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ON THE ORTHOGONAL PEXIDER DERIVATIONS IN ORTHOGONALITY BANACH ALGEBRAS

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Abstract. In the present paper, we introduce a new system of functional equations, known as orthogonal Pexider derivations. We investigate the stability and hyperstability of this class of functional equations, including the orthogonal Pexider ring derivation and the orthogonal Pexider Jordan ring derivation, by using the fixed point method.
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1. Introduction and preliminaries

An orthogonality space $(X, \perp)$ is a real vector space X with $\dim X \geq 2$ together with a binary relation $\perp$ satisfying some axioms similar to the ones in [11].

There are several orthogonality notions on a real normed space such as Birkhoff–James, Boussouis, (semi–)inner product, Singer, Carlsson, area, unitary–Boussouis, Roberts, Pythagorean, isosceles and Diminnie (see, e.g., [1, 2]). But here, we present the orthogonality concept introduced by J. Rätz [20]. This is given in the following definition.

Suppose that X is a real vector space (or an algebra) with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O_4) the Thalesian property: if P is a $2-$dimensional subspace (subalgebra) of X, $x \in P$ and $\lambda \in R_+$,
then there exists $u_x \in P$ such that $x \perp u_x$ and $x + u_x \perp \lambda x - u_x$.

The pair $(X, \perp)$ is called an orthogonality space (algebra). By an orthogonality normed space (normed algebra) we mean an orthogonality space (algebra) having a normed structure.

The first result on the stability of functional equations was given in 1941 by Hyers [12] who proved the following theorem:
Let X and Y be Banach spaces. If $\varepsilon > 0$ and $f : X \to Y$ be a mapping such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and some $\varepsilon > 0$, then, there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$. This was a first answer given to a question proposed by S.M. Ulam in a talk at a conference at the Wisconsin University in 1940 and it represents the starting point of the Hyers–Ulam stability theory of functional equations (see [23]). The subject was later strongly developed by many authors. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\varepsilon \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Th.M. Rassias [19] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for all $x \in X$. A generalization of the theorem of Th.M. Rassias was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function $\varphi : X \times X \to [0, \infty)$ in the spirit of Rassias approach.

There are cases in which each approximate mapping is actually a true mapping. In such cases, we say that the functional equation is hyperstable. Indeed, a functional equation is hyperstable if every solution satisfying the equation approximately is an exact solution of it. For the history and various aspects of this theory we refer the reader to papers [4, 5, 9, 10, 18, 21, 22].

The orthogonal Cauchy functional equation $f(x+y) = f(x)+f(y)$, $x \perp y$ in which $\perp$ is an abstract orthogonality relation was first investigated in [11]. A generalized version of Cauchy equation is the equation of Pexider type $f_1(x+y) = f_2(x)+f_3(y)$. Jun et. al. [14, 15] obtained the Hyers–Ulam stability of this Pexider equation.

Let $(A, \perp)$ be an orthogonality normed algebra and B be an A-bimodule. A mapping $d : A \to B$ is an orthogonally ring derivation if d is an orthogonally additive mapping satisfying

$$d(xy) = xd(y) + d(x)y \tag{1.1}$$

for all $x, y \in A$ with $x \perp y$. Moreover, a mapping $d : A \to B$ is said to be an orthogonally Jordan ring derivation, if d is an orthogonally additive mapping satisfying

$$d(xy + yx) = xd(y) + d(x)y + yd(x) + d(y)x \tag{1.2}$$

for all $x, y \in A$ with $x \perp y$. In particular, we may define orthogonally derivations associated to the Pexiderized Cauchy functional equation.

Definition 1.1. Let $(A, \perp)$ be an orthogonality normed algebra and B be an A-bimodule and let $f, g, h : A \to B$ be mappings satisfying the system

$$f(x+y) = g(x) + h(y),$$

$$f(xy) = xg(y) + h(x)y$$

for all $x, y \in A$ with $x \perp y$, then we call it an orthogonal Pexiderized ring derivation system of equations. Moreover, if the mappings f, g, h satisfy the system

$$f(x+y) = g(x) + h(y),$$

$$f(xy+yx) = xg(y) + h(x)y + yg(x) + h(y)x$$

for all $x, y \in A$ with $x \perp y$, we call it an orthogonal Pexiderized Jordan ring derivation system of equations.

