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AROUND PEROV'S FIXED POINT THEOREM FOR MAPPINGS ON GENERALIZED METRIC SPACES

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Abstract. We revisit Perov's fixed point theorem for selfmaps of a set endowed with a vector metric taking values in the Euclidean space $\mathbb{R}^m$. In particular, we show that this result is subsumed by the classical Banach contraction principle. We also obtain a generalization of Perov's theorem by considering mappings on K-metric spaces satisfying a nonlinear Lipschitz condition. Two applications are presented and some characterizations of convergence in K-metric spaces are given.

Key Words and Phrases: Generalized metric space, Perov's fixed point theorem, K-metric space, cone metric space, spectral radius, Cauchy initial value problem, solid cone, normal cone, nonlinear Lipschitz condition.

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1. Introduction

In 1964 Perov [20] obtained a generalization of the Banach contraction principle by considering mappings on the so-called generalized metric spaces. A generalized metric space is a pair (X, d), where X is a nonempty set and d is a function from the Cartesian product $X \times X$ to the Euclidean space $\mathbb{R}^m$ satisfying three well-known axioms of a metric with respect to the following partial ordering $\preceq$ in $\mathbb{R}^m$: $(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_m)$ iff $a_i \leq b_i$ for $i = 1, \ldots, m$. Then a classical contractive condition for a mapping $T\colon X \to X$ is replaced by

$$d(Tx, Ty) \preceq A\, d(x, y) \quad \text{for} \quad x, y \in X,$$

where A is an $m \times m$ matrix with nonnegative entries such that the spectral radius of A is less than one. Many authors, mainly from the former Soviet Union, extended Perov's result by considering yet more general class of spaces – the so-called K-metric spaces, in which a function d takes values in a cone in a Banach space. A survey paper of Zabreĭko [27] can serve as an excellent reference on this topic. In 2007 Huang and Zhang [9] rediscovered the notion of K-metric space under the name 'cone metric space', and they establish a fixed point theorem for mappings satisfying a contractive condition of the following form:

$$d(Tx, Ty) \preceq \lambda d(x, y) \quad \text{for} \quad x, y \in X,$$

where $\preceq$ is a partial ordering induced by a cone in a Banach space and $\lambda \in [0, 1)$. Since then a number of papers appeared containing various extensions of the Huang–Zhang theorem (see, e.g., a survey paper of Janković et al. [11] and 100 references therein). In 2010 Du [5] observed that the above theorem could be proved via the classical Banach contraction principle by introducing an appropriate scalar metric induced by a K-metric. Other constructions of such scalar metrics were given in [12, 13]. Yet another approach was presented in [11], where the authors showed that all earlier fixed point results obtained for mappings on K-metric spaces with a normal and solid cone K could be derived from the corresponding results for mappings on (scalar) metric spaces. Thus this part of fixed point theory may be interesting only if we work with K-metric spaces with a non-normal or non-solid cone K.

However, most of papers which appeared after 2007 deal with contractive conditions in which values of a vector metric d are multiplied by scalars, whereas in Perov's theorem there is a linear operator (induced by a matrix A), which acts on values of d. Thus the Huang–Zhang theorem for mappings on generalized metric spaces is identical with the special case of Perov's theorem in which $A = \lambda I$, where I denotes the identity operator on $\mathbb{R}^m$. However, the last condition for A is very restrictive. Actually, the power of Perov's theorem lies in a fact that even a norm of matrix A need not be less than one, which is very important for applications as presented in Section 4. A natural question arises whether Perov's theorem can also be derived from the Banach contraction principle. This question was posed by Jonathan Borwein during the "Workshop on Infinite Products of Operators and Their Applications" in Haifa in 2012 after the talk of Adrian Petruşel on extensions of Perov's theorem. With the help of our remetrization theorem presented on the same Workshop, we could answer this question in the affirmative. However, a few months later we found that such an observation had been made in 2002 by E. De Pascale and L. De Pascale [4]. Since rather a sketch of a proof was given in [4] and our argument is somewhat different, we decided to present our proof in this paper (see Section 3). We also extend that argument to obtain in Section 6 a generalization of Perov's theorem for mappings on K-metric spaces with a non-solid cone K. Section 5 contains characterizations of two notions of convergence and Cauchy's condition for sequences of elements of a K-metric space.

2. Generalized metric spaces in the sense of Perov

Though some results presented below can be established in a more general context of K-metrics or cone metrics, in this section we restrict our attention to vector metrics with values in the Euclidean space $\mathbb{R}^m$. This may be convenient for the reader who had no occasion yet to work with vector metrics. Moreover, it is natural to study first a finite dimensional case.

We start with the definition of a *generalized metric space* in the sense of Perov. Recently, this notion has been recalled by Petruşel et al. [22]. Actually, Perov [20] (see also Perov and Kibenko [21]) considered *generalized normed spaces*, i.e., linear spaces endowed with a vector norm taking values in the Euclidean space $\mathbb{R}^m$, which is equipped with the following partial ordering: for $a, b \in \mathbb{R}^m$, $a = (a_1, \ldots, a_m)$,

$b = (b_1, \ldots, b_m)$,

$$a \preceq b \quad \text{iff} \quad a_i \leq b_i \quad \text{for} \quad i = 1, \ldots, m. \tag{2.1}$$

