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COMMON FIXED POINT THEOREMS VIA MEASURE OF NONCOMPACTNESS

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Abstract. In this paper, we provide sufficient conditions for the existence of common fixed point for two commuting operators with the technique associated with measure of noncompactness in Banach spaces. Our results generalize Darbo's fixed point theorem and also some fixed point theorems which were recently proved by some authors [2, 13].
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1. Introduction

Measure of noncompactness is one of the fruitful tools in nonlinear analysis which extensively applied to investigate the existence of solutions to nonlinear integral equations via Darbo's fixed point theorem (see, for example, [3, 4, 6, 8, 14, 15]).

In a recent paper, Aghajani et al [2], by modeling Caristi's fixed point theorem with measure of noncompactness in Banach spaces, obtained some generalization of Darbo's theorem.

Also, Hajji [12, 13] proved some common fixed point theorems for a pair of commuting operators that generalize Darbo's, Sadovski's and Markoff-Kakutani's fixed point theorems.

Motivated by above works, in this paper, we consider some common fixed point results which include theorem [2, 13] as corollary.

This paper is organized as follows: in section (2), some facts and results about measure of noncompactness and related theorems are given and in section (3) in spire of [12] we prove the main result and some corollaries.

2. Preliminaries

In this section, we present some definitions and results which will be needed later.

Let $(E, \|.\|)$ be a Banach space. We write $B(x,r)$ to denote the closed ball centered at x with radius r. Moreover, let $\mathfrak{M}_E$ be the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ be its subfamily consisting of all relatively compact sets.
We mention the following definition of the measure of noncompactness, given in [7].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}$ is said to be a measure of noncompactness in E if it satisfies the following conditions:
(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_E$.
(2) $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(X)$.
(4) $\mu(convX) = \mu(X)$.
(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0,1]$.
(6) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, ...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty and $X_\infty \in \ker \mu$.

Now, we mention the following two theorems stated in [1, 7].

Theorem 2.2. *(Schauder's fixed point theorem) Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

Theorem 2.3. *(Darbo's fixed point theorem) Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that $\mu(TX) \leq k\mu(X)$ for all nonempty subset X of Ω, where $k \in [0,1)$ is a constant. Then T has a fixed point in the set Ω.*

Definition 2.4. [11] A mapping T of a convex set M is said to be affine if it satisfies the identity

$$T(kx + (1-k)y) = kTx + (1-k)Ty,$$

whenever $0 < k < 1$, and $x, y \in M$.

3. Main results

In this section, we prove some common fixed point theorems with the technique associated with measure of noncompactness for two commuting operators in Banach spaces. Our results extend Darbo's fixed point theorem and also some fixed point theorems which were recently proved by some authors [2, 13].

Theorem 3.1. *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and T, S be two continuous operators from Ω into Ω such that*

a) $TS = ST$.

b) *For any $M \subset \Omega$,*

$$T(\overline{conv}(M)) \subset \overline{conv}(T(M)).$$

c) *For any $M \subset \Omega$,*

$$\psi(\mu(S(M))) \leq \psi(\mu(T(M))) - \phi(\mu(T(M))),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous, monotone nondecreasing and* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is lower semicontinuous and monotone decreasing mapping such that* $\phi(0) = 0,\ \phi(t) > 0$ *for* $t > 0$.
Then,

1) *The set* $\{x \in \Omega : S(x) = x\}$ *is nonempty and compact.*
2) T *has a fixed point and the set* $\{x \in \Omega : T(x) = x\}$ *is closed and invariant by* S.
3) *If* T *is affine then* T *and* S *have a common fixed point and the set* $\{x \in \Omega : T(x) = S(x) = x\}$ *is compact.*

Remark 3.2. If T is the identity function, $\psi(t) = t$ and $\phi(t) = (1-k)t$ $(0 < k < 1)$, then theorem (3.1) is a generalization of Darbo's fixed point theorem.

