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THE STUDY OF A BOUNDARY VALUE PROBLEM WITH PARAMETER FOR A SYSTEM WITH ADVANCED AND RETARDED ARGUMENT

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Abstract. The purpose of this paper is to study the existence and uniqueness, data dependence of the solutions of a boundary value problem with parameter for a system of functional-differential equations with retarded and advanced arguments, by applying fixed point theory. Here is used the Perov's technique. In this paper we extend some results from [3] and [12].
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1. Introduction

We have the following problem: given $f \in C([a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$, $\phi \in C([a-h,a],\mathbb{R}^m)$ and $\psi \in C([b,b+h],\mathbb{R}^m)$, the problem is to determine

$$x \in (C(a-h,b+h)\cap C^1[a,b];\mathbb{R}^m)$$

and $\lambda \in \mathbb{R}^m$ such that:

$$x'(t) = \quad f(t,x(t),x(t-h),x(t+h))+\lambda, t\in[a,b]. \tag{1.1}$$

$$\begin{aligned} x(t)&=\phi(t), \quad t\in[a-h,a],\\ x(t)&=\psi(t), \quad t\in[b,b+h]. \end{aligned} \tag{1.2}$$

We suppose that:

(C$_1$) $a,b\in R$, $0<a<b$, $h>0$,
(C$_2$) $f\in C([a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$, $\phi\in C([a-h,a],\mathbb{R}^m)$ and $\psi\in C([b,b+h],\mathbb{R}^m)$,
(C$_3$) there exists a matrix $L_f\in M_{mm}(\mathbb{R}_+)$ such that

$$|f(s,u^1,u^2,u^3)-f(s,v^1,v^2,v^3)|\le L_f(|u^1-v^1|+|u^2-v^2|+|u^3-v^3|),$$

for all $t \in [a,b]$ and $u^i = (u^i_1, u^i_2, \cdots, u^i_m)$, $v^i = (v^i_1, v^i_2, \cdots, v^i_m) \in \mathbb{R}^m$, $i = 1, 2, 3$, and where

$$|w| = \begin{pmatrix} |w_1| \\ \vdots \\ |w_m| \end{pmatrix}$$

The equations of type (1.1) come from different fields of applications as in optimal control problem ([7]), in biomathematics ([13]) and others.

In 2004 I.A. Rus and V.A. Ilea studied in [12] the problem described above for first order functional-differential equations with advanced and retarded arguments, by applying fixed point theory and data dependence for (1.1)-(1.2). In 2007 V.A. Ilea and M.A. Serban studied in [3] the existence of the solutions for the first order functional differential equation with advanced and retarded argument, by applying Maia's type fixed point theorem, where the functions space is endowed with two metrics. Using this technique we improved the contraction condition (Theorem 3.1 (ii)) from I.A. Rus, V.A. Dârzu-Ilea [12].

In this paper we continue the research in this field (advanced and retarded arguments for functional differential equation) and develop the study to n populations, with the specification that the populations are - in the same environment - prade or predator. Existence, uniqueness for the Cauchy problem are obtained. Our results are essentially based on Perov fixed point theorem and weakly Picard operator technique, which will be presented below [11, 13, 17]. The use of the Perov's fixed point theorem [5], [6], [4] generates an efficient technique to approach systems of functional differential equations. We also continue the work of [3] and [12] with the study of systems of functional differential equations of mixed arguments.

2. Preliminaries

Next, we introduce notation, definitions, and preliminary facts which are used throughout this paper (see [8]–[12]).

Let $(X, \to)$ be a L-space and $A : X \to X$ an operator.

We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed points set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of nonempty invariant subsets of A;

$A^{m+1} := A \circ A^m,\ A^0 = 1_X,\ A^1 = A,\ m \in \mathbb{N}$.

Definition 2.1. *Let $(X, \to)$ be L-space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2. *Let $(X, \to)$ be L-space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *If A is weakly Picard operator then we consider the operator A^∞ defined by $A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x)$. It is clear that $A^\infty(X) = F_A$.*

Let now (X, d) be a metric space with $d(x, y) \in \mathbb{R}^m_+$ and $A : X \to X$ an operator. In what follow we denote by $\mathbb{R}^{m\times m}_+$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix.