If $a \preceq b$ and $a \neq b$, then we write $a \prec b$. The notation $a \ll b$ means that $a_i < b_i$ for $i = 1, \ldots, m$. A natural extension of Perov's notion of vector norms is the following definition. Let X be a nonempty set and $d\colon X \times X \to \mathbb{R}^m$ be a mapping satisfying conditions: for any $x, y, z \in X$,

$$d(x,y) \succeq \Theta, \quad \text{and} \quad d(x,y) = \Theta \quad \text{iff} \quad x = y; \tag{2.2}$$

$$d(x,y) = d(y,x); \tag{2.3}$$

$$d(x,y) \preceq d(x,z) + d(z,y). \tag{2.4}$$

(In the sequel we write simply 0 instead of Θ.) Then as in [22] we say that (X, d) is a generalized metric space (in short, g.m.s.) and d is called Perov's metric. Following [21] we say that a sequence (x_n) of elements of X is convergent to $x \in X$ if for any $c \in \mathbb{R}^m$ with $c \gg 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for $n \geq k$. Similarly, we define a Cauchy sequence. Then the completeness of (X, d) is understood in an analogous way as in the case of real-valued metrics. Let us note that Perov's metric is a very particular case of the so-called K-metric (see, e.g., [27] and the references therein), which in turn was rediscovered by Huang and Zhang [9] under the name '*cone metric*'. More information on it will be given in the next section.

Recall that a function $\rho\colon X \times X \to [0, \infty)$ is a *pseudometric* if for any $x, y, z \in X$, $\rho(x,x) = 0$, $\rho(x,y) = \rho(y,x)$ and $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$. We use this notion in the following characterization of Perov's metrics.

Proposition 2.1. *Let X be a nonempty set and $d\colon X \times X \to \mathbb{R}^m$ be a mapping, so that $d = (\rho_1, \ldots, \rho_m)$, where $\rho_k\colon X \times X \to \mathbb{R}$ for $k = 1, \ldots, m$. Then (X, d) is a g.m.s. iff $\{\rho_1, \ldots, \rho_m\}$ is a separating family of pseudometrics, i.e., for any $x, y \in X$, if $x \neq y$ then $\rho_i(x,y) > 0$ for some $i \in \{1, \ldots, m\}$.*

Proof. Implication ($\Leftarrow$) is noted in [26, Remark 6.1.1]. Also, it is easy to prove ($\Rightarrow$): conditions (2.3) and (2.4) for d imply, respectively, the symmetry and the triangle inequality for each ρ_i. By (2.2), all functions ρ_i are nonnegative. Moreover, if $x, y \in X$ and $x \neq y$, then by (2.2), $d(x,y) \neq 0$, which yields that $\rho_i(x,y) > 0$ for some $i \in \{1, \ldots, m\}$. □

We omit a straightforward proof of the following

Proposition 2.2. *Let (X, d) be a g.m.s. and $\rho_1, \ldots, \rho_m$ be pseudometrics associated with d. Let $x_n \in X$ for $n \in \mathbb{N}$ and $x \in X$. The following statements hold:*

1. *(x_n) is convergent to x iff $\lim_{n\to\infty} \rho_k(x_n, x) \to 0$ for each $k = 1, \ldots, m$;*
2. *(x_n) is a Cauchy sequence iff $\lim_{i,j\to\infty} \rho_k(x_i, x_j) = 0$ for each $k = 1, \ldots, m$.*

We say that two metrics (scalar-valued or vector-valued) d and ρ in X are *Cauchy equivalent* if they define the same notion of a Cauchy sequence, i.e., for any sequence (x_n) of elements of X, (x_n) is a Cauchy sequence in (X, d) iff (x_n) is a Cauchy sequence in (X, ρ).

Proposition 2.3. *Let (X, d) and (X, ρ) be (generalized) metric spaces such that d and ρ are Cauchy equivalent. The following statements hold:*

(1) *d and ρ are equivalent;*
(2) *(X, d) is complete iff (X, ρ) is complete.*

Proof. Let us consider only the case when d is a vector metric and ρ is a scalar metric. Assume that $x_n \in X$ for $n \in \mathbb{N}$, $x \in X$ and $\rho(x_n, x) \to 0$. For $n \in \mathbb{N}$, define $y_{2n-1} := x_n$ and $y_{2n} := x$. Clearly, $\rho(y_n, x) \to 0$, which by hypothesis implies that (y_n) is a Cauchy sequence in (X, d). Let $\rho_1, \ldots, \rho_m$ be pseudometrics associated with d. By Proposition 2.2, we infer that $\lim_{n\to\infty} \rho_k(y_{2n-1}, y_{2n}) = 0$ for $k = 1, \ldots, m$, i.e., $\lim_{n\to\infty} \rho_k(x_n, x) = 0$ for all such k, which again by Proposition 2.2 implies that (x_n) is convergent in (X, d).

A similar argument shows that if (x_n) converges to x in (X, d), then $\rho(x_n, x) \to 0$. The second statement is now obvious. □

Of course, there exist equivalent metrics, which are not Cauchy equivalent. (For example, define for $x, y \in \mathbb{R}$, $d(x, y) := |x - y|$ and $\rho(x, y) := |\arctan x - \arctan y|$.)

We say that two scalar metrics d and ρ are *Lipschitz equivalent* (see, e.g., [2]) if there exist positive constants α and β such that for any $x, y \in X$,

$$\alpha\rho(x, y) \leq d(x, y) \leq \beta\rho(x, y).$$

Clearly, if d and ρ are Lipschitz equivalent, then they are Cauchy equivalent, but in general the converse is not true. (Consider, e.g., metrics $d(x, y) := |x - y|$ and $\rho(x, y) := |x^3 - y^3|$ for $x, y \in \mathbb{R}$.)