Proof. Consider the sequence (Ω_n) by $\Omega_0 = \Omega$ and $\Omega_n = \overline{conv} S(\Omega_{n-1})$ for $n = 1, 2, \ldots$. By induction, we show that

$$T(\Omega_n) \subset \Omega_n,\ (n = 1, 2, \ldots). \tag{3.1}$$

For $n = 1$, $T(\Omega_1) \subset \overline{conv}(S(T(\Omega_0)))) \subset \overline{conv}(S(\Omega_0))) = \Omega_1$.
Now, suppose (3.1) satisfies for $(n \geq 1)$, then

$$T(\Omega_{n+1}) = T(\overline{conv}(S(\Omega_n))) \subset \overline{conv}(S(T\Omega_n))) \subset \overline{conv}(S(\Omega_n))) = \Omega_{n+1}.$$

If there exists a number $n \in \mathbb{N}$ such that $\mu(\Omega_n) = 0$, then Ω_n is compact and by theorem (2.2) S has a fixed point in Ω.
So suppose $\mu(\Omega_n) > 0$, by properties of two functions ψ, ϕ, we have

$$\begin{aligned}\psi(\mu(\Omega_{n+1})) &= \psi(\mu(\overline{conv}(S(\Omega_n)))) \\ &= \psi(\mu(S(\Omega_n))) \leq \psi(\mu(T(\Omega_n))) - \phi(\mu(T(\Omega_n))) \\ &\leq \psi(\mu(\Omega_n)) - \phi(\mu(\Omega_n)).\end{aligned} \tag{3.2}$$

It follows that the sequence $\mu(\Omega_n)$ is monotone decreasing and consequently there exists $r \geq 0$ such that $\mu(\Omega_n) \to r$.
Letting $n \to \infty$ in (3.2) we obtain $\psi(r) \leq \psi(r) - \phi(r)$, which is a contradiction unless $r = 0$. Hence $\mu(\Omega_n) \to 0$.
Also, since $\Omega_{n+1} \subset \Omega_n$, by property (6) of definition (2.1) $C = \cap_{n=1}^{\infty} \Omega_n$ is nonempty and compact.
Moreover, since each Ω_n is convex, C is convex, and

$$S(\Omega_n) \subset S(\Omega_{n-1}) \subset \overline{conv}(S(\Omega_{n-1}))) = \Omega_n.$$

Hence $S : \Omega_n \to \Omega_n$ for $n = 0, 1, 2\ldots$, and so $S : C \to C$. Now, by Theorem (2.2), it follows that S has a fixed point. Thus the set

$$F = \{x \in \Omega\ s.t\ Sx = x\}$$

is closed.
Also, by commutativity of two operator T, S we have $S(Tx) = T(Sx) = Tx$. Thus $T(F) \subset F$ and since

$$\psi(\mu(F)) = \psi(\mu(S(F))) \leq \psi(\mu(T(F))) - \phi(\mu(T(F))) \leq \psi(\mu(F)) - \phi(\mu(F)).$$

Thus $\mu(F) = 0$ and hence F is compact. So part (1) proved.

2) Similarly as for S in (1), T has a fixed point and by continuity of T, $K = \{x \in \Omega \ s.t \ Tx = x\}$ is closed. Also, we have Sx is a fixed point of T for each $x \in K$, Therefore, K is invariant by S.
3) First note that K is closed, bounded and since T is affine, K is convex. Also, $S(K) \subset K$ and $T(K) \subset K$ and for any $M \subset K$ and we have

$$\psi(\mu(S(M))) \leq \psi(\mu(T(M))) - \phi(\mu(T(M))).$$

Then by part 1) S has a fixed point in K, therefore S and T have a common fixed point, now since S is continuous and by the hypothesis (c), the set of common fixed point of S and T is a compact. So the proof complete. □

As a consequence of Theorem (3.1), we have the following result.

Corollary 3.3. [2] *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and S be a continuous operator from Ω into Ω such that*

$$\psi(\mu(S(M))) \leq \psi(\mu(M)) - \phi(\mu(M)), \ \forall M \subset \Omega.$$

Where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semicontinuous mapping such that $\phi(0) = 0, \phi(t) > 0$ for $t > 0$.
Then, the set $\{x \in \Omega : S(x) = x\}$ is nonempty and compact.

Theorem 3.4. *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and T, S be two continuous operators from Ω into Ω such that*

a) $TS = ST$.
b) *For any $M \subset \Omega$,*

$$T(\overline{conv}(M)) \subset \overline{conv}(T(M)).$$

c) *For any $M \subset \Omega$,*

$$\mu(S(M))) \leq \psi(\mu(T(M))),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone nondecreasing mapping such that for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$.
Then,

1) *The set $\{x \in \Omega : S(x) = x\}$ is nonempty and compact.*
2) *T has a fixed point and the set $\{x \in \Omega : T(x) = x\}$ is closed and invariant by S.*
3) *If T is affine then T and S have a common fixed point and the set $\{x \in \Omega : T(x) = S(x) = x\}$ is compact.*

Note 1. Note that $\lim_{n\to\infty} \psi^n(t) = 0$ implies that $\psi(t) < t$, indeed suppose $t \leq \psi(t)$ for some $t > 0$, then $\psi(t) \leq \psi(\psi(t))$ and therefore $t \leq \psi^2(t)$. By induction, $t \leq \psi^n(t)$ for $n \in \{1, 2, ...\}$. This is a contradiction. Thus $\psi(t) < t$ for each $t > 0$.