Definition 2.4. *Let A be a weakly Picard operator and $C \in \mathbb{R}^{m\times m}_+$. The operator A is C -weakly Picard operator if*

$$d(x, A^\infty(x)) \le Cd(x, A(x)),\ \forall x \in X.$$

Definition 2.5. *A square matrix Q with nonnegative elements is said to be convergent to zero if $Q^k \to 0$ as $k \to \infty$. It is known that the property of being convergent to zero is equivalent to each of the following three conditions (see* [8]*):*

(a) *$I - Q$ is nonsingular and $(I - Q)^{-1} = I + Q + Q^2 + \cdots$ (where I stands for the unit matrix of the same order as Q);*
(b) *the eigenvalues of Q are located inside the open unit disc of the complex plane;*
(c) *$I - Q$ is nonsingular and $(I - Q)^{-1}$ has nonnegative elements.*

We finish this section by recalling the following fundamental result.

Theorem 2.6. (Perov's fixed point theorem). *Let (X, d) with $d(x, y) \in \mathbb{R}^m_+$, be a complete generalized metric space and $A : X \to X$ an operator. We suppose that there exists a matrix $Q \in \mathbb{R}^{m\times m}_+$, such that*

(i) *$d(A(x), A(y)) \le Qd(x, y)$, for all $x, y \in X$;*
(ii) *$Q^n \to 0$ as $n \to \infty$.*

Then

(a) *$F_A = \{x^*\}$ and $A^n(x) \to x^*$ as $n \to \infty$, $\forall x \in X$;*
(b) *$d(A^n(x), x^*) \le (I - Q)^{-1}Q^n d(x_0, A(x_0))$ and $d(x, x^*) \le (I - Q)^{-1}d(x, A(x))$.*

3. Main result (Existence and uniqueness)

Let (x, λ) solution of (1.1)-(1.2).

We remark that the problem (1.1)-(1.2) is equivalent to the following equation:

$$x(t) = \begin{cases} \phi(t), & t \in [a - h, a] \\ \phi(a) + \int_a^t f(s, x(s), x(s - h), x(s + h))ds + \lambda(t - a), & t \in [a, b] \\ \psi(t), & t \in [b, b + h] \end{cases} \tag{3.1}$$

with

$$\lambda = \frac{\psi(b)-\phi(a)}{b-a} - \frac{1}{b-a} \int_a^b f(s, x(s), x(s - h), x(s + h))ds \tag{3.2}$$

So the problem (1.1)+(1.2) is equivalent with

$$x = B(x) \ \ \textit{and} \ \ \lambda \ = \ \textit{second part of} \ \ (3.2)$$

where the operator $B : C([a-h, b+h], \mathbb{R}^m) \to C([a-h, b+h], \mathbb{R}^m)$ has the form

$$B(x)(t) := \begin{cases} \phi(t), & t \in [a-h, a] \\ \phi(a) + \frac{t-a}{b-a}(\psi(b) - \phi(a)) - \\ -\frac{t-a}{b-a}\int_a^b f(s, x(s), x(s-h), x(s+h))ds + \\ +\int_a^t f(s, x(s), x(s-h), x(s+h))ds, & t \in [a, b] \\ \psi(t), & t \in [b, b+h] \end{cases} . \tag{3.3}$$

By a solution of the system (1.1)-(1.2) we understand a function $x \in C([a-h, b+h], R^n) \cap C^1([a, b; \mathbb{R}^m])$ that verifies the system.

We also remark that the equation (1.1) is equivalent to the following equation:

$$x(t) = \begin{cases} x(t), & t \in [a-h, a] \\ \phi(a) + \int_a^t f(s, x(s), x(s-h), x(s+h))ds + \lambda(t-a), & t \in [a, b] \\ x(t), & t \in [b, b+h] \end{cases} \tag{3.4}$$

and also

$$\lambda = \quad \frac{\psi(b)-\phi(a)}{b-a} - \frac{1}{b-a}\int_a^b f(s, x(s), x(s-h), x(s+h))ds \tag{3.5}$$

We consider the operator $E_f : C([a-h, b+h], \mathbb{R}^m) \to C([a-h, b+h], \mathbb{R}^m)$ with

$$E_f(x)(t) := \begin{cases} x(t), & t \in [a-h, a] \\ \phi(a) + \frac{t-a}{b-a}(\psi(b) - \phi(a)) - \\ -\frac{t-a}{b-a}\int_a^b f(s, x(s), x(s-h), x(s+h))ds + \\ +\int_a^t f(s, x(s), x(s-h), x(s+h))ds, & t \in [a, b] \\ x(t), & t \in [b, b+h] \end{cases} \tag{3.6}$$

Let $X := (C[a-h, b+h], \mathbb{R}^m)$ and $X_{\phi,\psi} := \{x \in X; x|_{[a-h,a]} = \phi,\ x|_{[b,b+h]} = \psi\}$

Here we use Perov's fixed point theorem to obtain existence and uniqueness theorem for the solution of the problem (1.1)-(1.2).