Now let us consider the following three standard norms in $\mathbb{R}^m$:
for $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$,

$$\| a \|_1 := \sum_{i=1}^{m} |a_i|, \quad \| a \|_2 := \left(\sum_{i=1}^{m} a_i^2\right)^{1/2} \quad \text{and} \quad \| a \|_\infty := \max_{1\leq i\leq m} |a_i|.$$

It is easily seen that each of these norms is monotone with respect to the partial ordering defined by (2.1) if we restrict to vectors in $\mathbb{R}^m$ with nonnegative coordinates, i.e.,

$$\text{if} \quad a, b \in \mathbb{R}^m \quad \text{and} \quad 0 \preceq a \preceq b \quad \text{then} \quad \| a \|_l \leq \| b \|_l \quad \text{for} \quad l = 1, 2, \infty.$$

Hence we may infer that if (X, d) is a g.m.s. and for $x, y \in X$ and $l = 1, 2, \infty$,

$$\rho^{(l)}(x, y) := \| d(x, y) \|_l,$$

then each $\rho^{(l)}$ is a metric in X. Moreover, Proposition 2.2 yields that d, $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(\infty)}$ are Cauchy equivalent. Consequently, we get the following

Corollary 2.4. *Let (X, d) be a g.m.s. and $\rho^{(1)}$, $\rho^{(2)}$, $\rho^{(\infty)}$ be metrics associated with d. If one of the four spaces is complete, then all of them are complete.*

Now we explain the reason for which relation '$\ll$' in the definition of convergence or the Cauchy condition should not be replaced by relation '$\prec$'.

Proposition 2.5. *Let (X, d) be a g.m.s. with $m \geq 2$ and (x_n) be a sequence of elements of X satisfying the following condition: for some $x \in X$ and for any $c \in \mathbb{R}^m$ with $c \succ 0$, there is $k \in \mathbb{N}$ such that*

$$d(x_n, x) \prec c \quad \textit{for each} \quad n \geq k.$$

Then $x_n = x$ for sufficiently large n, i.e., (x_n) converges to x in the discrete topology of X.

Proof. Set $c := (1, 0, \ldots, 0)$, $c \in \mathbb{R}^m$. Clearly, $c \succ 0$, so by hypothesis, there is $k \in \mathbb{N}$ such that for $n \geq k$, $d(x_n, x) \prec c$, which yields that $\rho_i(x_n, x) = 0$ for $i = 2, \ldots, m$ and $n \geq k$. Next set $c' := (0, 1, \ldots, 1)$, $c' \in \mathbb{R}^m$. There is $k' \in \mathbb{N}$ such that for $n \geq k'$, $d(x_n, x) \prec c'$, which implies $\rho_1(x_n, x) = 0$ for such n. So $x_n = x$ for $n \geq \max\{k, k'\}$ since family $\{\rho_1, \ldots, \rho_m\}$ is separating. □

Of course, an analogous modification of the definition of Cauchy's condition for sequences would also make this definition trivial.

3. Perov's fixed point theorem via the Banach contraction principle

We recall Perov's fixed point theorem (see [20, 21]), which originally was established for selfmaps of closed subsets of generalized Banach spaces. We denote by $M_m(\mathbb{R}_+)$ the set of all $m \times m$ matrices with nonnegative entries. Given $A \in M_m(\mathbb{R}_+)$, $r(A)$ denotes the *spectral radius* of A (see, e.g., [6, p. 149]), i.e.,

$$r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the set of all complex eigenvalues of A. It is well known that

$$r(A) = \lim_{n\to\infty} \| A^n \|^{1/n}$$

(see, e.g., [6, Theorem 14.16]). Hence, if $r(A) < 1$ then by Cauchy criterion, the series $\sum_{n=1}^{\infty} \| A^n \|$ converges; in particular, $\| A^n \| \to 0$. Actually, the latter condition is equivalent to '$r(A) < 1$' because of the inequality $r(A) \leq \| A^n \|^{1/n}$ (see [6, Lemma 14.14]).

The notion of the spectral radius can be considered in a more general setting. Namely, observe that each $m \times m$ matrix induces a linear selfmap of $\mathbb{R}^m$, which is automatically Lipschitzian. In fact, we may attribute a spectral radius to any Lipschitzian selfmap T of a metric space (X, ρ) in the following way. Denote by $L_\rho(T)$ the Lipschitz constant of T. Then the spectral radius $r_\rho(T)$ is defined by the formula:

$$r_\rho(T) := \lim_{n\to\infty} (L_\rho(T^n))^{1/n}.$$

(It is known that the above limit exists; see, e.g., [8, p. 10].) Moreover, the following result was proved by Goebel [7] (also, see [8, p. 11]).

Lemma 3.1 (Goebel)**.** *Let T be a Lipschitzian selfmap of a metric space (X, ρ). Then*

$$r_\rho(T) = \inf\{L_{\rho'}(T) : \rho' \textit{ is Lipschitz equivalent to } \rho\}.$$

Lemma 3.1 will be used in our proof of the following

Theorem 3.2 (Perov)**.** *Let (X, d) be a complete g.m.s. and $T \colon X \to X$ be a mapping. If there exists a matrix $A \in M_m(\mathbb{R}_+)$ such that $r(A) < 1$ and for any $x, y \in X$,*

$$d(Tx, Ty) \preceq A(d(x, y)), \tag{3.1}$$

then T has a unique fixed point $x_* \in X$ and for any $x_0 \in X$, $x_* = \lim_{n\to\infty} T^n x_0$.

