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AXIOMS FOR THE REDUCED LEFSCHETZ NUMBER OF SOME MULTIVALUED MAPS

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Abstract. Arkowitz and Brown [2] presented the system of four axioms characterizing the reduced Lefschetz number. We show that the number of axioms can be reduced to three. Then, we present an analogical system of axioms characterizing the reduced Lefschetz number in a category which morphisms are closely related to multivalued maps.
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1. Introduction

In recent years there appeared many papers characterizing the Lefschetz number and related invariants in an axiomatic way. Arkowitz and Brown [2] show that the Lefschetz number in the category of spaces of the homotopy type of finite, connected CW-complexes is the unique function which satisfies four natural axioms. Their approach is a generalization of the characterization of the Euler characteristic given by Watts in [17]. Later, Gonçalves and Weber [5] extend this approach to the case of the equivariant Lefschetz number. Moreover, they give also a similar axioms uniquely characterizing the Reidermeister trace and its equivariant version. Another paper by Furi, Pera and Spaldini [4] presents a system of axioms which characterizes the local fixed point index, a more general invariant which is a localized version of the Lefschetz number. Note that the axioms in [4] are simpler then the one presented in [2], but authors of [4] work in the category of differentiable manifolds. Staecker [14] and [15] generalized it to the coincidence theory and the Reidemeister trace on topological manifolds. There is also a paper [10] by Hadwiger, where the author characterizes the Euler characteristic in the context of lattice valuations. Recently, Staecker [16] generalized this approach to the case of the Lefschetz number.

In this paper we show that the number of axioms characterizing the Lefschetz number presented in [2] is not minimal. Only three axioms are needed in such a characterization. Suitable analog of our system of axioms also can be applied to the case of equivariant Lefschetz number, so the number of axioms presented in [5] can also be reduced to three.

The main goal of this paper is considering the case of multivalued maps. Precisely, we investigate the reduced Lefschetz number in a category which morphisms are equivalence classes of specific diagrams which are closely related to multivalued maps. Górniewicz and Granas [9] such a category call "category of morphisms". Let $\mathcal{D}$ denote the category of spaces of the homotopy type of finite, connected CW-complexes and morphisms in the sense of [9]. Given a morphism $\varphi \in M(X,X)$ in the category $\mathcal{D}$, we define the reduced Lefschetz number $\tilde{\mathcal{L}}_m(\varphi)$ and show that in $\mathcal{D}$ the system of axiom characterizing $\tilde{\mathcal{L}}_m$ is similar to the one characterizing the reduced Lefschetz number of single-valued maps.

For $k \geq 1$, denote by $\bigvee^k S^1$ the wedge of k copies of the 1-sphere S^1. Write $e_j \colon S^1_j \to \bigvee^k S^1$ for the inclusion map into the j-th summand and $p_j \colon \bigvee^k S^1 \to S^1$ for the projection map onto the j-th summand for $j = 1, \ldots, k$. Then, we characterize the reduced Lefschetz number $\tilde{\mathcal{L}}_m$ as follows.

Theorem 1.1. *The reduced Lefschetz number $\tilde{\mathcal{L}}_m$ is the unique function $\tilde{\lambda}$ from the set of self-morphisms of spaces in $\mathcal{D}$ to the integers that satisfies the following conditions:*
(i) (*Homotopy equivalence axiom*) *If $\varphi \in M(X,X)$ and $h \colon X \to Y$ is a homotopy equivalence with homotopy inverse $k \colon Y \to X$, then $\tilde{\lambda}(\varphi) = \tilde{\lambda}(h\varphi k)$;*
(ii) (*Cofibration axiom*) *Let A be a subcomplex of X. If $\varphi \in M(X,X)$ is such that $\varphi(A) \subseteq A$ and morphisms $\varphi' \in M(A,A)$ and $\bar{\varphi} \in M(X/A, X/A)$ are induced by φ,*

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{\pi} & X/A \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} \\ A & \xrightarrow{i} & X & \xrightarrow{\pi} & X/A \end{array}$$

then $\tilde{\lambda}(\varphi) = \tilde{\lambda}(\varphi') + \tilde{\lambda}(\bar{\varphi})$;
(iii) (*Wedge of circles axiom*) *If $\varphi \in M\left(\bigvee^k S^1, \bigvee^k S^1\right)$ for $k \geq 1$, then*

$$\tilde{\lambda}(\varphi) = -(\deg(\varphi_1) + \cdots + \deg(\varphi_k)),$$

where $\varphi_j = p_j \varphi e_j$.