At the first, the stability problem for derivations was studied by Šemrl in [21]. Then, the topic of approximate derivations, or the stability of the equations of derivation, was taken up by a number of mathematicians (see [3, 7]).

In 1991, J. Baker [6] used the Banach fixed point theorem for prove the Hyers–Ulam stability. The method was generalized in [17]. We recall this fundamental result as follows.

Theorem 1.2 (Banach contraction principle). *Let (X, m) be a complete generalized metric space and consider a mapping $T : X \to X$ as a strictly contractive mapping, that is*

$$m(Tx, Ty) \leq Lm(x, y)$$

for all $x, y \in X$ and for some (Lipschitz constant) $0 < L < 1$. Then

- *T has one and only one fixed point $x^* = T(x^*)$;*
- *x^* is globally attractive, that is, $\lim_{n\to\infty} T^n x = x^*$ for any starting point $x \in X$;*
- *One has the following estimation inequalities for all $x \in X$ and $n \geq 0$*
  $m(T^n x, x^*) \leq L^n m(x, x^*)$,
  $m(T^n x, x^*) \leq \frac{1}{1-L} L^n m(T^n x, T^{n+1} x)$,
  $m(x, x^*) \leq \frac{1}{1-L} m(x, Tx)$.

Theorem 1.3 (The Alternative of Fixed Point [16]). *Suppose that we are given a complete generalized metric space (X, m) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either $m(T^n x, T^{n+1} x) = +\infty$ for all nonnegative integers n or there exists a positive integer n_0 such that $m(T^n x, T^{n+1} x) < +\infty$ for all $n \geq n_0$. If the second alternative holds, then*

$\star$ *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
$\star$ *y^* is the unique fixed point of T in the set $Y = \{y \in X, m(T^{n_0} x, y) < +\infty\}$;*
$\star$ *$m(y, y^*) \leq \frac{1}{1-L} m(y, Ty)$, $y \in Y$.*

In this paper, we apply the above–mentioned fixed point method to prove the Hyers–Ulam stability property for the orthogonal derivations in orthogonality Banach algebras associated to the Pexiderized Cauchy functional equation. Throughout this paper, let $\mathcal{A}$ be an orthogonality Banach algebra and $\mathcal{B}$ a Banach $\mathcal{A}$-bimodule.

2. The orthogonal Pexider ring derivation

In the following theorem, by applying the fixed point method (Theorem 1.3), we will prove the Hyers–Ulam stability and hyperstability properties for the orthogonal Pexider ring derivation.

Theorem 2.1. *Suppose that $f, g, h : \mathcal{A} \to \mathcal{B}$ are mappings fulfilling the system of functional inequalities*

$$\|f(x+y) - g(x) - h(y)\| \leq \varphi(x, y), \tag{2.1}$$

$$\|f(xy) - xg(y) - h(x)y\| \leq \phi(x, y), \tag{2.2}$$

where $\varphi, \phi : X \times X \to [0, \infty)$ are mappings such that

$$\lim_{n\to\infty} \frac{\varphi(2^{nj}x, 2^{nj}y)}{2^{nj}} = 0, \tag{2.3}$$

$$\lim_{n\to\infty} \frac{\phi(2^{nj}x, y)}{2^{nj}} = \lim_{n\to\infty} \frac{\phi(x, 2^{nj}y)}{2^{nj}} = 0 \tag{2.4}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$, where $j \in \{-1, 1\}$. If f is an odd mapping, $\varphi(0,0) = \phi(0,0) = 0$ and there exists $0 < L = L(j) < 1$ such that for any fixed $x \in \mathcal{A}$ and some $u_x \in \mathcal{A}$ with $x \perp u_x$, the mapping