It turns out that Perov's theorem can be derived from the classical Banach contraction principle in view of the following

Theorem 3.3. *Under the assumptions of Perov's theorem, for any* $\lambda \in (r(A), 1)$ *there exists a scalar metric* ρ_λ*, which has the following properties:*

(1) ρ_λ *and* $\rho^{(1)}$ *are Lipschitz equivalent, hence* ρ_λ *and* d *are Cauchy equivalent;*
(2) (X, ρ_λ) *is complete;*
(3) $\rho_\lambda(Tx, Ty) \le \lambda\rho_\lambda(x, y)$ *for any* $x, y \in X$.

Proof. Since the matrix A has nonnegative entries, we may infer that A is nondecreasing with respect to $\preceq$ on the set of all vectors in $\mathbb{R}^m$ having nonnegative coordinates, i.e., if $a, b \in \mathbb{R}^m$ and $0 \preceq a \preceq b$ then $0 \preceq Aa \preceq Ab$. Hence (3.1) implies that for any $n \in \mathbb{N}$ and $x, y \in X$,

$$d(T^n x, T^n y) \preceq A^n(d(x, y)).$$

Since norm $\| \cdot \|_1$ is monotone with respect to $\preceq$ and A^n is a linear bounded operator on $\mathbb{R}^m$, we obtain that

$$\| d(T^n x, T^n y) \|_1 \le \| A^n(d(x, y)) \|_1 \le \| A^n \|_{1,1} \| d(x, y) \|_1 .$$

(Here $\| A \|_{1,1} = \max_{1 \le j \le m} \sum_{i=1}^m |a_{ij}|$; see, e.g., [6, Lemma 7.14].) Hence $L_{\rho^{(1)}}(T^n) \le \| A^n \|_{1,1}$ for any $n \in \mathbb{N}$, which yields $r_{\rho^{(1)}}(T) \le r(A)$. Now let $\lambda \in (r(A), 1)$. Then $r_{\rho^{(1)}}(T) < \lambda$, so by Lemma 3.1,

$$\inf\{L_\rho(T) : \rho \ \text{ is Lipschitz equivalent to } \rho^{(1)}\} < \lambda.$$

Hence there exists a metric ρ_λ, which is Lipschitz equivalent to $\rho^{(1)}$ and $L_{\rho_\lambda}(T) < \lambda$. Thus properties 1 and 3 hold, whereas 2 follows from 1 in view of Proposition 2.3. □

Finally, we give the following equivalent formulation of Perov's theorem, which will be useful in the next section.

Theorem 3.4 (Perov)**.** *Let* (X, d) *be a complete g.m.s. and* $T \colon X \to X$ *be a mapping. Let* $\rho_1, \ldots, \rho_m$ *be pseudometrics associated with* d*. If there exists a matrix* $A = [a_{ij}] \in M_m(\mathbb{R}_+)$ *such that* $r(A) < 1$ *and for any* $x, y \in X$,

$$\rho_i(Tx, Ty) \le \sum_{j=1}^m a_{ij}\rho_j(x, y) \quad \textit{for} \quad i = 1, \ldots, m, \tag{3.2}$$

then T *has a unique fixed point* $x_* \in X$ *and for any* $x_0 \in X$, $x_* = \lim_{n\to\infty} T^n x_0$.

4. Two applications of Perov's fixed point theorem

We start with recalling Matkowski's [18] fixed point theorem for selfmaps of the Cartesian product of metric spaces. It is known (but rather not well-known) that Matkowski's result can be derived from Perov's theorem. In particular, it was noticed without any details in [15, p. 40], where only the reference to the book of Krasnoselskiĭ et al. [14] was given. Here we would like to explain more exactly the connections between the two theorems. Let us note that Păvăloiu [19] and Rus [25] obtained the following result in case $m = 2$ by using different methods.

Theorem 4.1 (Matkowski). *Let $m \in \mathbb{N}$ and (X_1, d_1),..., (X_m, d_m) be complete metric spaces, $X := X_1 \times \cdots \times X_m$, and $T_i \colon X \to X_i$ be mappings for $i = 1, \ldots, m$. If there exists a matrix $A = [a_{ij}] \in M_m(\mathbb{R}_+)$ such that $r(A) < 1$ and*

$$d_i(T_i(x_1, \ldots, x_m), T_i(y_1, \ldots, y_m)) \leq \sum_{j=1}^{m} a_{ij} d_j(x_j, y_j) \tag{4.1}$$

for any $i = 1, \ldots, m$ and $(x_1, \ldots, x_m)$, $(y_1, \ldots, y_m) \in X$, then mapping $T := (T_1, \ldots, T_m)$ has a unique fixed point $x^* = (x_1^*, \ldots, x_m^*) \in X$, i.e., $x_i^* = T_i(x_1^*, \ldots, x_m^*)$ for $i = 1, \ldots, m$, and for any $x_0 \in X$, $x^* = \lim_{n\to\infty} T^n x_0$.