The paper is organized as follows. In Section 2 we show that only three axioms are needed in a characterization of the reduced Lefschetz number of single-valued maps. Section 3 takes into account basic definitions related to multivalued maps. We precisely explain what the category $\mathcal{D}$ is and recall its main properties. Then, in Section 4 we present the proof of Theorem 1.1. In Section 5 we make a comment about an extension of the system of axioms defining the Lefschetz number to a category of some spaces of finite type.

2. Axioms for the Lefschetz number of single-valued maps

Let $\mathcal{C}$ be the category of spaces of the homotopy type of finite, connected CW-complexes and continuous single-valued maps. Arkowitz and Brown show in [2] that the reduced Lefschetz number is characterized by four axioms. That number of axioms is not minimal. We show that it is possible to use only three other axioms in such a characterization. Namely we prove

Theorem 2.1. *The reduced Lefschetz number $\tilde{\mathcal{L}}$ is the unique function $\tilde{\lambda}$ from the set of self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the following conditions:*
(i) (*Homotopy equivalence axiom*) *If $f\colon X \to X$ is a map and $h\colon X \to Y$ is a homotopy equivalence with a homotopy inverse $k\colon Y \to X$, then $\tilde{\lambda}(f) = \tilde{\lambda}(hfk)$;*
(ii) (*Cofibration axiom*) *If A is a subcomplex of X and $A \to X \to X/A$ is the resulting cofiber sequence, and if there exists a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

then $\tilde{\lambda}(f) = \tilde{\lambda}(f') + \tilde{\lambda}(\bar{f})$;
(iii) (*Wedge of circles axiom*) *If $f\colon \bigvee^k S^1 \to \bigvee^k S^1$ is a map, $k \geq 1$, then*

$$\tilde{\lambda}(f) = -(\deg(f_1) + \cdots + \deg(f_k)),$$

where $f_j = p_j f e_j$.

Recall that the famous Lefschetz Fixed Point Theorem states that if X is a sufficiently nice space and $f\colon X \to X$ is a continuous map, such that $\mathcal{L}(f) \neq 0$, where $\mathcal{L}(f)$ denotes the Lefschetz number of f, then f has a fixed point. Note that the reduced Lefschetz number is the Lefschetz number minus 1. Therefore, as an easy consequence of Theorem 2.1, we get a characterization of the Lefschetz number by three axioms.

Corollary 2.2. *The Lefschetz number $\mathcal{L}$ is the unique function λ from the set of self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the following conditions:*
(i) (*Homotopy equivalence axiom*) *If $f\colon X \to X$ is a map and $h\colon X \to Y$ is a homotopy equivalence with a homotopy inverse $k\colon Y \to X$, then $\lambda(f) = \lambda(hfk)$;*
(ii) (*Cofibration axiom*) *If A is a subcomplex of X and $A \to X \to X/A$ is the resulting cofiber sequence, and if there exists a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

then $\lambda(f) = \lambda(f') + \lambda(\bar{f}) - 1$;
(iii) (*Wedge of circles axiom*) *If $f\colon \bigvee^k S^1 \to \bigvee^k S^1$ is a map, $k \geq 1$, then*

$$\lambda(f) = 1 - (\deg(f_1) + \cdots + \deg(f_k)),$$

where $f_j = p_j f e_j$.

The difference between the system of axioms listed in Theorem 2.1 and the one from [2] is that we replace the commutativity and homotopy axioms by the homotopy equivalence axiom. To prove Theorem 2.1 we need to do some modifications in the Arkowitz and Brown approach [2]. First observe that the commutativity axiom is not used directly in the proof of [2, Theorem 1.1], but it plays an important role in proofs of [2, Lemmas 3.1, 3.2 and 3.3]. Because of the above observation, to prove Theorem 2.1 it is enough to make some changes in lemmas from [2] and then show that the homotopy axiom is a consequence of our axioms.

Observe that [2, Lemma 3.1] is the homotopy equivalence axiom. The result [2, Lemma 3.2] is needed to prove [2, Lemma 3.3]. We can rewrite it in the following form.

Lemma 2.3. *If X is contractible and $f\colon X \to X$, then $\tilde{\lambda}(f) = 0$.*

Proof. Let $\{*\}$ be a single point space and $\mathrm{id}_{\{*\}}\colon \{*\} \to \{*\}$ the unique identity map. Then by the cofibration axiom $\tilde{\lambda}(\mathrm{id}_{\{*\}}) = 0$ (see the proof of [2, Lemma 3.2]). The space X is contractible, so the constant map $h\colon X \to \{*\}$ is a homotopy equivalence. Given $x \in X$, the map $k\colon \{*\} \to X$ defined by $k(*) = x$ is a homotopy inverse of h. Now, the homotopy equivalence axiom implies $\tilde{\lambda}(f) = \tilde{\lambda}(hfk) = \tilde{\lambda}(\mathrm{id}_{\{*\}}) = 0$, because $hfk = \mathrm{id}_{\{*\}}$. ☐

The proof of [2, Lemma 3.3] needs only easy modifications, so we omit it. Now, we show that the homotopy axiom is a consequence of our axioms.