Theorem 3.1. *We suppose that:*

(i) *the conditions (C_1)-(C_3) are satisfied;*
(ii) $Q^n \to 0$ *as* $n \to \infty$, *where* $Q := 6(b-a)L_f$

Then:

(a) *the problem (1.1)-(1.2) has a unique solution* $x^* \in C^1([a, b], \mathbb{R}^1)$;
(b) *for all* $x^0 \in C^1([a, b], \mathbb{R}^1)$, *the sequence* $(x^n)_{n\in\mathbb{N}}$ *defined by* $x^{n+1} = A(x^n)$, *converges uniformly to* x^*, *for all* $t \in [a, b]$, *and*

$$\|x^n - x^*\| \le (I-Q)^{-1}Q^n \left\|x^0 - x^1\right\| ;$$

(c) *the operator* B *is Picard operator in* $(C([a-h, b+h], \mathbb{R}^m), \xrightarrow{unif})$;
(d) *the operator* E_f *is weakly Picard operator in* $(C([a-h, b+h], \mathbb{R}^m), \xrightarrow{unif})$.

Proof. Consider on the generalized Banach space $X := C([a-h, b+h], \mathbb{R}^m)$ the norm

$$\|u\| := \begin{pmatrix} \max_{[a-h,b+h]} |u_1| \\ \vdots \\ \max_{[a-h,b+h]} |u_m| \end{pmatrix},$$

which endows X with the uniform convergence.

Let $X_{\phi,\psi} := \{x \in X; x|_{[a-h,a]} = \phi,\ x|_{[b,b+h]} = \psi\}$.

Then $X = \cup_{x \in C([a-h,b+h],\mathbb{R}^n)} X_{\phi,\psi}$ is a partition of X and from [11] we have

(1) $B(X) \subset X_{\phi,\psi}, B(X_{\phi,\psi}) \subset X_{\phi,\psi}$;
(2) $B|_{X_{\phi,\psi}} = E_f|_{X_{\phi,\psi}}$.

On the other hand, for $t \in [a-h, a] \cup [a, b] \cup [b, b+h]$

$$\|B(x_1) - B(x_2)\| \leq 6(b-a)L_f \|x_1 - x_2\|,$$

whence B is a contraction in $(X, \|\cdot\|)$ with $Q = 6(b-a)L_f$. Applying Perov theorem we have (a), (b) and (c). Moreover the operator B is c-PO and E_f is c-WPO with $c = [1 - 6(b-a)L_f]^{-1}$. □

4. Data dependence: continuity

Consider the problem (1.1)-(1.2) with the dates $f^i \in C([a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $i = 1, 2$ and suppose that f^i satisfy the conditions from Theorem 3.1 with the same Lipshitz constants.

Theorem 4.1. *Let $f^i \in C([a, b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$, $\phi^i \in C([a-h, a], R^n)$ and $\psi^i \in C([b, b+h], R^n)$, $i = 1, 2$ be as in Theorem 3.1. We suppose that*

(i) *there exists $\eta_1, \eta_2 > 0$ such that*

$$\left| \begin{pmatrix} \phi^1 \\ \psi^1 \end{pmatrix}(t) - \begin{pmatrix} \phi^2 \\ \psi^2 \end{pmatrix}(t) \right| \leq \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},\ \forall t \in [a-h, a] \cup [b, b+h];$$

(ii) *there exists $\eta_3 > 0$ such that*

$$\left| f^1(t, u_1, u_2, u_3, u_4) - f^2(t, u_1, u_2, u_3, u_4) \right| \leq \eta_3, \forall t \in [a, b], u_i \in \mathbb{R}, i=\overline{1,4}.$$

Then

$$\left\| x^*(\cdot, \phi^1, \psi^1, f^1) - x^*(\cdot, \phi^2, \psi^2, f^2) \right\| \leq (I - Q)^{-1} \max_{[a-h,b+h]} (\eta_1, (\eta_2 + (b-a)\eta_3),$$

where $x^*(\cdot, \phi, \psi, f)$ denote the unique solution of (1.1)-(1.2).

Proof. Consider the operators B_{ϕ^i, f^i}, with $i = 1, 2$. From Theorem 3.1 it follows

$$\left\| B_{\phi^1,f^1} x_1 - B_{\phi^1,f^1} x_2 \right\| \leq Q \|x_1 - x_2\|, \forall x_i \in X.$$

where $Q := 6(b-a)L_f$.