Proof. For $x, y \in X$, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ and $i = 1, \ldots, m$, define

$$\rho_i(x, y) := d_i(x_i, y_i).$$

It is easily seen that $\{\rho_1, \ldots, \rho_m\}$ is a separating family of pseudometrics in X and (4.1) can be written in the following form:

$$\rho_i(Tx, Ty) \leq \sum_{j=1}^{m} a_{ij} \rho_j(x, y) \quad \text{for} \quad i = 1, \ldots, m.$$

Moreover, since each (X_i, d_i) is complete, so is $(X, \rho^{(1)})$. (Here $\rho^{(1)}(x, y) = \sum_{i=1}^{m} d_i(x_i, y_i)$ for $x, y \in X$.) Set $d := (\rho_1, \ldots, \rho_m)$. By Proposition 2.1 and Corollary 2.4, (X, d) is a complete g.m.s., so it suffices to apply Theorem 3.4. $\square$

Remark 4.2. Clearly, with the help of Perov's theorem, we could get some versions of Matkowski's theorem by considering other pseudometrics on the Cartesian product. For example, let $m = 3$ and for $x, y \in X_1 \times X_2 \times X_3$,

$$\rho_1(x, y) := \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \quad \text{and} \quad \rho_2(x, y) := d_3(x_3, y_3).$$

Let $A = [a_{ij}] \in M_2(\mathbb{R}_+)$ and $r(A) < 1$. Now, if we replace (4.1) in Theorem 4.1 by

$$d_i(T_i x, T_i y) \leq a_{11} \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} + a_{12} d_3(x_3, y_3) \quad \text{for} \quad i = 1, 2, \quad \text{and}$$
$$d_3(T_3 x, T_3 y) \leq a_{21} \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} + a_{22} d_3(x_3, y_3),$$

then, by Theorem 3.4 (here with $m = 2$!), such mapping $T = (T_1, T_2, T_3)$ has a unique fixed point.

Now we present a proof of the classical Picard–Lindelöf theorem by using Perov's fixed point theorem. Most certainly, our proof is not new and, probably, it can be found in [20]. However, Perov's paper [20] is hardly available (in particular, it was unavailable for us), so it seems that the proof presented below is not well-known. Actually, it is inspired by the argument used by E. De Pascale and L. De Pascale [4] in their proof of Lou's [17] fixed point theorem. In fact, the Picard–Lindelöf theorem can be proved in many ways: in particular, three different approaches were presented in [8, pp. 14–18].

Theorem 4.3 (Picard–Lindelöf). *Let a be a positive real and $f \colon [0, a] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that for some $L > 0$,*

$$|f(t, u) - f(t, v)| \leq L|u - v| \quad \textit{for any} \quad t \in [0, a] \quad \textit{and} \quad u, v \in \mathbb{R}.$$

Then for any $u_0 \in \mathbb{R}$, the Cauchy initial value problem

$$x'(t) = f(t, x(t)) \quad \textit{for} \quad t \in [0, a], \quad \textit{and} \quad x(0) = u_0 \tag{4.2}$$

has a unique solution.

Proof. (Via Perov's theorem.) Given $u_0 \in \mathbb{R}$, we define

$$(Tx)(t) := u_0 + \int_0^t f(s, x(s))ds \quad \text{for} \quad t \in [0, a] \quad \text{and} \quad x \in C[0, a].$$

There exist $m \in \mathbb{N}$ and a partition $(t_i)_{i=0}^m$ of interval $[0, a]$, i.e., $0 = t_0 < t_1 < \cdots < t_m = a$ such that

$$\max_{1 \le i \le m} (t_i - t_{i-1}) \le \frac{1}{2L}.$$

For $x, y \in C[0, a]$ and $i = 1, \ldots, m$, set

$$\rho_i(x, y) := \max\{|x(t) - y(t)| : t_{i-1} \le t \le t_i\}.$$

Then $\{\rho_1, \ldots, \rho_m\}$ is a separating family of pseudometrics in $C[0, a]$, so by Proposition 2.1, $d := (\rho_1, \ldots, \rho_m)$ is Perov's metric in $C[0, a]$. Moreover, metric $\rho^{(\infty)}$ associated with d coincides with the standard maximum metric in $C[0, a]$. Since $(C[0, a], \rho^{(\infty)})$ is complete, so is $(C[0, a], d)$ in view of Corollary 2.4. Thus by Theorem 3.4, it suffices to show that condition (3.2) holds. Let $i \in \{1, \ldots, m\}$, $x, y \in C[0, a]$ and $t \in [t_{i-1}, t_i]$. Then we have

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &= |\textstyle\int_0^t (f(s, x(s)) - f(s, y(s))ds| \le \int_0^{t_i} |f(s, x(s)) - f(s, y(s))| \\
&= \textstyle\sum_{j=1}^i \int_{t_{j-1}}^{t_j} |f(s, x(s)) - f(s, y(s))|ds \\
&\le L \textstyle\sum_{j=1}^i \int_{t_{j-1}}^{t_j} |x(s) - y(s)|ds \\
&\le L \textstyle\sum_{j=1}^i (t_j - t_{j-1})\rho_j(x, y) \le \sum_{j=1}^i \frac{1}{2}\rho_j(x, y).
\end{aligned}$$

Hence we infer that

$$\rho_i(Tx, Ty) \le \sum_{j=1}^i \frac{1}{2}\rho_j(x, y).$$

For $i, j = 1, \ldots, m$, set $a_{ij} := 1/2$ if $j \le i$, and $a_{ij} := 0$ if $j > i$. Then $A := [a_{ij}]_{i,j=1,\ldots,m}$ is a triangular matrix with $1/2$ on the diagonal, so $1/2$ is the only eigenvalue of A, and hence $r(a) = 1/2$. By Theorem 3.4, T has a unique fixed point, which implies that the Cauchy problem (4.2) has a unique solution. □