Lemma 2.4. *If $f, g\colon X \to X$ are homotopic then $\tilde{\lambda}(f) = \tilde{\lambda}(g)$.*

Proof. Let $h\colon X \times \mathbb{I} \to X \times \mathbb{I}$ be a fat homotopy between f and g, where $\mathbb{I} = [0, 1]$ is the unit interval. Write $CX = X \times \mathbb{I}/X \times \{0\}$ for the cone of X and $\bar{h}\colon CX \to CX$ for the map induced on CX by h. Then we have a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X \times \mathbb{I} & \longrightarrow & CX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{h}} \\ X & \longrightarrow & X \times \mathbb{I} & \longrightarrow & CX. \end{array}$$

The cone CX is contractible, so Lemma 2.3 leads to $\tilde{\lambda}(\bar{h}) = 0$ and we have $\tilde{\lambda}(f) = \tilde{\lambda}(h)$ by the cofibration axiom. Similarly, we show that $\tilde{\lambda}(g) = \tilde{\lambda}(h)$. Summing up we get $\tilde{\lambda}(f) = \tilde{\lambda}(g)$ and the proof is completed. ☐

Now, the proof of Theorem 2.1 can be done similarly to that of [2, Theorem 1.1]. Moreover, we can make appropriate changes in Gonçalves and Weber approach, to get the analog of [5, Theorem 3.1]. Let G be a finite group.

Theorem 2.5. *Let $\tilde{\lambda}_G$ be a function from the set of G-equivariant endomorphisms of finite based G-CW-complexes to the Burnside ring $A(G)$ that satisfies the following axioms:*

(i) (*G-Homotopy equivalence axiom*) *If* $f\colon X \to X$ *is an equivariant map and* $h\colon X \to Y$ *is a G-homotopy equivalence with G-homotopy inverse* $k\colon Y \to X$*, then* $\tilde{\lambda}_G(f) = \tilde{\lambda}_G(hfk)$*;*
(ii) (*Cofibration axiom*) *If* A *is a sub-G-CW-complex of* X *and* $A \to X \to X/A$ *is the resulting cofiber sequence, and if there exists a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

then $\tilde{\lambda}_G(f) = \tilde{\lambda}_G(f') + \tilde{\lambda}_G(\bar{f})$*;*
(iii) (*Wedge of circles axiom*) *If* $f\colon \bigvee_{i=1}^{k} G/H_i \times S^1 \to \bigvee_{i=1}^{k} G/H_i \times S^1$ *for* $k \geq 1$*, then*

$$\tilde{\lambda}_G(f) = -(\mathrm{inc}(f, c_1)[G/H_1] + \cdots + \mathrm{inc}(f, c_k)[G/H_k]),$$

where $\mathrm{inc}(f, c_i)$ *is the incidence number for the cell* c_i *with* $i = 1, \ldots, n$*. Then the function* $\tilde{\lambda}_G$ *coincides with the reduced equivariant Lefschetz number* $\tilde{\mathcal{L}}_G$*.*

3. The category of morphisms

Now we recall some basic information about the category of morphisms. More details one can find in [8] or [11]. In this section we consider only paracompact spaces.

Let X be a paracompact space. Denote by $H^k(X)$ the k-th Čech cohomology group with compact carriers of space X with coefficients in the field of rational numbers $\mathbb{Q}$. Write $H^*(X) = \{H^k(X)\}$.

A space X is called *acyclic*, if:
(i) $H^k(X) = 0$ for all $k \geq 1$;
(ii) $H^0(X) = \mathbb{Q}$.

A continuous map $p\colon X \to Y$ is called *perfect*, provided p is closed and $p^{-1}(y)$ is compact for all $y \in Y$. The following two propositions will be usefull later.

Proposition 3.1 ([12])**.** *Let* $p\colon X \to Y$ *be a perfect map. If* Y *is compact, then* X *is compact.*

Proposition 3.2 ([1])**.** *Every continuous mapping of a compact space* X *into Hausdorff space* Y *is closed.*

A continuous map $p\colon X \to Y$ is called a *Vietoris map*, provided the following conditions hold:
(i) $p\colon X \to Y$ is a perfect surjection;
(ii) the set $p^{-1}(y)$ is acyclic for all $y \in Y$.

Now, we recall some important properties of Vietoris maps.