$$\begin{aligned} x \mapsto \psi(x, u_x) =& \varphi\left(\frac{x+u_x}{2}, \frac{x-u_x}{2}\right) + \varphi\left(0, \frac{x-u_x}{2}\right) \\ &+ \varphi\left(\frac{x+u_x}{2}, 0\right) + \varphi\left(\frac{x}{2}, \frac{u_x}{2}\right) + \varphi\left(\frac{x}{2}, \frac{-u_x}{2}\right) \\ &+ 2\varphi\left(\frac{x}{2}, 0\right) + \varphi\left(0, \frac{u_x}{2}\right) + \varphi\left(0, \frac{-u_x}{2}\right) \end{aligned} \tag{2.5}$$

has the property

$$\psi(x, u_x) \leq L2^j \psi(\frac{x}{2^j}, \frac{u_x}{2^j}), \tag{2.6}$$

then there exists a unique orthogonally ring derivation $d : \mathcal{A} \to \mathcal{B}$ such that

$$\begin{aligned} \|f(x) - d(x)\| &\leq \frac{L^{\frac{1+j}{2}}}{1-L} \psi(x, u_x), \\ \|g(x) - g(0) - d(x)\| &\leq \frac{L^{\frac{1+j}{2}}}{1-L} \psi(x, u_x) + \varphi(x, 0), \\ \|h(x) - h(0) - d(x)\| &\leq \frac{L^{\frac{1+j}{2}}}{1-L} \psi(x, u_x) + \varphi(0, x). \end{aligned} \tag{2.7}$$

Proof. Let $E = \{e : \mathcal{A} \to \mathcal{B} \mid e(0) = 0\}$. For any fixed $x \in \mathcal{A}$ and some $u_x \in \mathcal{A}$ with $x \perp u_x$, define $m : E \times E \to [0, \infty]$ by

$$m(e_1, e_2) = inf\Big\{K \in \mathbb{R}_+ : \|e_1(x) - e_2(x)\| \leq K\psi(x, u_x)\Big\}.$$

As usual $\inf O = \infty$. It is easy to see that (E, m) is a complete generalized metric space. Let us consider the linear mapping $T : E \to E$, $Te(x) = \frac{1}{2^j} e(2^j x)$ for all $x \in \mathcal{A}$. T is a strictly contractive mapping with the Lipschitz constant L. Indeed, for given e_1 and e_2 in E such that $m(e_1, e_2) < \infty$ and any $K > 0$ satisfying $m(e_1, e_2) < K$ and any fixed $x \in \mathcal{A}$ and some $u_x \in \mathcal{A}$ with $u_{\alpha x} = \alpha u_x$ $(\alpha \in \mathbb{R})$ and $x \perp u_x$, we have

$$\begin{aligned}
&\|e_1(x) - e_1(x)\| \leq K\psi(x, u_x) \\
&\Rightarrow \|\frac{1}{2^j} e_1(2^j x) - \frac{1}{2^j} e_2(2^j x)\| \leq \frac{1}{2^j} K\psi(2^j x, 2^j u_x) \\
&\Rightarrow \|\frac{1}{2^j} e_1(2^j x) - \frac{1}{2^j} e_2(2^j x)\| \leq LK\psi(x, u_x) \\
&\Rightarrow m(Te_1, Te_2) \leq LK.
\end{aligned}$$

Put $K = m(e_1, e_2) + \frac{1}{n}$ for positive integers n. Then $m(Te_1, Te_2) \leq L(m(e_1, e_2) + \frac{1}{n})$. Letting $n \to \infty$ gives

$$m(Te_1, Te_2) \leq Lm(e_1, e_2)$$

for all $e_1, e_2 \in E$.

