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FIXED POINT THEOREMS VIA CONE-NORMS AND CONE-VALUED MEASURES OF NONCOMPACTNESS

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Abstract. In this paper, we obtain an extension of the Krasnoselskii fixed point theorem for sum of two operators to the case of cone normed spaces. We also prove a variant of the Darbo-Sadovskii theorem on fixed points for operators condensing with respect to a cone-valued measure of noncompactness and apply it to the Cauchy problem with deviating argument.
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1. Introduction

Cone metric and cone normed spaces were introduced in the middle of the 20^{th} century by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric [9,10,13]. These spaces have applications in approximation theory [9,10], in fixed point theory and theory of differential equations in Banach spaces [1,13]. In recent years, starting from the work by L.G.Huang and X.Zhang [7], the investigation of fixed point theory in cone metric spaces (in most cases for contractive mappings) has again attracted much attention from mathematicians. We refer to the papers [2,3,6,12,13] for some historical notes, discussion on obtained results and further references.

The aim of this paper is to establish the fixed point theorems by using cone norms and cone-valued measures of noncompactness. In our opinion, the advantage of using cone-valued metrics and norms or measures of noncompactness is that we will have more useful information from the relation between their two values, such as the relation between two elements of an ordered space. For example, in Section 3 of the paper, our condition for a mapping f to be condensing with respect to cone-valued measure of noncompactness φ is $\varphi[f(\Omega)] \leq A[\varphi(\Omega)]$,where $\varphi(\Omega), \varphi[f(\Omega)]$ are elements of an order cone K and $A : K \longrightarrow K$ is an increasing operator. The relation $\varphi(\Omega) \leq \varphi[f(\Omega)]$ implies $\varphi(\Omega) \leq A[\varphi(\Omega)]$ and then we can use different tools, for instance, the Gronwall inequality, fixed point theorems for increasing operators, to prove $\varphi(\Omega) = \theta$.

The paper is organized as follows. In the next section, we prove an extension of the Krasnoselskii fixed point theorem for sum of two operators in cone normed spaces. We consider two cases. If the underlying cone is normal, we use the Minkowskii functional to reduce to the case of usual normed spaces. In the case of nonnormal cone, we introduce a kind of weak topology. In Section 3, we obtain a variant of the Darbo-Sadovskii fixed point theorem for mappings condensing with respect to a cone-valued measure of noncompactness. An application to the Cauchy problem with deviating argument is also given to see the advantage of using cone-valued measures of noncompactness.

2. Main results

2.1. **A fixed point theorem of the Krasnoselskii type in cone normed spaces.** Let $E = (E, \|.\|)$ be a real Banach space and $K \subset E$ be a cone, that is, K is a closed convex subset such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{\theta\}$. If in E we define a partial ordering by $x \leq y$ iff $y - x \in K$ then the pair (E, K) is called an ordered Banach space. The cone K is said to be normal if there exists a number $N > 0$ such that

$$\theta \leq u \leq v \text{ implies } \|u\| \leq N \|v\| . \tag{2.1}$$

A mapping $A : M \subset E \to E$ is said to be positive if $A(x) \geq \theta$ for all $x \in M$, $x \geq \theta$; it is called increasing if $x, y \in M$ and $x \leq y$ implies $A(x) \leq A(y)$. Clearly, if $A : E \longrightarrow E$ is linear and positive then it is increasing. The set

$$K^* = \{f \in E^* : f(x) \geq 0 \ \forall x \in K\}$$

is called the dual wedge of K. It is proved that $x \in K$ iff $f(x) \geq 0 \ \forall f \in K^*$, and so if $x \in K \setminus \{\theta\}$ then there exists $f \in K^*$ such that $f(x) > 0$.

The following lemma allows us to choose $N = 1$ in (2.1).