Proposition 3.3 ([11])**.** *If* $p\colon X \to Y$ *is a Vietoris map and* $B \subseteq Y$*, then the map* $p'\colon p^{-1}(B) \to B$ *given by* $p'(x) = p(x)$ *for every* $x \in p^{-1}(B)$ *is a Vietoris map too.*

Proposition 3.4 ([11]). *If $p_1\colon X \to Y$ and $p_2\colon Y \to Z$ are Vietoris maps, then so is the composition $p_2p_1\colon X \to Z$.*

Theorem 3.5 (Vietoris, Bagle, [11]). *If X and Y are paracompact spaces and $p\colon X \to Y$ is a Vietoris map, then the induced homomorphism $p^*\colon H^*(Y) \to H^*(X)$ is an isomorphism.*

Consider a diagram

$$X_1 \xrightarrow{q} Y \xleftarrow{p} X_2$$

and its pullback

$$X_1 \xleftarrow{\bar{p}} X_1 \times_Y X_2 \xrightarrow{\bar{q}} X_2,$$

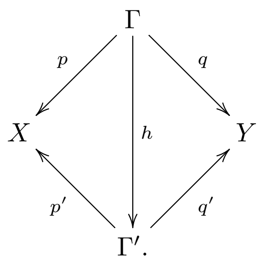
where $X_1 \times_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid q(x_1) = p(x_2)\}$ and $\bar{p}(x_1, x_2) = x_1$, $\bar{q}(x_1, x_2) = x_2$ for $(x_1, x_2) \in X_1 \times X_2$.

Proposition 3.6 ([11]). *In the situation above, if p is a Vietoris map, then $\bar{p}$ is a Vietoris map too.*

Let $D(X, Y)$ denote the set of all diagrams of the form

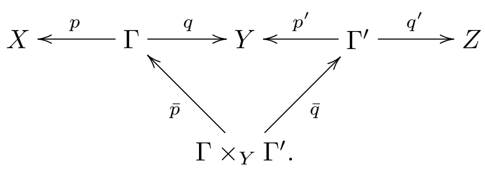
$$X \xleftarrow{p} \Gamma \xrightarrow{q} Y,$$

where p is a Vietoris map and q is a continuous map. A diagram like the above we denote by (p, q). Given two diagrams $(p, q), (p', q') \in D(X, Y)$ we write $(p, q) \simeq (p', q')$ if there exists a homeomorphism $h\colon \Gamma \to \Gamma'$ such that the following diagram is commutative



The relation $\simeq$ is an equivalence relation and elements of the quotient $M(X, Y) = D(X, Y)/ \simeq$ are called *morphisms*. If $\varphi \in M(X, Y)$ is represented by the diagram (p, q), then we write $\varphi = [p, q]$.

If $\varphi \in M(X, Y)$ and $\psi \in M(Y, Z)$ are morphisms represented by pairs (p, q) and (p', q') respectively, then we have a commutative diagram



According to Propositions 3.4 and 3.6 the composition $p\bar{p}$ is a Vietoris map. Define $\psi\varphi \in M(X, Z)$ by $\psi\varphi = [p\bar{p}, q'\bar{q}]$. This definition does not depend on the choice of the diagrams (p, q) and (p', q').

Paracompact spaces and morphisms considered above constitute a category. The cohomology functor extends over this category. A morphism $\varphi \in M(X,Y)$ induces a homomorphism $H^*(\varphi) = \varphi^*\colon H^*(Y) \to H^*(X)$ given by $\varphi^* = (p^*)^{-1}q^*$. The homomorphism φ^* does not depend on the choice of the diagram (p,q). If $\varphi \in M(X,Y)$ and $\psi \in M(Y,Z)$, then $(\psi\varphi)^* = \varphi^*\psi^*$.

Let $\mathcal{D}$ be a category of spaces of the homotopy type of finite, connected CW-complexes and morphisms in the above sense. If $X \in \mathcal{D}$ and $\varphi \in M(X,X)$, then we define the reduced Lefschetz number of φ by $\tilde{\mathcal{L}}_m(\varphi) = \tilde{\mathcal{L}}((p^*)^{-1}q^*)$, where $\tilde{\mathcal{L}}((p^*)^{-1}q^*)$ denotes the reduced Lefschetz number of the homomorphism $(p^*)^{-1}q^*$. This definition does not depend on the choice of the diagram (p,q) representing φ.

Recall that if X, Y and Γ are metric spaces, then a morphism $\varphi \in M(X,Y)$ determines a multivalued u.s.c. map $\varphi\colon X \multimap Y$ given by $\varphi(x) = q(p^{-1}(x))$, where p^{-1} denotes the multivalued inverse function of p.

