Fixed Point Theory, 17(2016), No. 2, 341-348
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FIXED-POINT-FREE FUNCTIONS GENERATING COUNTEREXAMPLES TO ROLLE'S THEOREM IN ℓ_2

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Abstract. The map $T : \ell_2 \to \ell_2$ such that $T(x) = \left(\frac{1}{2} - \|x\|^2\right) \cdot e_1 + Rx$, as well as the map $N(x) = \sqrt{1 - \|x\|^2} \cdot e_1 + Rx$, were used by this author to produce two corresponding counterexamples to the classical Rolle's theorem in the closed unit ball of ℓ_2. In this paper we introduce a class of maps, containing the before mentioned examples, which can be used to generate counterexamples to Rolle's theorem in the unit ball of ℓ_2.

Key Words and Phrases: Fixed-point-free maps, Rolle's theorem counterexamples.
2010 Mathematics Subject Classification: 49J50, 49J52, 47H10.

1. Introduction

Back in 1995, M. Furi and M. Martelli posed the problem of finding an explicit counterexample to the classical Rolle's theorem in the closed unit ball of ℓ_2, see [3], that is, a real-valued function f continuous in the closed unit ball of ℓ_2, Fréchet differentiable in its interior, with $f = 0$ in the sphere, such that $f'(x) \neq 0$ in the open ball. We gave an answer to this problem in [1] by means of the function

$$f(x) = \frac{1 - \|x\|^2}{\|x - T(x)\|^2},$$

where $x \in \ell_2$, $\|\cdot\|$ is the Euclidean norm and $T : \ell_2 \to \ell_2$ is the mapping given by

$$T(x) = \left(\frac{1}{2} - \|x\|^2\right) \cdot e_1 + Rx,$$

with e_1 being the first unit vector and R standing for the right-shift operator in ℓ_2. In fact, the function f is similar to the one used by Furi and Martelli, only with no square in the denominator, in order to show that a real-valued function may be continuous in the closed unit ball of ℓ_2 and yet be unbounded. We just put the square in the denominator to make the function Fréchet-differentiable, then proving that $f'(x) \neq 0$, whenever $\|x\| < 1$. The continuity of this function relies on the fact that the mapping

The author has been partially supported by grant MTM2014-57838-C2-2-P..

T is fixed-point-free. Some time later, in [2], we gave a different counterexample to Rolle's theorem, namely

$$f_N(x) = \frac{1-\|x\|^2}{\|x-N(x)\|^2},$$

where $N(x) = \sqrt{1-\|x\|^2}\cdot e_1 + Rx$. Even though both functions f and f_N are quite similar, the argument by contradiction used to show that their derivatives do not vanish in the open unit ball works much faster in the second example. We would like to recall that both maps T and N share the property of having no fixed points. In fact, T is a variant of the so-called Kakutani's map $K(x) = (1-\|x\|)\cdot e_1 + Rx$, see [5], while N is shown not to be pseudo-contractive for any renorming of ℓ_2 in [4]. In this paper, we introduce a class of maps, containing the former mentioned examples, which will be used to obtain some necessary conditions in order to satisfy Rolle's theorem. These necessary conditions turn out useful to obtain some explicit counterexamples to Rolle's theorem which contain the two original ones. We shall always be working in the closed unit ball B_{ℓ_2} of the Hilbert space ℓ_2, its interior, i.e., the open unit ball, will be denoted by U_{ℓ_2}.

2. A class of maps with no fixed points

We consider the family $\mathbf{C}$ formed by all real functions $\varphi : [0,1] \to \mathbf{R}$ satisfying the following conditions

i) $\varphi(0) \neq 0$,
ii) φ is continuous in $[0,1]$,
iii) φ is differentiable in $]0,1[$,
iv) the function $\psi(t) := \varphi(t^2)$ is also differentiable at $t=0$ and $\psi'(0) = 0$..

For a function $\varphi \in \mathbf{C}$, we define the map $T_\varphi : B_{\ell_2} \to \ell_2$ such that

$$T_\varphi(x) := \varphi(\|x\|^2)\cdot e_1 + Rx,$$

where e_1 and R are, as usual, the first unit vector and the right-shift operator, respectively. Clearly, the maps N and T mentioned before are of the type T_φ for some function φ in the class $\mathbf{C}$ just introduced by simply taking

$$\left.\begin{array}{l} \varphi_1(t) := \sqrt{1-t} \\ \varphi_2(t) := \frac{1}{2} - t \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \varphi_1, \varphi_2 \in \mathbf{C} \\ N = T_{\varphi_1}\ T = T_{\varphi_2}. \end{array}\right.$$

Notice that, although Kakutani's map can be written as T_φ, with $\varphi(t) = 1-\sqrt{t}$, φ does not belong to class $\mathbf{C}$, since condition iv) is not satisfied; in anyway, this map K is out of our scope of interest given that the corresponding function f_φ is not Fréchet differentiable at zero.