Lemma 2.1. [11] *Let Banach space $(E, \|.\|)$ be ordered by the cone K and $\|.\|_*$ be the Minkowskii functional of the set $[B(\theta, 1) - K] \cap [B(\theta, 1) + K]$. Then*

1) $\|.\|_*$ is a norm in E satisfying $\|u\|_* \leq \|u\| \ \forall u \in E$ and $\|u\|_* \leq \|v\|_*$ if $\theta \leq u \leq v$,

2) $\|.\|_* \sim \|.\|$ if *K is normal.*

Definition 2.2. [13] Let (E, K) be an ordered Banach space and X be a real linear space. A mapping $p : X \longrightarrow E$ is called a cone norm (or K-norm) if

(i) $p(x) \in K$ or equivalently $p(x) \geq \theta_E \ \forall x \in X$ and $p(x) = \theta_E$ iff $x = \theta_X$, where θ_E, θ_X are the zero elements of E and X respectively,

(ii) $p(\lambda x) = |\lambda| p(x) \ \ \forall \lambda \in \mathbb{R}, \forall x \in X$,

(iii) $p(x + y) \leq p(x) + p(y) \ \forall x, y \in X$.

If p is a cone norm in X then the pair (X, p) is called a cone normed space (or K-normed space). The cone normed space (X, p) endowed with a topology τ will be denoted by (X, p, τ).

We shall use the following two topologies on a cone normed space.

Definition 2.3. Let (E, K) be an ordered Banach space and (X, p) be a K-normed space.

1) We define $\lim\limits_{n\to\infty} x_n = x$ iff $\lim\limits_{n\to\infty} p(x_n - x) = \theta$ in E and we call a subset $A \subset X$ closed if whenever $\{x_n\} \subset A$, $\lim\limits_{n\to\infty} x_n = x$ then $x \in A$.

Clearly, $\tau_1 = \{G \subset X \ : X\backslash G \text{ is closed}\}$ is a topology on X.

2) We denote by τ_2 the topology on X, defined by the family of seminorms $\{f \circ p : f \in K^*\}$. Thus (X, τ_2) is a locally convex topological vector space such that the sets

$$\left\{x \in X : \max_{1\le i\le n} f_i \circ p(x) < \varepsilon\right\}, f_i \in K^*, n \in \mathbb{N}^*, \varepsilon > 0$$

form a neighborhood base of zero and a net $\{x_\alpha\} \subset X$ converges to x in τ_2 iff

$$\lim f(p(x_\alpha - x)) = 0 \quad \forall f \in K^*.$$

Definition 2.4. [13] Let (E, K) be an ordered Banach space, (X, p) be a K-normed space, and τ be a topology on X

1) We say that (X, p, τ) is complete in the sense of Weierstrass if whenever $\{x_n\} \subset X$, $\sum\limits_{n=1}^{\infty} p(x_{n+1} - x_n)$ converges in E then $\{x_n\}$ converges in (X, p, τ).

2) We say that (X, p, τ) is complete in the sense of Kantorovich if any sequence $\{x_n\}$ satisfies

$$p(x_k - x_l) \le a_n \ \ \forall k, l \ge n, \text{ with } \{a_n\} \subset K, \lim_{n\to\infty} a_n = \theta_E \tag{2.2}$$

then $\{x_n\}$ converges in (X, p, τ).

The following lemmas will state some relations between the defined notions of completeness.

Lemma 2.5. *Let the Banach space $(E, \|.\|)$ be ordered by the normal cone K with $N = 1$ in (2.1) and (X, p) be a K-normed space. Then the mapping $q : X \longrightarrow \mathbb{R}$, $q(x) = \|p(x)\|$ is a norm on X, having the following properties.*

1) The topology τ_1 coincides with the topology of normed space (X, q).

2) If (X, p, τ_1) is complete in the sense of Weierstrass then (X, q) is complete.