If $A \subseteq X$ and $\varphi \in M(X,Y)$, then by $\varphi(A)$ we denote a subset of Y given by $\varphi(A) = q(p^{-1}(A))$.

Observe that, we can identify a single-valued map $f\colon X \to Y$ with a morphism $f \in M(X,Y)$ represented by the diagram

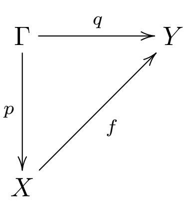
$$X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y.$$

This implies that for two singlevalued maps $f\colon Z \to X$, $g\colon Y \to Z$ and a morphism $\varphi \in M(X,Y)$, we have well defined compositions $\varphi f \in M(Z,Y)$ and $g\varphi \in M(X,Z)$.

Morphisms $\varphi_0, \varphi_1 \in M(X,Y)$ are called *homotopic* if there exists a morphism $\varphi \in M(X \times \mathbb{I}, Y)$, called a *homotopy*, such that $\varphi i_j = \varphi_j$, where $i_j\colon X \to X \times \mathbb{I}$ is given by $i_j(x) = (x,j)$ for $j = 0,1$ and $x \in X$. Then a homotopy relation is an equivalence relation in $M(X,Y)$. Denote by $M[X,Y]$ the set of all homotopy classes of $M(X,Y)$.

We have very useful properties which connect the sets $M[X,Y]$ and $[X,Y]$, where $[X,Y]$ denotes the set of all homotopy classes of singlevalued maps from X to Y.

Proposition 3.7 ([11])**.** *Let $X,Y \in \mathcal{D}$ and $\varphi \in M(X,Y)$, $\varphi = [p,q]$. If there exists a singlevalued map $f\colon X \to Y$ such that the following diagram*



is homotopically commutative (*i.e., fp and q are homotopic as single-valued maps*), *then morphisms φ and f are homotopic.*

Proposition 3.8 ([11])**.** *Let $X,Y \in \mathcal{D}$ and Y be homotopically simple* (*i.e., Y is n-simple for any $n \geq 1$*)*. If $\varphi \in M(X,Y)$, then there exists a map $f\colon X \to Y$ such that φ and f are homotopic. Therefore, there is a one-to-one correspondence $M[X,Y] \leftrightarrow [X,Y]$.*

Denote by ΣX the suspension of a space X. As an easy consequence of Propositions 3.1 and 3.2 we get the following proposition.

Proposition 3.9. *If Y is a compact space and $p\colon X \to Y$ is a Vietoris map, then $\Sigma p\colon \Sigma X \to \Sigma Y$ is a Vietoris map too.*

Let X be a compact space and $\varphi \in M(X,Y)$ be represented by a diagram

$$X \xleftarrow{\;p\;} \Gamma \xrightarrow{\;q\;} Y.$$

Then we define a *suspension* of φ by $\Sigma\varphi \in M(\Sigma X, \Sigma Y)$ which is a morphism represented by a diagram

$$\Sigma X \xleftarrow{\;\Sigma p\;} \Sigma\Gamma \xrightarrow{\;\Sigma q\;} \Sigma Y.$$

This definition does not depend on the choice of the diagram (p,q) representing φ.

4. The Proof of Theorem 1.1

The main goal of this section is to prove Theorem 1.1. First, we show that the reduced Lefschetz number on morphisms of $\mathcal{D}$ has required properties.

Proposition 4.1. *If $\varphi \in M(X,X)$ and $h\colon X \to Y$ is a homotopy equivalence with homotopy inverse $k\colon Y \to X$, then $\tilde{\mathcal{L}}_m(\varphi) = \tilde{\mathcal{L}}_m(h\varphi k)$.*

Proof. We have

$$\tilde{\mathcal{L}}_m(h\varphi k) = \tilde{\mathcal{L}}(k^*(p^*)^{-1}q^*h^*) = \tilde{\mathcal{L}}((p^*)^{-1}q^*h^*k^*) = \tilde{\mathcal{L}}((p^*)^{-1}q^*) = \tilde{\mathcal{L}}_m(\varphi).$$

Here, the second equality is a consequence of the commutativity of the trace function in the category of graded vector spaces and the third one follows from the equality $h^*k^* = \mathrm{id}_{H^*(Y)}$. □

To state the cofibration property we need the following lemma.