Additionally,

$$\left\| B_{\phi^1,f^1} x - B_{\phi^2,f^2} x \right\| \leq \eta_2 + (b-a)\eta_3, t \in [a, b],$$

$\left\| B_{\phi^1,f^1} x - B_{\phi^2,f^2} x \right\| \leq \eta_1, t \in [a-h, a]$ and $\left\| B_{\phi^1,f^1} x - B_{\phi^2,f^2} x \right\| \leq \eta_2, t \in [b, b+h]$.

Thus

$$\left\|B_{\phi^1,f^1}x - B_{\phi^2,f^2}x\right\| \le \max_{[a-h,b+h]}(\eta_1, \eta_2 + (b-a)\eta_3).$$

So

$$\begin{aligned}
&\left\|x^*(\cdot,\phi^1,\psi^1,f^1) - x^*(\cdot,\phi^2,\psi^2,f^2)\right\| \\
&= \left\|B_{\phi^1,f^1}(x^*(\cdot,\phi^1,\psi^1,f^1)) - B_{\phi^2,f^2}(x^*(\cdot,\phi^2,\psi^2,f^2))\right\| \\
&\le \left\|B_{\phi^1,f^1}x^*(\cdot,\phi^1,\psi^1,f^1) - B_{\phi^1,f^1x^*}(\cdot,\phi^2,\psi^2,f^2)\right\| \\
&\quad + \left\|B_{\phi^1,f^1}x^*(\cdot,\phi^2,\psi^2,f^2) - B_{\phi^2,f^2}x^*(\cdot,\phi^2,\psi^2,f^2)\right\| \\
&\le Q\left\|x^*(\cdot,\phi^1,\psi^1,f^1) - x^*(\cdot,\phi^2,\psi^2,f^2)\right\| + \max(\eta_1,\eta_2+(b-a)\eta_3)
\end{aligned}$$

and because $Q^n \to \infty$ as $n \to \infty$ imply that $(I-Q)^{-1} \in \mathbb{R}_+^{m\times m}$, we have the conclusion of the theorem

$$\|x^*(\cdot,\phi_1,\psi_1,f_1) - x^*(\cdot,\phi_2,\psi_2,f_2)\| \le (I-Q)^{-1}\eta$$

where $\eta \in \mathbb{R}_+^m$ is such that $\eta_1^1 \le \eta$ and $\eta^2 + (b-a)\eta^3 \le \eta$. □

5. Data dependence: comparison results

In order to establish the Čaplygin type inequalities we need the following abstract result.

Lemma 5.1. *(see [9]) Let $(X, \to, \le)$ be an ordered L-space and $A : X \to X$ an operator. Suppose that A is increasing and WPO. Then the operator A^∞ is increasing.*

Now we consider the operators B and E_f on the ordered Banach space $(C([a-h, b+h], \mathbb{R}^m), \|\cdot\|, \le)$ where we consider the following order relation on $\mathbb{R}^m$: $x \le y \Leftrightarrow x_i \le y_i,\ i = \overline{1,m}$.

Theorem 5.2. *Suppose that:*

(a) *the conditions of Theorem 3.1 are satisfied;*
(b) *$Q^n \to 0$, as $n \to \infty$, where $Q := 6(b-a)L_f$;*
(c) *$f(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m, g(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ are increasing, $\forall t \in [a,b]$.*

Let x be a solution of the system (1.1) and y a solution of the inequality

$$y'(t) \le f(t, y(t), y(t-h), y(t+h)) + \lambda, t \in [a,b].$$

Then $y(a) \le x(a)$ implies that $y \le x$.

Proof. From 3.1(d) we have that E_f is WPO. On the other hand, from the condition (c) and Lemma 5.1 we get that the operator E_f^∞ is increasing. If $x_0 \in \mathbb{R}^m$, then we denote by $\widetilde{x}_0$ the following function

$$\widetilde{x}_0 : [a,b] \to \mathbb{R}^m, \widetilde{x}_0(t) = x_0,\ \forall t \in [a,b].$$

Hence $y \le E_f(y) \le E_f^2(y) \le \ldots \le E_f^\infty(y) = E_f^\infty(\widetilde{y}(a)) \le E_f^\infty(\widetilde{x}(a)) = x$. □

In order to study the monotony of the solution of the problem (1.1)–(1.2) with respect to x_0, f, g we need the following result from WPOs theory.