From the condition $\varphi(0) \neq 0$, it is straightforward to check that, for each $\varphi \in \mathbf{C}$, the map T_φ has no fixed points. We shall now proceed to calculate the Fréchet derivative of T_φ. Identifying in the usual fashion ℓ_2 with its dual space, it is clear that the map T_φ is Fréchet differentiable in every point of the open unit ball except possibly at zero. We show first that T_φ is also Fréchet differentiable at $x = 0$ by proving that

$T'_\varphi(0) = R$, and then evaluate its derivative in any arbitrary point of U_{ℓ_2}.

$$\lim_{\|y\|\to 0} \frac{T_\varphi(y) - T_\varphi(0) - Ry}{\|y\|} = \lim_{\|y\|\to 0} \frac{\varphi(\|y\|^2)e_1 + Ry - \varphi(0)e_1 - Ry}{\|y\|}$$

$$= \lim_{\|y\|\to 0} \frac{\varphi(\|y\|^2) - \varphi(0)}{\|y\|} \cdot e_1 = \lim_{t\to 0} \frac{\varphi(t^2) - \varphi(0)}{t} \cdot e_1$$

$$= \lim_{t\to 0} \frac{\psi(t) - \psi(0)}{t} \cdot e_1 = \psi'(0) \cdot e_1 = 0.$$

If $x \in U_{\ell_2} \setminus \{0\}$, then, using the notation $\langle \cdot, \cdot \rangle$ for the inner product of ℓ_2, and $T'_\varphi(x)$ for the Fréchet derivative of T_φ at x,

$$T'_\varphi(x)y = 2\varphi'(\|x\|^2)\langle x, y\rangle \cdot e_1 + Ry, y \in \ell_2,$$

which gives

$$T'_\varphi(x) = 2\varphi'(\|x\|^2)\langle x, \cdot\rangle \cdot e_1 + R. \tag{2.1}$$

Therefore, the expression in (2.1) gives the value of the derivative T'_φ for any point in U_{ℓ_2}. Next we are going to introduce the type of functions which we will later use to obtain counterexamples to Rolle's theorem in order to first get hold of its Fréchet derivative. For each $\varphi \in \mathbf{C}$, we consider the function $f_\varphi : B_{\ell_2} \to \mathbf{R}$ defined as

$$f_\varphi(x) = \frac{1 - \|x\|^2}{\|x - T_\varphi(x)\|^2}.$$

To evaluate the Fréchet derivative of f_φ, we introduce the real-valued function

$$g(x) := \|x - T_\varphi(x)\|^2, \ x \in B_{\ell_2}.$$

Hence we have

$$f_\varphi(x) = \frac{1 - \|x\|^2}{g(x)}.$$

Now, noticing that $\langle x, Ry\rangle = \langle Lx, y\rangle$, where L denotes the left-shift operator, the Fréchet derivative of g at $x \in U_{\ell_2}$ is given by, for $y \in \ell_2$,

$$\langle g'(x), y\rangle = 2 \cdot \langle x - T_\varphi(x), y - T'_\varphi(x)y\rangle = 2 \cdot [\langle x - T_\varphi(x), y\rangle - \langle x - T_\varphi(x), T'_\varphi(x)y\rangle]$$

$$= 2 \cdot [\langle x - T_\varphi(x), y\rangle - \langle x - T_\varphi(x), 2\varphi'(\|x\|^2)\langle x, y\rangle \cdot e_1 + Ry\rangle]$$

$$= 2 \cdot [\langle x - T_\varphi(x), y\rangle - 2\varphi'(\|x\|^2)\langle x, y\rangle\langle x, e_1\rangle - \langle x, Ry\rangle$$
$$+ 2\varphi'(\|x\|^2)\langle x, y\rangle\varphi(\|x\|^2) + \langle T_\varphi(x), Ry\rangle]$$