Proof. Clearly, q is a norm in X and $\lim\limits_{n\to\infty} x_n = x$ in (X, p, τ_1) iff $\lim\limits_{n\to\infty} x_n = x$ in (X, q). Consequently, a set $A \subset X$ is closed in (X, p, τ_1) iff it is closed in (X, q) and the first conclusion of the lemma holds. To show completeness of (X, q) we consider a sequence $\{x_n\} \subset X$ such that $\sum\limits_{n=1}^{\infty} q(x_n) < \infty$ and we have to prove the convergence of the series $\sum\limits_{n=1}^{\infty} x_n$ in (X, q). In fact, for $s_n = x_1 + x_2 + ... + x_n$, $n \in \mathbb{N}^*$ we have

$$\sum_{n=1}^{\infty} \|p(s_n - s_{n-1})\| = \sum_{n=1}^{\infty} q(x_n) < \infty,$$

which implies convergence of $\sum\limits_{n=1}^{\infty} p(s_n - s_{n-1})$ in $(E, \|.\|)$. Since (X, p, τ_1) is complete in the sense of Weierstrass, we obtain the convergence of $\{s_n\}$ in (X, p, τ_1) and in (X, q).

Lemma 2.6. *Let (E, K) be an ordered Banach space and (X, p) be a K-normed space, τ be a topology on X.*

1) If (X, p, τ) is complete in the sense of Kantorovich then it is complete in the sense of Weierstrass.

2) If K is normal and (X, p, τ_1) is complete in the sense of Weierstrass then (X, p, τ_1) is complete in the sense of Kantorovich.

Proof. 1. Let $\{x_n\} \subset X$ be such that $\sum_{n=1}^{\infty} p(x_{n+1} - x_n)$ converges in E and let s, s_n be the sum and n-th partial sum of the series respectively . For $l > k \geq n$ we have $p(x_l - x_k) \leq s_{k-1} - s_{l-1} \leq s - s_n$ with $\lim_{n\to\infty} (s - s_n) = \theta$ in E. Therefore, $\{x_n\}$ converges by completeness of (X, p, τ) in the sense of Kantorovich.

2. Consider a sequence $\{x_n\}$ satisfying (2.2). By normality of K we have $\|p(x_l - x_k)\| \leq N \|a_n\|$, hence $\{x_n\}$ is a Cauchy sequence in (X, q) and therefore it converges in (X, q) and in (X, p, τ_1) by Lemma 2.5.

Theorem 2.7. *Let (E, K)* be an ordered *Banach space,* (X, p) be a *K-normed space and $\tau = \tau_1$* or *$\tau = \tau_2$. Assume that C is a convex closed subset in (X, p, τ) and S,$T : C \longrightarrow X$ are operators such that*

(i) $T(x) + S(y) \in C$ $\forall x, y \in C$;

(ii) *S is continuous and $\overline{S(C)}$ is compact with respect to the topology τ;*

(iii) *there is a positive continuous linear operator $Q : E \longrightarrow E$ with the spectral radius $r(Q) < 1$ such that*

$$p(T(x) - T(y)) \leq Q[p(x - y)] \quad \textit{for all } x, y \in C.$$

Then the operator $T + S$ has a fixed point in the following cases.

(C_1) $\tau = \tau_1$, K is normal, (X, p, τ_1) is complete in the sense of Weierstrass.

(C_2) $\tau = \tau_2$, (X, p, τ_2) is complete in the sense of Kantorovich.

Proof. First we observe by hypothesis (i) and closedness of C that $T(x) + y \in C$ $\forall x \in C, \forall y \in \overline{S(C)}$. Fix $y \in \overline{S(C)}$, we define the operator $T_y : C \longrightarrow C$ by $T_y(x) = T(x) + y$. Then, starting with element $x_0 \in C$ we construct the sequence $x_n = T_y(x_{n-1})$. Putting $u = p(x_1 - x_0)$ we easily deduce that $p(x_{n+1} - x_n) \leq Q^n(u)$. We know that $\sum_{n=0}^{\infty} Q^n(u) = (I - Q)^{-1}(u)$; let s_n be the n-th partial sum of the series, then for $l > k \geq n$ we obtain

$$\begin{aligned} p(x_l - x_k) &\leq \sum_{i=k}^{l-1} p(x_{i+1} - x_i) \leq \sum_{i=k}^{l-1} Q^i(u) \\ &\leq (I - Q)^{-1}(u) - s_n \longrightarrow \theta \textit{ as } n \longrightarrow \infty. \end{aligned}$$