Since $\varphi(0, 0) = \phi(0, 0) = 0$, putting $x, y = 0$ in (2.1) and (2.2), we get

$$f(0) = 0, \;\; g(0) + h(0) = 0. \tag{2.8}$$

For every $x, y \in \mathcal{A}$, $x, y \perp 0$. So we can put $y = 0$ and $x = 0$ in (2.1), respectively, to obtain

$$\|f(x) - g(x) - h(0)\| \leq \varphi(x, 0),$$

$$\|f(y) - g(0) - h(y)\| \leq \varphi(0, y)$$

and by (2.8), we conclude that

$$\|f(x) - \big(g(x) - g(0)\big)\| \leq \varphi(x, 0), \tag{2.9}$$

$$\|f(y) - \big(h(y) - h(0)\big)\| \leq \varphi(0, y) \tag{2.10}$$

for all $x, y \in \mathcal{A}$.

Let $x \in \mathcal{A}$ be fixed. By (O_4) there exists $u_x \in \mathcal{A}$ such that $x \perp u_x$, $x + u_x \perp x - u_x$ and $u_{\alpha x} = \alpha u_x$ for all $\alpha \in \mathbb{R}$. Hence

$$\|f(x + u_x) - g(x) - h(u_x)\| \leq \varphi(x, u_x). \tag{2.11}$$

By (O_3), $x \perp -u_x$ and so

$$\|f(x - u_x) - g(x) - h(-u_x)\| \leq \varphi(x, -u_x). \tag{2.12}$$

Replacing x and y by $x + u_x$ and $x - u_x$ in (2.1), we have

$$\|f(2x) - g(x + u_x) - h(x - u_x)\| \leq \varphi(x + u_x, x - u_x). \tag{2.13}$$

Substituting $x + u_x$ for x in (2.9) and $x - u_x$ for y in (2.10), respectively, one gets the inequalities

$$\|f(x + u_x) - \big(g(x + u_x) - g(0)\big)\| \leq \varphi(x + u_x, 0), \tag{2.14}$$

$$\|f(x - u_x) - \big(h(x - u_x) - h(0)\big)\| \leq \varphi(0, x - u_x). \tag{2.15}$$

Thus the triangle inequality and inequalities (2.13), (2.14) and (2.15) yield

$$\begin{aligned}\|f(2x)-f(x+u_x)-f(x-u_x)\| &\le \|f(2x)-g(x+u_x)-h(x-u_x)\| \\ &\quad + \|f(x+u_x)-\big(g(x+u_x)-g(0)\big)\| \\ &\quad + \|f(x-u_x)-\big(h(x-u_x)-h(0)\big)\| \\ &\le \varphi(x+u_x,x-u_x)+\varphi(x+u_x,0)+\varphi(0,x-u_x).\end{aligned} \tag{2.16}$$

It follows from (2.1), (2.10), (2.11), (2.12), oddness of f and triangle inequality that

$$\begin{aligned}&\|2f(x)-f(x+u_x)-f(x-u_x)\| \\ &\le \|f(x+u_x)-g(x)-h(u_x)\|+\|f(x-u_x)-g(x)-h(-u_x)\| \\ &+2\|f(x)-\big(g(x)-g(0)\big)\|+\|f(u_x)-g(0)-h(u_x)\| \\ &+\|f(-u_x)-g(0)-h(-u_x)\| \\ &\le \varphi(x,u_x)+\varphi(x,-u_x)+2\varphi(x,0)+\varphi(0,u_x)+\varphi(0,-u_x).\end{aligned} \tag{2.17}$$

Now, combining (2.16) and (2.17), we have

$$\begin{aligned}\|f(2x)-2f(x)\| &\le \varphi(x+u_x,x-u_x)+\varphi(x+u_x,0) \\ &\quad +\varphi(0,x-u_x)+\varphi(x,u_x)+\varphi(x,-u_x) \\ &\quad +2\varphi(x,0)+\varphi(0,u_x)+\varphi(0,-u_x).\end{aligned} \tag{2.18}$$

Using (2.5) and (2.6), we can reduce (2.18) to

$$\|f(x)-\frac{1}{2}f(2x)\| \le \frac{1}{2}\psi(2x,2u_x) \le L\psi(x,u_x),$$

that is, $m(f,Tf) \le L = L^1 < \infty$. Moreover, replacing x in (2.18) by $\frac{x}{2}$ implies the appropriate inequality for $j=-1$