Proof. Consider the sequence (Ω_n) by $\Omega_0 = \Omega$ and $\Omega_n = \overline{conv}S(\Omega_{n-1})$ for $n = 1, 2, ...$. By induction, for $n = 1, 2, ...$ we have

$$T(\Omega_n) \subset \Omega_n, \ \mu(\Omega_n) \leq \psi^n(\mu(\Omega_0)). \tag{3.3}$$

The relation $T(\Omega_n) \subset \Omega_n$ was proved in theorem (3.1), but for the second part we have

$$\begin{aligned}\mu(\Omega_1) &= \mu(\overline{conv}(S(\Omega_0))) \\ &= \mu(S(\Omega_0)) \leq \psi(\mu(T(\Omega_0))) \leq \psi(\mu(\Omega_0)).\end{aligned}$$

Suppose that (3.3) is true for $n-1$, Thus

$$\begin{aligned}\mu(\Omega_n) &= \mu(\overline{conv}(S(\Omega_{n-1}))) = \mu(S(\Omega_{n-1})) \\ &\leq \psi(\mu(T(\Omega_{n-1}))) \leq \psi(\mu(\Omega_{n-1})) \leq \psi(\psi^{n-1}(\mu(\Omega_0))) = \psi^n(\mu(\Omega_0)).\end{aligned}$$

Taking into account our assumptions we have $\mu(\Omega_n) \to 0$.
Also, since $\Omega_{n+1} \subset \Omega_n$, by property (6) of definition (2.1) $C = \cap_{n=1}^{\infty}\Omega_n$ is nonempty and compact.
Moreover, since each Ω_n is convex, C is convex, and

$$S(\Omega_n) \subset S(\Omega_{n-1}) \subset \overline{conv}(S(\Omega_{n-1}))) = \Omega_n.$$

Hence $S : \Omega_n \to \Omega_n$ for $n = 0, 1, 2...$, and so $S : C \to C$. Now, by Theorem (2.2), it follows that S has a fixed point. Thus the set

$$F = \{x \in \Omega \ s.t \ Sx = x\}$$

is closed.
Also, by commutativity of two operator T, S we have $S(Tx) = T(Sx) = Tx$. Thus $T(F) \subset F$ and since

$$\mu(F) = \mu(S(F)) \leq \psi(\mu(T(F))) \leq \psi(\mu(F)).$$

Thus by Note (1), $\mu(F) = 0$ and hence F is compact. So part (1) proved.
The other parts can be proved as in the proof of theorem (3.1). □

Theorem 3.5. *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and T, S be two continuous operators from Ω into Ω such that:*

a) *$TS = ST$.*
b) *T is affine.*
c) *For any $M \subset \Omega$,*

$$\psi(\mu(ST(M))) \leq \psi(\mu(T(M))) - \phi(\mu(T(M))),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone nondecreasing and
$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous and monotone decreasing mapping such that $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$.
Then T and S have a common fixed point.

Proof. Consider the operator $H(x) = ST(x)$. It is clear that H maps Ω into Ω and $HT = TH$ and H is continuous.
Also, by hypothesis,

$$\psi(\mu(H(M))) = \psi(\mu(ST(M))) \leq \psi(\mu(T(M))) - \phi(\mu(T(M))),$$

so by theorem (3.1) H and T have a common fixed point i.e

$$F = \{x \in \Omega \mid Hx = Tx = x\}$$

is nonempty and compact, thus for $x \in F$ we have

$$x = H(x) = ST(x) = S(x),$$

hence, S and T have a common fixed point. □

Corollary 3.6. [13] *Let E be a Banach space, Ω be a nonempty, closed, convex and bounded subset of E and T, S be two continuous operators from Ω into Ω such that:*

a) $TS = ST$.
b) *T is affine.*
c) *For any $M \subset \Omega$,*

$$\mu(ST(M)) \leq k\mu(T(M))),$$

where $k \in [0, 1)$ is a constant. Then T and S have a common fixed point.

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