Since (X, p, τ) is complete in the sense of Kantorovich, it follows that there exists $x_* = \lim_{n\to\infty} x_n$. We have

$$\begin{aligned} p[T_y(x_*) - x_*] &\leq p[T_y(x_*) - T_y(x_n)] + p(x_{n+1} - x_*) \\ &\leq Q[p(x_* - x_n)] + p(x_{n+1} - x_*) \end{aligned} \tag{2.3}$$

and

$$f(p[T_y(x_*) - x_*]) \leq f \circ Q[p(x_* - x_n)] + f \circ p(x_{n+1} - x_*) \quad \forall f \in K^*, \tag{2.4}$$

By letting $n \to \infty$ in (2.3) we deduce that $T_y(x_*) = x_*$ for the case (C_1). For the case (C_2) we use (2.4) and that $f \circ Q \in K^*$ to get the same conclusion. This fixed point of T_y will be unique. Indeed, if we also have $T_y(a) = a$ then $p(a - x_*) = p[T_y(a) - T_y(x_*)] \leq Q[p(a - x_*)]$. Since $(I - Q)^{-1}$ is positive and linear we conclude that $p(a - x_*) = \theta_E$ and $a = x_*$.

Since the operator $T_y(x) = T(x) + y$ has a unique fixed point $\forall y \in \overline{S(C)}$ then there exists operator $(I - T)^{-1} : \overline{S(C)} \longrightarrow C$. We shall prove its continuity. In fact, let a net $\{y_\alpha\} \subset \overline{S(C)}$ be convergent to $y \in \overline{S(C)}$ in topology τ. Putting $x_\alpha = (I - T)^{-1}(y_\alpha)$, $x = (I - T)^{-1}(y)$ we have

$$\begin{aligned} p(x_\alpha - x) &\leq p[T(x_\alpha) - T(x)] + p(y_\alpha - y) \\ &\leq Q[p(x_\alpha - x)] + p(y_\alpha - y), \end{aligned}$$

which implies

$$p(x_\alpha - x) \leq (I - Q)^{-1}[p(y_\alpha - y)] \tag{2.5}$$

and

$$f \circ p(x_\alpha - x) \leq f \circ (I - Q)^{-1}[p(y_\alpha - y)] \quad \forall f \in K^*. \tag{2.6}$$

In the case (C_1) we deduce from (2.5) and the normality of the cone K that $\{x_\alpha\}$ converges to x in the topology τ_1. For the case (C_2), it follows from (2.6) and $f \circ (I - Q)^{-1} \in K^*$ that the net $\{x_\alpha\}$ converges to x in the topology τ_2. The operator $(I - T)^{-1} \circ S : C \longrightarrow C$ is continuous, the set $\overline{(I - T)^{-1} \circ S(C)}$ is equal to compact set $(I - T)^{-1}\left(\overline{S(C)}\right)$ since $(I - T)^{-1}$ is isomorphism. By the Tychonoff theorem there exists $x \in C$ such that $x = (I - T)^{-1} \circ S(x)$ or equivalently $x = T(x) + S(x)$.

2.2. A fixed point theorem via cone-valued measures of noncompactness and application.

2.2.1. **Cone-valued measures of noncompactness.** In this subsection we use some definitions and statements from [4]. For the convenience of the reader we recall the relevant material without proofs.