$$= 2 \cdot [\langle x - T_\varphi(x), y\rangle - 2\varphi'(\|x\|^2)\langle x, y\rangle\langle x, e_1\rangle - \langle Lx, y\rangle + 2\varphi'(\|x\|^2)\langle x, y\rangle\varphi(\|x\|^2) + \langle x, y\rangle]$$

$$= 2 \cdot \langle x - T_\varphi(x) - 2\varphi'(\|x\|^2)\langle x, e_1\rangle \cdot x - Lx + 2\varphi(\|x\|^2)\varphi'(\|x\|^2) \cdot x + x, y\rangle$$

$$= 2 \cdot \langle 2[1 + \varphi'(\|x\|^2) \cdot (\varphi(\|x\|^2) - \langle x, e_1\rangle)] \cdot x - T_\varphi(x) - Lx, y\rangle],$$

i.e.

$$g'(x) = 2 \cdot [2(1 + \varphi'(\|x\|^2) \cdot (\varphi(\|x\|^2) - \langle x, e_1\rangle)) \cdot x - T_\varphi(x) - Lx]. \tag{2.2}$$

Hence, the derivative of f_φ at $x \in U_{\ell_2}$ is, for $y \in \ell_2$,

$$\langle f'_\varphi(x), y\rangle = \frac{1}{g(x)^2}[-2g(x)\langle x, y\rangle - (1 - \|x\|^2)\langle g'(x), y\rangle],$$

i.e.

$$f'_\varphi(x) = \frac{-1}{g(x)^2}[2g(x)\cdot x + (1-\|x\|^2)g'(x)]. \tag{2.3}$$

Consequently, since from (2.2)

$$g'(0) = -2\cdot T_\varphi(0) = -2\cdot\varphi(0)\cdot e_1,$$

it is plain that, from (2.3),

$$f'_\varphi(0) = \frac{-g'(0)}{g(0)^2} = \frac{2\cdot\varphi(0)}{\varphi(0)^4}\cdot e_1 = \frac{2}{\varphi(0)^3}\cdot e_1 \neq 0.$$

Thus, if there is a point $x \in U_{\ell_2}$ such that $f'_\varphi(x) = 0$, then $x \neq 0$.

3. Generating counterexamples to Rolle's theorem

In the previous section, given a function $\varphi \in \mathbf{C}$, we have calculated the Fréchet derivative of f_φ. We proceed next to study under what additional conditions on φ we can assure that $f'_\varphi(x) \neq 0$, whenever $x \in U_{\ell_2}$.

Assuming there is a point $x \in U_{\ell_2}$ such that $f'_\varphi(x) = 0$, from what we did before we know that $x \neq 0$, and, after (2.2) and (2.3), we have

$$\frac{g(x)}{1-\|x\|^2}\cdot x = -2[1+\varphi'(\|x\|^2)(\varphi(\|x\|^2) - \langle x, e_1\rangle)]\cdot x + T_\varphi(x) + Lx.$$

Setting

$$\lambda := 2[1+\varphi'(\|x\|^2)(\varphi(\|x\|^2) - \langle x, e_1\rangle)] + \frac{g(x)}{1-\|x\|^2} \in \mathbf{R},$$

we obtain the equation

$$T_\varphi(x) + Lx = \lambda x, \tag{3.1}$$

which by applying the operator L, noticing that $LT_\varphi(x) = x$, we have

$$L^2x - \lambda\cdot Lx + x = 0. \tag{3.2}$$

Given that $x = (x_1, x_2, \ldots) \in \ell_2$, we may use general notions of finite difference equations to recognize (3.2) as a finite difference equation of second order. The discriminant of the characteristic equation being $\lambda^2 - 4$ gives three different possibilities:

One. $|\lambda| = 2$. In this case, the general solution of the finite difference equation has the form $x = Au + Bv$, with

$$u = \left(1, \frac{\lambda}{2}, \left(\frac{\lambda}{2}\right)^2, \ldots\right);\ v = \left(0, \frac{\lambda}{2}, 2\left(\frac{\lambda}{2}\right)^2, 3\left(\frac{\lambda}{2}\right)^3, \ldots\right).$$

Hence

$$x_n = A\left(\frac{\lambda}{2}\right)^{n-1} + B(n-1)\left(\frac{\lambda}{2}\right)^{n-1},\ n \geq 1.$$

But $\lim_{n\to\infty} x_n = 0$ implies that $A = B = 0$, i.e., $x = 0$, which is a contradiction.