5. K-metric spaces and some their properties

Actually, a more general notion than the concept of generalized metric spaces was known much earlier and was first introduced by Kurepa [16] in 1934, who used the term *pseudodistance*. However, now this term is used in a different sense and following, e.g., Zabreĭko [27] we substitute the term 'K-metric' for 'pseudodistance'. Instead of $\mathbb{R}^m$ as in Section 2, we consider now an arbitrary Banach space E and a cone $K \subseteq E$ (for the definition, see, e.g., [3, p. 218]). We denote by $\preceq$ the partial ordering induced by K by the formula: for $a, b \in E$,

$$a \preceq b \quad \text{iff} \quad b - a \in K.$$

We write $a \prec b$ if $a \preceq b$ and $a \neq b$. If K is *solid*, i.e., $\operatorname{int} K \neq \emptyset$, then $a \ll b$ stands for $b - a \in \operatorname{int} K$. Clearly, orderings $\preceq$, $\prec$ and $\ll$ considered in Section 2 were induced by the cone

$$K := \{(a_1, \dots, a_m) : a_i \geq 0 \quad \text{for} \quad i = 1, \dots, m\}.$$

Now if X is a nonempty set, then a function $d\colon X \times X \to E$ is called a K-*metric* in X if d satisfies conditions (2.2), (2.3) and (2.4) with respect to the partial ordering $\preceq$ induced by K. Given a solid cone K, we define the convergence, Cauchy's condition for sequences and the completeness of (X, d) in the same way as in Section 2. If K is not solid, then we may consider the convergence with respect to the function

$$X \times X \ni (x, y) \mapsto \| \, d(x, y) \, \|,$$

which however, in general, is not a metric. Similarly, a counterpart of Cauchy's condition for a sequence (x_n) of elements of X is the following: $\lim_{m,n\to\infty} \| \, d(x_n, x_m) \, \| = 0$. Then we may consider another notion of completeness of (X, d) as given in [14, p. 92]. To distinguish the two notions, we say that (X, d) is $\| \cdot \|$-*complete* if for any sequence (x_n) such that $\lim_{m,n\to\infty} \| \, d(x_n, x_m) \, \| = 0$, there exists $x \in X$ such that $\lim_{n\to\infty} \| \, d(x_n, x) \, \| = 0$. In general, the two notions of convergence and completeness are not equivalent. Now we discuss on relations between them. Given $c \in E$ and $r > 0$, we denote by $B(c, r)$ the open ball centered at c with radius r. If $A \subseteq E$ then we define $c - A := \{c - a : a \in A\}$.

Proposition 5.1. *Let (X, d) be a K-metric space with a solid cone K in a Banach space E. Let $c \in E$, $x_n \in X$ for $n \in \mathbb{N}$ and $x \in X$. The following statements hold:*

(1) *$c \gg 0$ iff there exists $r > 0$ such that for any $a \in E$, $\| \, a \, \| < r$ implies $a \ll c$.*
(2) *If $\| \, d(x_n, x) \, \| \to 0$ then (x_n) is convergent to x.*
(3) *If $\lim_{m,n\to\infty} \| \, d(x_n, x_m) \, \| = 0$ then (x_n) is a Cauchy sequence.*

Proof. We prove $(\Rightarrow)$ of statement 1. Let $c \gg 0$, i.e., $c \in \operatorname{int} K$. Then there is $r > 0$ such that $B(c, r) \subseteq K$. Hence $c - B(c, r) \subseteq c - K$, i.e., $B(0, r) \subseteq c - K$ since $c - B(c, r) = B(0, r)$. The latter inclusion is equivalent to $c - B(0, r) \subseteq K$, and hence we get

$$c - B(0, r) = \operatorname{int}(c - B(0, r)) \subseteq \operatorname{int} K,$$

which means that if $\| \, a \, \| < r$ then $a \ll c$. Conversely, the last condition implies that $c - B(0, r) \subseteq \operatorname{int} K$, so $B(c, r) \subseteq \operatorname{int} K$ since $c - B(0, r) = B(c, r)$, and thus $c \gg 0$. So statement 1 is proved.

Now statements 2 and 3 are immediate consequences of 1. Indeed, if for example,

$$\lim_{m,n\to\infty} \| \, d(x_n, x_m) \, \| = 0 \quad \text{and} \quad c \gg 0,$$

then there is $r > 0$ as in statement 1. Then there exists $k \in \mathbb{N}$ such that $\| \, d(x_n, x_m) \, \| < r$ for all $m, n \geq k$, which implies that $d(x_n, x_m) \ll c$ by property of r. So (x_n) is a Cauchy sequence. □

We will need the following folklore result a proof of which is left to the reader.

Lemma 5.2. *Let K be a cone in a Banach space. Then*

$$K + \operatorname{int} K = \operatorname{int} K \quad \textit{and} \quad \lambda \operatorname{int} K = \operatorname{int} K \quad \textit{for any} \quad \lambda > 0.$$

Lemma 5.2 improves an observation from [24, p. 720] by which $\operatorname{int} K + \operatorname{int} K \subseteq \operatorname{int} K$ and $\lambda \operatorname{int} K \subseteq \operatorname{int} K$ for $\lambda > 0$. As an immediate consequence of Lemma 5.2, we get the following result, which is also mentioned in [11, p. 2598].