Lemma 5.3. *(Abstract comparison lemma,* [10]*) Let* $(X, \to, \leq)$ *be an ordered L-space and* $A, B, C : X \to X$ *be such that:*

(i) *the operator* A, B, C *are WPOs;*
(ii) $A \leq B \leq C$;
(iii) *the operator* B *is increasing.*

Then $x \leq y \leq z$ *imply that* $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

From this abstract result we obtain the following result:

Theorem 5.4. *Let* $f^j, g^j \in C([a, b] \times \mathbb{R}^m, \mathbb{R}^m), j = \overline{1, 3}$, *and suppose that conditions the conditions from Theorem 3.1 holds. Furthermore suppose that:*

(i) $f^1 \leq f^2 \leq f^3, g^1 \leq g^2 \leq g^3$;
(ii) $f^2(t, \cdot) : \mathbb{R}^m \to \mathbb{R}^m, g^2(t, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ *are increasing.*

Let x^j *be a solution of the equation*

$$x^{j\prime}(t) = f(t, x^{j\prime}(t), x^{j\prime}(t - h), x^{j\prime}(t + h)) + \lambda, t \in [a, b] \text{ and } j = \overline{1, 3}.$$

Then $x^1(a) \leq x^2(a) \leq x^3(a)$, *implies* $x^1 \leq x^2 \leq x^3$, *i.e. the unique solution of the problem (1.1)–(1.2) is increasing with respect to* x_0, f *and* g.

Proof. From 3.1, the operators $E_{f_j}, j = \overline{1, 3}$, are WPOs. From the condition (ii) the operator E_{f_2} is monotone increasing. From the condition (i) it follows that $E_{f_1} \leq E_{f_2} \leq E_{f_3}$. Let $\widetilde{x}^j(a) \in (C[a, b], \mathbb{R}^m)$ be defined by $\widetilde{x}^j(a) = x^j(a),\ \forall t \in [a, b]$. We notice that

$$\widetilde{x}^1(a)(t) \leq \widetilde{x}^2(a)(t) \leq \widetilde{x}^3(a)(t),\ \forall t \in [a, b].$$

From Lemma 5.3 we have that $E_{f_1}^\infty(\widetilde{x}^1(a)) \leq E_{f_2}^\infty(\widetilde{x}^2(a)) \leq E_{f_3}^\infty(\widetilde{x}^3(a))$. But $x^j = E_{f_j}^\infty(\widetilde{x}^j(a))$, so $x^1 \leq x^2 \leq x^3$. □

6. Gronwall lemmas

In this section we study the following inequalities

$$x'(t) \leq f(t, x(t), x(t - h), x(t + h)) + \lambda, t \in [a, b], \tag{6.1}$$

and

$$x'(t) \geq f(t, x(t), x(t - h), x(t + h)) + \lambda, t \in [a, b], \tag{6.2}$$

using the Abstract Gronwall Lemma and Abstract Gronwall-comparison Lemma.

Lemma 6.1. *(I.A. Rus* [10]*) (Abstract Gronwall lemma) Let* $(X, \to, \leq)$ *be an ordered L-space and* $A : X \to X$ *be an operator. We assume that A is PO and it is increasing.*

If we denote by x_A^* *the unique fixed point of A, then* $x \leq A(x) \Rightarrow x \leq x_A^*$ *and* $x \geq A(x) \Rightarrow x \geq x_A^*$.

Lemma 6.2. *(I.A. Rus* [10]*) (Abstract Gronwall-comparison lemma) Let* $(X, \to, \leq)$ *be an ordered L-space and* $A_1, A_2 : X \to X$ *be two operator. We assume that A and B are PO, A is increasing and* $A \leq B$.

If we denote by x_B^* *the unique fixed point of B, then* $x \leq A(x) \Rightarrow x \leq x_B^*$.

Theorem 6.3. *We consider the equation (1.1) with conditions (1.2). We assume that:*

(i) *f, g satisfy the condition (i) and (ii) from Theorem 3.1;*
(ii) *$f(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m, g(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ are increasing functions , $\forall t \in [a,b]$.*

Then we have:

(a) *If x is a solution of (6.1) then $x \leq x^*$, where x^* is the unique solution of (1.1)-(1.2);*
(b) *If x is a solution of (6.2) then $x \geq x^*$, where x^* is the unique solution of (1.1)-(1.2).*

Proof. We consider the operator B defined by (3.3). From Theorem 3.1 we have that B is PO, if x^* is the unique solution of (1.1)-(1.2) then $F_B = \{x^*\}$. Conditions (i) and (ii) imply that B is increasing. In terms of the operator B the inequality (6.1) is $x \leq B(x)$, $\forall x \in (C[a,b], \mathbb{R}^m)$ and the inequality (6.2) is $x \geq B(x)$, $\forall x \in (C[a,b], \mathbb{R}^m)$. The conclusion is obtained from Abstract Gronwall lemma. $\square$

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