Two. $|\lambda| < 2$. Now the characteristic equation has the complex roots

$$z_1 = \cos\theta + i\cdot\sin\theta,\ z_2 = \cos\theta - i\cdot\sin\theta, \sin\theta \neq 0.$$

Consequently, there are complex constants A, B such that

$$x_n = A(\cos\theta + i\cdot\sin\theta)^{n-1} + B(\cos\theta - i\cdot\sin\theta)^{n-1}, n \geq 1.$$

Thus, for suitable real constants C, D, we obtain

$$x_n = C\cos(n-1)\theta + D\sin(n-1)\theta, n \geq 1.$$

But, since $\sin\theta \neq 0$, again $\lim_{n\to\infty} x_n = 0$ implies that $C = D = 0$, i.e., $x = 0$, again a contradiction.

Three. $\mid\lambda\mid > 2$. Here, the characteristic equations has the real roots $r, \frac{1}{r}$, such that one of them has absolute value strictly smaller than one. We assume that $0 <\mid r\mid< 1$. Then, since

$$x_n = Ar^{n-1} + B\left(\frac{1}{r}\right)^{n-1},\ n \geq 1,$$

again $\lim\limits_{n\to\infty} x_0 = 0$ implies that $B = 0$, and so x is the geometric progression

$$x = (x_1, x_2, x_3, \ldots) = (x_1, x_1 r, x_1 r^2, \ldots), x_1 \neq 0.$$

We do now some more calculations in order to obtain some necessary conditions.

$$\left.\begin{array}{l} \|x\|^2 = \frac{x_1^2}{1-r^2} \\ \|T_\varphi(x)\|^2 = \varphi(\|x\|^2)^2 + \|x\|^2 \end{array}\right\} \Rightarrow \|T_\varphi(x)\|^2 = \varphi(\|x\|^2)^2 + \frac{x_1^2}{1-r^2}.$$

But, from (3.1),

$$\varphi(\|x\|^2) + x_1 r = \lambda\cdot x_1,$$

from where

$$\varphi(\|x\|^2) = (\lambda - r)x_1 = \frac{x_1}{r},$$

and so

$$\|T_\varphi(x)\|^2 = \left(\frac{x_1}{r}\right)^2 + \frac{x_1^2}{1-r^2} = \frac{x_1^2}{r^2(1-r^2)} = \frac{\|x\|^2}{r^2}.$$

Still more patient calculations give us

$$g(x) = \|x - T_\varphi(x)\|^2 = \|x\|^2 + \|T_\varphi(x)\|^2 - 2\cdot\langle x, T_\varphi(x)\rangle,$$

which, using our former evaluations, lead us to

$$g(x) = \|x\|^2 + \frac{\|x\|^2}{r^2} - 2\cdot\langle x, T_\varphi(x)\rangle. \tag{3.3}$$

But

$$\langle x, T_\varphi(x)\rangle = \varphi(\|x\|^2)\cdot x_1 + x_1^2 r(1 + r^2 + r^4 + \ldots) = \varphi(\|x\|^2)\cdot x_1 + \frac{x_1^2 r}{1-r^2},$$

i.e.

$$\langle x, T_\varphi(x)\rangle = \varphi(\|x\|^2)\cdot x_1 + r\|x\|^2.$$

Consequently

$$\frac{g(x)}{1-\|x\|^2} = \frac{(1-r)^2\|x\|^2}{r^2(1-\|x\|^2)}.$$

Now, from the definition of the value λ

$$r+\frac{1}{r}=\lambda=2[1+\varphi'(\|x\|^2)(\varphi(\|x\|^2)-x_1)]+\frac{(1-r)^2\|x\|^2}{r^2(1-\|x\|^2)}.$$

Multiplying by r^2

$$r^3+r=2r^2[1+\varphi(\|x\|^2)\varphi'(\|x\|^2)-x_1\cdot\varphi'(\|x\|^2)]+\frac{(1-r)^2\|x\|^2}{1-\|x\|^2},$$

and so

$$r(1-r)=2r^2\cdot\varphi(\|x\|^2)\varphi'(\|x\|^2)+\frac{(1-r)\|x\|^2}{1-\|x\|^2}.$$