Lemma 4.2. *Let A be a subcomplex of X and $\varphi \in M(X,X)$ be represented by a diagram (p,q). If $\varphi(A) \subseteq A$, then φ induces morphisms $\varphi' \in M(A,A)$ and $\bar{\varphi} \in M(X/A, X/A)$ given by diagrams*

$$A \xleftarrow{\;p'\;} p^{-1}(\Gamma) \xrightarrow{\;q'\;} A,$$

$$X/A \xleftarrow{\;\bar{p}\;} \Gamma/p^{-1}(A) \xrightarrow{\;\bar{q}\;} X/A,$$

where p' and q' are restrictions of p and q respectively and $\bar{p}$ and $\bar{q}$ are maps induced by p and q on the quotient spaces.

Proof. We have to show that p' and $\bar{p}$ are Vietoris maps. Proposition 3.3 implies that p' is Vietoris. It is easy to see that $\bar{p}^{-1}(x)$ is acyclic and compact for all $x \in X/A$

(because $p^{-1}(x)$ is acyclic and compact for all $x \in X$). Now, we show that $\bar{p}$ is closed. Consider a commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{p} & X \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ \Gamma/p^{-1}(A) & \xrightarrow{\bar{p}} & X/A, \end{array}$$

where π and $\bar{\pi}$ are quotient maps. Let K be a closed subset of $\Gamma/p^{-1}(A)$. Then $Z = \bar{\pi}^{-1}(K)$ is closed in Γ because $\bar{\pi}$ is continuous. We have $\bar{p}(K) = \bar{p}\bar{\pi}(Z) = \pi p(Z)$. The set $p(Z)$ is compact because p is perfect and X is compact CW-complex. This implies that $\pi p(Z)$ is compact, so $\bar{p}(K) = \pi p(Z)$ is closed in X/A and the proof is finished. □

Proposition 4.3. *Let A be a subcomplex of X. If $\varphi \in M(X,X)$ is such that $\varphi(A) \subseteq A$, then $\tilde{\mathcal{L}}_m(\varphi) = \tilde{\mathcal{L}}_m(\varphi') + \tilde{\mathcal{L}}_m(\bar{\varphi})$, where $\varphi' \in M(A,A)$ and $\bar{\varphi} \in M(X/A, X/A)$ are induced by φ.*

Proof. If $\varphi = [p,q]$, $\varphi' = [p', q']$ and $\bar{\varphi} = [\bar{p}, \bar{q}]$, then we have a commutative diagram

$$\begin{array}{ccccc} A & \xhookrightarrow{i} & X & \xrightarrow{\pi} & X/A \\ \uparrow{\scriptstyle p'} & & \uparrow{\scriptstyle p} & & \uparrow{\scriptstyle \bar{p}} \\ p^{-1}(A) & \xhookrightarrow{\bar{i}} & \Gamma & \xrightarrow{\bar{\pi}} & \Gamma/p^{-1}(A) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \bar{q}} \\ A & \xhookrightarrow{i} & X & \xrightarrow{\pi} & X/A, \end{array}$$

where $i, \bar{i}$ are inclusions and $\pi, \bar{\pi}$ are quotient maps. This diagram induced the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \tilde{H}^0(X/A) & \to & H^0(X) & \to & H^0(A) & \to & H^1(X/A) \to \dots \\ & & \downarrow{\scriptstyle \bar{\varphi}^0} & & \downarrow{\scriptstyle \varphi^0} & & \downarrow{\scriptstyle \varphi'^0} & & \downarrow{\scriptstyle \bar{\varphi}^1} \\ 0 & \to & \tilde{H}^0(X/A) & \to & H^0(X) & \to & H^0(A) & \to & H^1(X/A) \to \dots \end{array}$$

$$\begin{array}{ccccccc} \dots \to & H^N(X/A) & \to & H^N(X) & \to & H^N(A) & \to 0 \\ & \downarrow{\scriptstyle \bar{\varphi}^N} & & \downarrow{\scriptstyle \varphi^N} & & \downarrow{\scriptstyle \varphi'^N} & \\ \dots \to & H^N(X/A) & \to & H^N(X) & \to & H^N(A) & \to 0, \end{array}$$

where N is the dimension of X. The result [2, Theorem 2.1] implies that $\tilde{\mathcal{L}}(\varphi^*) = \tilde{\mathcal{L}}(\varphi'^*) + \tilde{\mathcal{L}}(\bar{\varphi^*})$ and as a consequence we get $\tilde{\mathcal{L}}_m(\varphi) = \tilde{\mathcal{L}}_m(\varphi') + \tilde{\mathcal{L}}_m(\bar{\varphi})$. □

If $\varphi \in M(S^1, S^1)$, then we define a degree of φ by $\deg(\varphi) = \deg((p^*)^{-1}q^*)$ provided φ is represented by the pair (p, q). The proof of the following wedge of circles property is similar to the one for the single-valued case presented in [2].