Definition 2.8. [4] Let (E, K) be an ordered Banach space, X be a Banach space, $\mathcal{F}$ be a family of bounded subsets of X such that: if $\Omega \in \mathcal{F}$ then $\overline{co}(\Omega) \in \mathcal{F}$. A mapping $\varphi : \mathcal{F} \longrightarrow K$ is called a measure of noncompactness if

$$\varphi[\overline{co}(\Omega)] = \varphi(\Omega) \quad \forall \Omega \in \mathcal{F}.$$

A measure of noncompactness φ is said to be

1) regular if $\varphi(\Omega) = 0 \Longleftrightarrow \Omega$ is relatively compact,

2) semi-homogeneous if $\varphi(t\Omega) = |t|\,\varphi(\Omega)$ for $\Omega \in \mathcal{F}$ such that $t\Omega \in \mathcal{F}$,

3) algebraic semi-additive if $\varphi(\Omega_1 + \Omega_2) \leq \varphi(\Omega_1) + \varphi(\Omega_2)$ for $\Omega_1, \Omega_2 \in \mathcal{F}$ such that $\Omega_1 + \Omega_2 \in \mathcal{F}$,

4) invariant under translations if $\varphi(x + \Omega) = \varphi(\Omega)$ whenever Ω, $x + \Omega \in \mathcal{F}$,

5) continuous with respect to the Hausdorff metric ρ if

$$\forall \varepsilon > 0, \forall \Omega \in \mathcal{F} \;\; \exists \delta > 0 : \forall \Omega' \in \mathcal{F}, \rho(\Omega', \Omega) < \delta \Longrightarrow \|\varphi(\Omega') - \varphi(\Omega)\| < \varepsilon,$$

where

$$\rho(\Omega_1, \Omega_2) = \inf\{\varepsilon > 0 : \Omega_1 + \varepsilon B \supset \Omega_2, \Omega_2 + \varepsilon B \supset \Omega_1\}$$

and $B = \{x \in X : \|x\|_X < 1\}$.

Example. [4] Consider a Banach space $(Y, |.|)$ and a real-valued measure of noncompactness φ defined for all bounded subsets of Y. In $X = C([a,b];Y)$ we consider the norm $\|x\| = \sup\{|x(t)| : t \in [a,b]\}$. For each bounded subset $\Omega \subset X$ we set $\Omega(t) = \{x(t) : x \in \Omega\}$ and define a function $\varphi_c(\Omega) : [a,b] \longrightarrow \mathbb{R}$ by $\varphi_c(\Omega)(t) = \varphi[\Omega(t)]$. If the measure φ is continuous and a set $\Omega \subset X$ is equicontinuous then the function $\varphi_c(\Omega)$ is continuous. Consequently, there exists the mapping φ_c from the family $\digamma$ of all equicontinuous subsets of X into the cone of nonnegative functions in $C([a,b];\mathbb{R})$. This mapping φ_c is a measure of noncompactness in the sense of Definition 2.8 and if φ has a property in Definition 2.8 then φ_c has the same property.

Definition 2.9. [4] Let (E, K) be an ordered Banach space, X be a Banach space and $\varphi : \digamma \subset 2^X \longrightarrow K$ be a cone-valued measure of noncompactness. A continuous mapping $f : D \subset X \longrightarrow X$ is called condensing if for $\Omega \subset D$ such that $\Omega \in \digamma$, $f(\Omega) \in \digamma$ and $\varphi[f(\Omega)] \geq \varphi(\Omega)$ then Ω is relatively compact.

Theorem 2.10. [4, Generalization 1.5.12] *Assume that (E, K) is an ordered Banach space, X is a Banach space and $\varphi : \digamma \subset 2^X \longrightarrow K$ is a cone-valued measure of noncompactness such that $\digamma$ contains any bounded sequence and*

$$\varphi(\{x_n : n \geq 1\}) = \varphi(\{x_n : n \geq 2\}). \tag{2.7}$$

Let $D \subset X$ be a nonempty convex closed subset and let $f : D \longrightarrow D$ be a condensing mapping. Then f has a fixed point in D.