Corollary 5.3. *Let K be a solid cone ina Banach space E, and let $a, b, c \in E$.*

(1) *If $a \preceq b$ and $b \ll c$, or $a \ll b$ and $b \preceq c$, then $a \ll c$.*
(2) *If $c \gg 0$ then $\lambda c \gg 0$ for any $\lambda > 0$.*

Now we give a characterization of the convergence in a K-metric space with a solid cone.

Proposition 5.4. *Let (X, d) be a K-metric space with a solid cone K in a Banach space E. Let $x_n \in X$ for $n \in \mathbb{N}$ and $x \in X$. The following statements are equivalent:*

(i) *(x_n) converges to x;*
(ii) *for any $c \gg 0$, there exists $k \in \mathbb{N}$ such that for any $n \geq k$, $d(x_n, x) \preceq c$;*
(iii) *there exists $c \gg 0$ such that for any $\lambda \in (0, 1)$, there exists $k \in \mathbb{N}$ such that $d(x_n, x) \preceq \lambda c$ for all $n \geq k$;*
(iv) *there exists a sequence (c_n) such that $c_n \gg 0$ for any $n \in \mathbb{N}$, $\| c_n \| \to 0$ and for any $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $d(x_m, x) \preceq c_n$ for each $m \geq k$.*

Proof. (i)⇒(ii) is obvious since $a \ll b$ implies $a \preceq b$.

(ii)⇒(iii): Choose any $c_0 \gg 0$ and $\lambda \in (0, 1)$. By Corollary 5.3, $\lambda c_0 \gg 0$, so by (ii) there is $k \in \mathbb{N}$ such that $d(x_n, x) \preceq \lambda c_0$ for $n \geq k$, which proves (iii).

(iii)⇒(iv): It suffices to set $c_n := (1/n)c$, where c is as in (iii).

(iv)⇒(i): Fix $c \gg 0$. Since $\| c_n \| \to 0$, Proposition 5.1 yields that $c_j \ll c$ for some $j \in \mathbb{N}$. By (iv), there is $k \in \mathbb{N}$ such that for $i \geq k$, $d(x_i, x) \preceq c_j$. Hence, by Corollary 5.3, we infer that $d(x_i, x) \ll c$ for each $i \geq k$, which proves (i). □

Recall that a cone K is *normal* if

$$\inf\{\| x + y \| : x, y \in K \quad \text{and} \quad \| x \| = \| y \| = 1\} > 0.$$

A norm $\| \cdot \|$ on E is called *semi-monotone* if there exists $\gamma > 0$ such that for any $a, b \in E$,

$$0 \preceq a \preceq b \quad \text{implies} \quad \| a \| \leq \gamma \| b \| .$$

$\| \cdot \|$ is said to be *monotone* if it is semi-monotone with $\gamma = 1$.

The following characterization of normal cones is known (see, e.g., [1, Theorem 2.38]).

Lemma 5.5. *Let K be a cone in a Banach space $(E, \| \cdot \|)$. The following statements are equivalent:*

(i) *K is normal;*
(ii) *the norm $\| \cdot \|$ is semi-monotone;*
(iii) *E admits an equivalent monotone norm.*

We may add the following characterization of normal cones, which strengthens a result of Huang and Zhang [9, Lemma 1]. It is inspired by [23, Example 9.2].

Proposition 5.6. *Let K be a solid cone in a Banach space E. The following statements are equivalent:*

(i) *K is normal;*
(ii) *for any K-metric space (X, d) and any sequence (x_n) of elements of X, (x_n) converges to some $x \in X$ iff $\| d(x_n, x) \| \to 0$.*

Proof. (i)$\Rightarrow$(ii) was proved in [9].

(ii)$\Rightarrow$(i): Suppose, on the contrary, K is not normal. By Lemma 5.5 ((ii)$\Rightarrow$(i)), there exist sequences (a_n) and (b_n) of elements of E such that for any $n \in \mathbb{N}$,

$$0 \preceq a_n \preceq b_n \quad \text{and} \quad \| a_n \| > n \| b_n \| .$$

In particular, $\| a_n \| > 0$, so $a_n \neq 0$, and hence $b_n \neq 0$ since $0 \preceq a_n \preceq b_n$. Thus we may define:

$$a'_n := \frac{1}{n \| b_n \|} a_n \quad \text{and} \quad b'_n := \frac{1}{n \| b_n \|} b_n \quad \text{for} \quad n \in \mathbb{N}.$$

Then $\| a'_n \| > 1$, $\| b'_n \| = 1/n$ and $0 \preceq a'_n \preceq b'_n$. Now set $X := K$ and for $a, b \in K$,

$$d(a, b) := a + b \quad \text{if} \quad a \neq b, \quad \text{and} \quad d(a, b) := 0 \quad \text{if} \quad a = b.$$

It is easy to verify that (X, d) is a K-metric space. Observe that since $a'_n \neq 0$,

$$\| d(a'_n, 0) \| = \| a'_n \| > 1,$$

so $\| d(a'_n, 0) \| \nrightarrow 0$. We show that (a'_n) converges to 0, which will yield a contradiction. Fix $c \gg 0$. Since also $b'_n \neq 0$,

$$\| d(b'_n, 0) \| = \| b'_n \| = \frac{1}{n} \to 0,$$

so by Proposition 5.1, (b'_n) converges to 0. Hence there is $k \in \mathbb{N}$ such that $b'_n \preceq c$ for $n \geq k$. Since $a'_n \preceq b'_n$, we get that

$$d(a'_n, 0) = a'_n \preceq c \quad \text{for} \quad n \geq k,$$

so $a'_n \to 0$, which gives a contradiction. □

6. An extension of Perov's theorem

It is known that Perov's theorem can be extended by substituting a K-metric space for a generalized metric space, and replacing a matrix A by a linear bounded operator Λ. The following theorem is taken from [14, p. 92]. It is not clear who discovered this result; see comments in Zabreĭko's paper [27, p. 841].