Proposition 4.4. *If $\varphi \in M\left(\bigvee^k S^1, \bigvee^k S^1\right)$, then*

$$\tilde{\mathcal{L}}_m(\varphi) = -(\deg(\varphi_1) + \cdots + \deg(\varphi_k)),$$

where $\varphi_j = p_j \varphi e_j$.

Let $\mathcal{D}'$ be a full subcategory of $\mathcal{D}$ which consists of finite connected CW-complexes. Proofs of the following lemmas are similar to the suitable ones considered in [2] and Section 2.

Lemma 4.5. *If $X \in \mathcal{D}'$ is contractible and $\varphi \in M(X, X)$, then $\tilde{\lambda}(\varphi) = 0$.*

Lemma 4.6. *If $X \in \mathcal{D}'$ and $\varphi \in M(X, X)$, then $\tilde{\lambda}(\Sigma\varphi) = -\tilde{\lambda}(\varphi)$.*

Lemma 4.7. *If $X \in \mathcal{D}'$ and $\varphi, \psi \in M(X, X)$ are homotopic, then $\tilde{\lambda}(\varphi) = \tilde{\lambda}(\psi)$.*

We showed that the reduced Lefschetz number of morphisms satisfies the axioms (i)–(iii) of Theorem 1.1. As a consequence we get the following two propositions.

Proposition 4.8. *If $X \in \mathcal{D}'$ and $\varphi, \psi \in M(X, X)$ are homotopic, then $\tilde{\mathcal{L}}_m(\varphi) = \tilde{\mathcal{L}}_m(\psi)$.*

Proposition 4.9. *If $X \in \mathcal{D}'$ and $\varphi \in M(X, X)$, then $\tilde{\mathcal{L}}_m(\Sigma\varphi) = -\tilde{\mathcal{L}}_m(\varphi)$.*

Now, we are able to present the proof of the main theorem.

Proof. (**Theorem 1.1**) Propositions 4.1, 4.3 and 4.4 show that the reduced Lefschetz number of morphisms satisfies axioms (i)–(iii).

Now, suppose λ is a function from self-morphisms of spaces in $\mathcal{D}$ to the integers that satisfies axioms (i)–(iii). Because of the homotopy equivalence axiom it is enough to consider spaces from $\mathcal{D}'$. We have two cases.
(i) If X is homotopically simple, then by Proposition 3.8 there exists a singlevalued continuous map $f\colon X \to X$ which is homotopic to $\varphi\colon X \to X$. Lemma 4.7 gives $\lambda(\varphi) = \lambda(f)$. Then $\lambda(f) = \tilde{\mathcal{L}}(f)$ by axioms from [2] for singlevalued maps. Moreover, $\tilde{\mathcal{L}}(f) = \tilde{\mathcal{L}}_m(f) = \tilde{\mathcal{L}}_m(\varphi)$ by Proposition 4.8. Summing up we get $\lambda(\varphi) = \tilde{\mathcal{L}}_m(\varphi)$.
(ii) Let X be not homotopically simple. The space X is path-connected, so by a well known property we have that ΣX is 1-connected. This implies that ΣX is homotopically simply. Now, by Lemma 4.6 we have $\lambda(\varphi) = -\lambda(\Sigma\varphi)$. Using (i) we get $\lambda(\Sigma\varphi) = \tilde{\mathcal{L}}_m(\Sigma\varphi)$. Moreover, we have $\tilde{\mathcal{L}}_m(\varphi) = -\tilde{\mathcal{L}}_m(\Sigma(\varphi))$ by Proposition 4.9. Summing up we get $\lambda(\varphi) = \tilde{\mathcal{L}}_m(\varphi)$ and the proof is completed. □

5. A comment about AANRs

There is a natural question if we can characterize in an axiomatic way the Lefschetz number of self-maps in larger categories then $\mathcal{C}$. It is rather impossible to do it for all spaces of finite type, because in general there are no natural candidates of spaces which can replace spheres in the wedge of circles axiom. Nevertheless, some extensions are possible. In this section we recall the definition and basic properties of absolute approximative neighbourhood retracts (AANRs) and show that in a category $\mathcal{E}$ of spaces of the homotopy type of compact AANRs and single-valued continuous maps there is a system of axioms uniqly characterizing the reduced Lefschetz number. Note that in this case the commutativity axiom is used directly in the proof of the main theorem, so it cannot be removed. Moreover, the homotopy axiom is replaced by a stronger property, which we called the induced homomorphism axiom. In this section we use Čech homology with coefficients in the field of rational numbers.

Recall that absolute approximative neighbourhood retracts were introduced by Noguchi [13] and later generalized by Clapp [3]. Here we use the notation from [8].