Theorem 2.11. *Let (E, K) be an ordered Banach space, X be a Banach space and $\varphi : \digamma \subset 2^X \longrightarrow K$ be a regular measure of noncompactness having property* (2.7). *Assume that $D \subset X$ is a nonempty convex closed subset and $f : D \longrightarrow D$ is a continuous mapping such that there exists a mapping $A : K \longrightarrow K$ satisfying*

(H_1) $\varphi[f(\Omega)] \leq A[\varphi(\Omega)]$ *whenever* $\Omega \subset D, \Omega \in \digamma$, $f(\Omega) \in \digamma$,

(H_2) *if* $x_0 \in K$, $x_0 \leq A(x_0)$ *then* $x_0 = \theta$.

Then f has a fixed point in D.

Proof. We shall show that f is condensing and then apply Theorem 2.10. Indeed, if $\Omega \subset D$ is such that $\Omega \in \digamma$, $f(\Omega) \in \digamma$ and $\varphi(\Omega) \leq \varphi[f(\Omega)]$ then by the hypothesis (H_1) we get $\varphi(\Omega) \leq A[\varphi(\Omega)]$ which implies $\varphi(\Omega) = 0$ by (H_2). Consequently, Ω is relatively compact.

Corollary 2.12. *Suppose the measure φ and the mapping f satisfy hypothesis* (H_1) *and*

(H_2') *The mapping A is increasing and* $\lim_{n\to\infty} A^n(x) = \theta \ \forall x \in K$.

Then f has a fixed point in D.

Proof. Let us prove that (H_2') implies (H_2). In fact, if $x_0 \in K$, $x_0 \leq A(x_0)$ then $x_0 \leq A^n(x_0)$ by monotonicity of A and so $x_0 = \theta$ by using (H_2').

Corollary 2.13. *Suppose the measure φ and the mapping f satisfy hypothesis* (H_1) *and* (H_2'')

1) *The mapping A is increasing, the sequence $\{A(x_n)\}$ converges whenever $\{x_n\}$ is an increasing sequence in D,*

2) *A does not have fixed points in $K \backslash \{\theta\}$.*

Then f has a fixed point in D.

Proof. We shall prove that the hypothesis (H_2) holds. Assume the contrary that

$$\exists x_0 \in K\backslash\{\theta\},\ x_0 \le A(x_0). \tag{2.8}$$

From 1) in ($H_2^{''}$) and (2.8) and using a result of [5,8] about fixed points of increasing mappings we deduce that A has a fixed point $x \ge x_0$ which contradicts the hypothesis 2) in ($H_2^{''}$).

2.2.2. **Application.** Let $(Y, |.|)$ be a Banach space and φ be a real-valued measure of noncompactness, defined for all bounded subsets of Y. We assume that φ satisfies all of properties 1-5 in Definition 2.8. We shall use the corresponding cone-valued measure of noncompactness φ_c, which is introduced in Example.

Let $B(x_0, r)$ be a ball in Y, $f : [0, b] \times B(x_0, r) \times B(x_0, r) \longrightarrow Y$ be a uniformly continuous bounded mapping and $h : [0, b] \longrightarrow \mathbb{R}$ be a continuous function, satisfying

(f_1) $\exists m, l > 0$, $\exists \alpha \in (0, 1] : \varphi[f(t, L, M)] \le l\varphi(L) + m[\varphi(M)]^\alpha$ for all subsets $L, M \subset B(x_0, r)$;

(f_2) $0 \le h(t) \le t^{1/\alpha}$.

Let us consider the Cauchy problem

$$x^{/}(t) = f[t, x(t), x(h(t))], \qquad x(0) = x_0. \tag{2.9}$$

In the case that f does not depend on second variable, (2.9) has been studied in [4]. Here we also follow the method of [4] by using the cone-valued measure φ_c.