Theorem 6.1. *Let K be a normal (not necessarily solid) cone in a Banach space E and $\Lambda \colon E \to E$ be a linear bounded operator, which is positive, i.e., $a \succeq 0$ implies $\Lambda a \succeq 0$ for any $a \in E$. Let (X, d) be a $\| \cdot \|$-complete K-metric space and $T \colon X \to X$ be such that*

$$d(Tx, Ty) \preceq \Lambda(d(x, y)) \quad \textit{for all} \quad x, y \in X.$$

If the spectral radius of Λ is less than 1, then T has a unique fixed point $x_* \in X$ and for any $x_0 \in X$, $x_* = \lim_{n \to \infty} T^n x_0$.

Actually, it is known that Theorem 6.1 can be derived from the classical Banach contraction principle (see [4], where a sketch of a proof is given). Here we show that also a more general version of Theorem 6.1 can be proved via Banach's contraction principle. We consider a mapping T satisfying a *nonlinear Lipschitz condition* with operator Λ, i.e., we allow Λ to be nonlinear. Such mappings were also studied (see [27, Theorem 2] and references therein), but it seems that the following result may be new.

Theorem 6.2. *Let K be a normal cone in a Banach space $(E, \|\cdot\|_0)$ and $\Lambda\colon K \to E$ be a monotone operator, i.e., $a \preceq b$ implies $\Lambda a \preceq \Lambda b$ for $a, b \in K$, and $\Lambda 0 = 0$. Let (X, d) be a $\|\cdot\|_0$-complete K-metric space and $T\colon X \to X$ be such that*

$$d(Tx, Ty) \preceq \Lambda(d(x, y)) \quad \text{for all} \quad x, y \in X. \tag{6.1}$$

If Λ is continuous and for some $p \in \mathbb{N}$, Λ^p is Lipschitzian with $L(\Lambda^p) < 1$, then T has a unique fixed point $x_* \in X$ and for any $x_0 \in X$, $\lim_{n\to\infty} \| d(T^n x_0, x_*) \|_0 = 0$.

Proof. By Lemma 5.5, since K is normal, E admits an equivalent monotone norm $\|\cdot\|$. Then the function

$$\rho(x, y) := \| d(x, y) \| \quad \text{for} \quad x, y \in X$$

is a metric. Since (X, d) is $\|\cdot\|_0$-complete and norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent, we may infer that (X, ρ) is complete. Since (6.1) holds, the monotonicity of Λ yields

$$d(T^n x, T^n y) \preceq \Lambda^n(d(x, y)) \quad \text{for any} \quad n \in \mathbb{N} \quad \text{and} \quad x, y \in X.$$

Hence, since $\|\cdot\|$ is monotone, we obtain that $\rho(T^n x, T^n y) \leq \| \Lambda^n(d(x, y)) \|$. In particular,

$$\rho(Tx, Ty) \leq \| \Lambda(d(x, y)) \|, \tag{6.2}$$

so if $\rho(x_n, x) \to 0$, i.e., $\| d(x_n, x) \| \to 0$, then by continuity of Λ, $\Lambda(d(x_n, x)) \to \Lambda 0 = 0$ which, by (6.2), implies that $\rho(Tx_n, Tx) \to 0$. Thus T is continuous with respect to ρ. Moreover, since Λ^p is Lipschitzian and $\Lambda^p 0 = 0$, we have

$$\rho(T^p x, T^p y) \leq \| \Lambda^p(d(x, y)) - \Lambda^p 0 \| \leq L(\Lambda^p) \| d(x, y) \| = L(\Lambda^p)\rho(x, y),$$

so $L_\rho(T^p) \leq L(\Lambda^p) < 1$. By [10, Theorem 2.1], there exists a metric ρ' equivalent to ρ such that (X, ρ') is complete and $L_{\rho'}(T) < 1$. By Banach's contraction principle, T has a unique fixed point $x_* \in X$ and for any $x_0 \in X$, $\rho'(T^n x_0, x_*) \to 0$, which implies that

$$\| d(T^n x_0, x_*) \| = \rho(T^n x_0, x_*) \to 0,$$

and hence also $\| d(T^n x_0, x_*) \|_0 \to 0$ since the two norms are equivalent. □

Let us note that under the assumptions of Theorem 6.1, Λ is monotone since $a \preceq b$ means $b - a \succeq 0$, so $\Lambda(b - a) \succeq 0$ and hence $\Lambda a \preceq \Lambda b$ because of linearity of Λ. Moreover, $r(\Lambda) < 1$ implies that $\| \Lambda^p \| < 1$ for some $p \in \mathbb{N}$ since $r(\Lambda) = \lim_{n\to\infty} \|\Lambda^n\|^{1/n}$. So indeed, Theorem 6.2 is a generalization of Theorem 6.1.

In a forthcoming paper we are going to present an application of theorems given in this section to obtain a result on the existence of solutions of the Cauchy initial value problem, which act on a half-line and have a limit at the infinity.

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