Let A be a subset of X and let d be a metric in X. A map $r_\epsilon\colon X \to A$ is said to be an *ϵ-retraction*, $\epsilon > 0$, if for every $x \in A$ we have $d(x, r_\epsilon(x)) < \epsilon$. Note that the retraction $r\colon X \to A$ is an ϵ-retraction for every $\epsilon > 0$.

A subset $A \subseteq X$ is called an *approximative retract* of X provided for every $\epsilon > 0$ there exists an ϵ-retraction $r_\epsilon\colon X \to A$; A is called an *approximative neighbourhood retract* of X provided there exists an open neighbourhood U of A in X such that A is an approximative retract of U.

A compact space X is called an *absolute approximative retract* (written $X \in \mathrm{AAR}$) provided that for every embedding $h\colon X \to Y$ the set $h(X)$ is approximative retract of Y; X is called an *absolute approximative neighbourhood retract* (written $X \in \mathrm{AANR}$) provided that for every embedding $h\colon X \to Y$ the set $h(X)$ is approximative neighbourhood retract of Y.

Recall some usefull properties of AANRs and spaces of finite type.

Proposition 5.1 ([8]). *Assume that $X \in AANR$. Then there exists a compact ANR Y such that X is homeomorphic to an approximative retract of Y.*

Proposition 5.2 ([8]). *Let (X, d) be a compact metric space of a finite type. Then there exists $\epsilon > 0$ such that for every compact space Y and for every two maps $f, g\colon X \to Y$, if $d(f(x), g(x)) < \epsilon$, for all $x \in X$, then $f_* = g_*$.*

Proposition 5.3 ([8]). *If X is a compact AANR, then X is a space of finite type.*

Note that because of Proposition 5.3 the Lefschetz number is well defined for self-maps in $\mathcal{E}$. Now we are to able formulate the main theorem of this section.

Theorem 5.4. *The reduced Lefschetz number $\tilde{\mathcal{L}}$ is the unique function $\tilde{\lambda}$ from the set of self-maps of spaces in $\mathcal{E}$ to the integers that satisfies the following conditions:*
(i) (*Induced homomorphism*) *If $f, g\colon X \to X$ are such that $f_* = g_*$, then $\tilde{\lambda}(f) = \tilde{\lambda}(g)$;*
(ii) (*Cofibration axiom*) *If A is a subcomplex of X and $A \to X \to X/A$ is the resulting*

cofiber sequence, and if there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

then $\tilde{\lambda}(f) = \tilde{\lambda}(f') + \tilde{\lambda}(\bar{f})$;
(iii) (*Wedge of circles axiom*) *If* $f\colon \bigvee^k S^1 \to \bigvee^k S^1$ *is a map,* $k \geq 1$, *then*

$$\tilde{\lambda}(f) = -(\deg(f_1) + \dots + \deg(f_k)),$$

where $f_j = p_j f e_j$;
(iv) (*Commutativity axiom*) *If* $f\colon X \to Y$ *and* $g\colon Y \to X$ *are maps, then* $\tilde{\lambda}(gf) = \tilde{\lambda}(fg)$.

Proof. (Sketch) Of course, the Lefschetz number satisfies axioms (i)–(iv). Note, that the induced homomorphism axiom and the commutativity axiom implies the homotopy equivalence property. Let $X \in \mathcal{E}$ and $f\colon X \to X$. Because of Proposition 5.1 it is enough to consider the case where X is a compact AANR and there exists $Y \in \mathcal{C}$ such that X is an approximative retract of Y. Let $s\colon X \to Y$ be an inclusion. Propositions 5.2 and 5.3 imply that there exist $\epsilon > 0$ and an ϵ-retraction $r_\epsilon\colon Y \to X$ such that $(r_\epsilon s)_* = (\mathrm{id}_Y)_*$. Now using the induced homomorphism axiom and the commutativity axiom we get

$$\tilde{\lambda}(f) = \tilde{\lambda}(\mathrm{id}_Y f) = \tilde{\lambda}(r_\epsilon s f) = \tilde{\lambda}(s f r_\epsilon) = \tilde{\mathcal{L}}(s f r_\epsilon) = \tilde{\mathcal{L}}(r_\epsilon s f) = \tilde{\mathcal{L}}(\mathrm{id}_Y f) = \tilde{\mathcal{L}}(f),$$

where $\tilde{\lambda}(s f r_\epsilon) = \tilde{\mathcal{L}}(s f r_\epsilon)$, because $Y \in \mathcal{C}$ and axioms (i)–(iv) uniquely characterize the Lefschetz number in $\mathcal{C}$. □

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