Theorem 2.14. *Let the hypotheses* (f_1),(f_2) *be satisfied. Then there exists a number* $b_1 \in [0, b]$ *such that* (2.9) *has a solution on* $[0, b_1]$.

Proof. First we observe that if $\Omega \subset C([0, b], Y)$ is an equicontinuous subset then by using properties 2, 3, 5 of the measure φ and that the value $\int_0^t x(s)\,ds$ can be uniformly approximated by the integral sums we deduce that

$$\varphi\left(\left\{\int_0^t x(s)\,ds | x \in \Omega\right\}\right) \le \int_0^t \varphi[\Omega(s)]\,ds. \tag{2.10}$$

By boundedness of f and that $\alpha \le 1$ we may choose $b_1 \in [0, b)$ small enough so that $b_1^{1/\alpha} \le b_1$ and

$$|f(t, x, y)| \le \frac{r}{b_1} \qquad \forall (t, x, y) \in [0, b] \times B(x_0, r) \times B(x_0, r),$$

Then we shall prove that the operator

$$Fx(t) = x_0 + \int_0^t f[s, x(s), x(h(s))]\,ds$$

has a fixed point in the set

$$D = \left\{x \in C([0, b_1], Y) : x(0) = x_0,\ x \text{ is Lipschitz with constant } \frac{r}{b_1}\right\}.$$

Let $E = C([0, b_1], \mathbb{R})$ and $K \subset E$ be the cone of nonnegative functions and φ_c be the K-valued measure of noncompactness, which is defined in Example. Let us define the operators $B : E \longrightarrow E$, $C : K \longrightarrow K$ by

$$Bu(t) = l\int_0^t u(s)\,ds, \qquad Cu(t) = \int_0^t (u[h(s)])^\alpha\,ds.$$

Clearly, B is positive linear with spectral radius $r(B) = 0$ and C is increasing. For $\Omega \subset D$, by using (2.10) and (f_1) we have

$$\begin{aligned}\varphi[F(\Omega)(t)] &= \varphi\left(\left\{\int_0^t f[s, x(s), x(h(s))]\,ds \ : x \in \Omega\right\}\right)\\ &\le \int_0^t \varphi(f[s, \Omega(s), \Omega(h(s))])\,ds\\ &\le l\int_0^t \varphi[\Omega(s)]\,ds + m\int_0^t (\varphi[\Omega(h(s))])^\alpha\,ds.\end{aligned}$$

Consequently, $\varphi_c(F(\Omega)) \le (B + mC)(\varphi_c(\Omega))$.

Consider an element $x_0 \in K$ satisfying $x_0 \le (B + mC)(x_0)$ or equivalently

$$x_0(t) \le l\int_0^t x_0(s)\,ds + m\int_0^t (x_0[h(s)])^\alpha\,ds. \tag{2.11}$$

By the Gronwall inequality, if

$$x_0(t) \le l\int_0^t x_0(s)\,ds + g(t)$$

and $g(t)$ is a nondecreasing function, then we have $x_0(t) \le e^{lt}g(t)$. Hence, we deduce from (2.11) that

$$x_0(t) \le k\int_0^t (x_0[h(s)])^\alpha\,ds = kC(x_0)(t) \tag{2.12}$$

for some $k > 0$. From (2.12) we can prove by induction that

$$\begin{aligned}x_0(t) &\le (kC)^n(x_0)(t)\\ &\le k^{1+\alpha+\ldots+\alpha^n}\cdot\|x_0\|^{\alpha^n}\cdot t^n\cdot\left[2^{\alpha^{n-2}}\cdot 3^{\alpha^{n-3}}\ldots(n-1)^\alpha\cdot n\right]^{-1}\end{aligned}$$

which implies $x_0 = \theta$. Thus, the operator $A = B + mC$ satisfies conditions (H_1), (H_2) of Theorem 3. Therefore, the operator F has a fixed point in D.

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