] L. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal., **5**(2002), 367–386.

[22] L. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[23] H.M. Srivastava, R.K. Saxena, *Operators of fractional integration and their applications*, Appl. Math. Comput., **118**(2001), 1–52.

[24] G. Wang, B. Ahmad, L. Zhang, J.J. Nieto, *Comments on the concept of existence of solution for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., **19**(2014), no. 3, 401-403.

[25] W. Wei, X. Xiang, Y. Peng, *Nonlinear impulsive integrodifferential equation of mixed type and optimal controls,* Optimization, **55**(2006), 141-156.

Received: November 23, 2013; Accepted: March 10, 2014.