Consequently, the following result has been obtained

Proposition 3.1 *Let f_φ be the real-valued function defined in the previous section, where $\varphi\in\mathbf{C}$. A necessary condition for f_φ to satisfy Rolle's theorem in the closed unit ball of ℓ_2 is that there exist real numbers r, x_1 and $x\in\ell_2$, with $0<|\ r\ |<1$, $0<|\ x_1\ |<1$, $x=(x_1,x_1r,x_1r^2,\ldots)$, such that*

$$\|x\|^2=\frac{x_1^2}{1-r^2}<1;\varphi(\|x\|^2)=\frac{x_1}{r};r(1-r)=2r^2\cdot\varphi(\|x\|^2)\varphi'(\|x\|^2)+\frac{(1-r)\|x\|^2}{1-\|x\|^2}.$$

4. Explicit counterexamples to Rolle's theorem

In this section, we introduce a subclass of the class $\mathbf{C}$ such that, for each element φ of this subclass, the corresponding function f_φ does not satisfy Rolle's theorem in the closed unit ball of ℓ_2. This subclass, which clearly contains the two original counterexamples given in [1] and [2], is formed by the functions

$$\{\varphi(t)=a\sqrt{1-t}:a\neq0\}\cup\{\varphi(t)=b-t:0<b\leq1\}.$$

In order to show that, for φ in this subclass, the function f_φ does not accomplish Rolle's theorem it suffices to prove that the necessary condition of the former proposition leads to a contradiction. With this in mind, let $\varphi(t)=a\sqrt{1-t}$, $a\neq0$. From the conditions above, since

$$\varphi(\|x\|^2)\varphi'(\|x\|^2)=-\frac{a^2}{2},$$

we have that

$$r(1-r)(1-\|x\|^2)=-a^2r^2(1-\|x\|^2)+(1-r)\|x\|^2. \tag{4.1}$$

Now, using again the conditions in the proposition above, we obtain

$$a^2(1-\|x\|^2)=\varphi(\|x\|^2)^2=\frac{x_1^2}{r^2}=\frac{1-r^2}{r^2}\|x\|^2,$$

which leads us to

$$(1-r^2)\|x\|^2=a^2r^2(1-\|x\|^2). \tag{4.2}$$

From (4.1) and (4.2), we deduce

$$0=r(1-r)(1-\|x\|^2+\frac{1+r}{r}\|x\|^2-\frac{1}{r}\|x\|^2)=r(1-r),$$

a contradiction.

For $\varphi(t) = b - t$, $0 < b \leq 1$, to show that the function f_φ is another counterexample to our theorem will be a little more complicated. As done before, since now

$$\varphi(\|x\|^2)\varphi'(\|x\|^2) = \|x\|^2 - b,$$

we obtain

$$r(1-r) = 2r^2\left(\|x\|^2 - b + \frac{1}{2}\right) - r^2 + \frac{(1-r)\|x\|^2}{1-\|x\|^2},$$

leading to

$$2\left(\|x\|^2 - b + \frac{1}{2}\right)(1-\|x\|^2) = \frac{1}{r} - \frac{\|x\|^2}{r^2}. \tag{4.3}$$

But

$$(b-\|x\|^2)^2 = \varphi(\|x\|^2)^2 = \frac{x_1^2}{r^2} = \frac{1-r^2}{r^2}\|x\|^2 = \frac{\|x\|^2}{r^2} - \|x\|^2. \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$\frac{1}{r} = \|x\|^2(2-\|x\|^2) + (1-b)^2. \tag{4.5}$$

This implies that $0 < r < 1$ and so

$$\|x\|^2(2-\|x\|^2) + (1-b)^2 > 1,$$

which yields that, since $b \leq 1$,

$$1 - b > 1 - \|x\|^2,$$

i.e.

$$\|x\|^2 > b. \tag{4.6}$$

Again making use of the equality of the former proposition, we have

$$r(1-r) = 2r^2(\|x\|^2 - b) + \frac{(1-r)\|x\|^2}{1-\|x\|^2},$$

which leads to

$$1 + (2b-1)r = r\cdot\|x\|^2\cdot[2 + \frac{1-r}{r^2(1-\|x\|^2)}].$$

From here, after (4.5), we get

$$2\left(\|x\|^2 - b + \frac{1}{2}\right) + \frac{(1-r)\|x\|^2}{r^2(1-\|x\|^2)} = \frac{1}{r} = \|x\|^2(2-\|x\|^2) + (1-b)^2,$$

which takes us to

$$b^2 - \|x\|^4 = \frac{(1-r)\|x\|^2}{r^2(1-\|x\|^2)} > 0,$$

a contradiction, in light of (4.6) since $b > 0$.

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Received: January 15, 2014; Accepted: March 8, 2014.