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \le \psi(x,u_x),$$

that is, $m(f,Tf) \le 1 = L^0 < \infty$. By Theorem 1.3, there exists a mapping $d:\mathcal{A}\to\mathcal{B}$ which is the fixed point of T and satisfies

$$d(x)=\lim_{n\to\infty}\frac{f(2^{nj}x)}{2^{nj}},$$

since $\lim_{n\to\infty} m(T^n f,d)=0$. The mapping d is the unique fixed point of T in the set $M=\{e\in E: m(f,e)<\infty\}$. Using Theorem 1.3 we get

$$m(f,d)\le\frac{1}{1-L}m(f,Tf)$$

which yields

$$\|f(x)-d(x)\|\le\frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x).$$

Further, inequalities (2.9) and (2.10) imply that

$$\begin{aligned}\|g(x)-g(0)-d(x)\| &\le \|f(x)-\big(g(x)-g(0)\big)\|+\|f(x)-d(x)\|\\ &\le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x)+\varphi(x,0),\end{aligned}$$

$$\begin{aligned}\|h(x)-h(0)-d(x)\| &\le \|f(x)-\big(h(x)-h(0)\big)\|+\|f(x)-d(x)\|\\ &\le \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x,u_x)+\varphi(0,x)\end{aligned}$$

as desired.

It follows from the inequalities (2.9) and (2.10) that

$$\|2^{-nj}f(2^{nj}x)-2^{-nj}\big(g(2^{nj}x)-g(0)\big)\| \le 2^{-nj}\varphi(2^{nj}x,0),$$

$$\|2^{-nj}f(2^{nj}x)-2^{-nj}\big(h(2^{nj}x)-h(0)\big)\| \le 2^{-nj}\varphi(0,2^{nj}x)$$

for all $x\in\mathcal{A}$ and $n\in\mathbb{N}$, whence

$$d(x)=\lim_{n\to\infty}\frac{g(2^{nj}x)-g(0)}{2^{nj}}=\lim_{n\to\infty}\frac{h(2^{nj}x)-h(0)}{2^{nj}}. \tag{2.19}$$

Let $x,y\in\mathcal{A}$ with $x\perp y$. (O_3) ensures $2^{nj}x\perp 2^{nj}y$ for all $n\in\mathbb{N}$ and from (2.1), (2.3) and (2.19), we obtain

$$\begin{aligned}\|2^{-nj}f\big(2^{nj}(x+y)\big)&-2^{-nj}\big(g(2^{nj}x)-g(0)\big)-2^{-nj}\big(h(2^{nj}y)-h(0)\big)\|\\ &=\|2^{-nj}f\big(2^{nj}(x+y)\big)-2^{-nj}g(2^{nj}x)-2^{-nj}h(2^{nj}y)\|\\ &\le 2^{-nj}\varphi(2^{nj}x,2^{nj}y).\end{aligned}$$

Therefore from $n\to\infty$, one establishes $d(x+y)-d(x)-d(y)=0$. Hence d is orthogonally additive.

In addition, we claim that the mapping d satisfies the functional equation (1.1). Define $r:\mathcal{A}\times\mathcal{A}\to\mathcal{B}$ by $r(x,y)=f(xy)-xg(y)-h(x)y$ for all $x,y\in\mathcal{A}$ with $x\perp y$. Condition (2.4) implies that

$$\lim_{n\to\infty}\frac{r(2^{nj}x,y)}{2^{nj}}=0. \tag{2.20}$$

Utilizing the relations (2.19) and (2.20), one obtains

$$\begin{aligned}d(xy)&=\lim_{n\to\infty}\frac{f\big(2^{nj}(xy)\big)}{2^{nj}}=\lim_{n\to\infty}\frac{f\big((2^{nj}x)y\big)}{2^{nj}}\\ &=\lim_{n\to\infty}\frac{2^{nj}xg(y)+h(2^{nj}x)y+r(2^{nj}x,y)}{2^{nj}}\\ &=\lim_{n\to\infty}\left(xg(y)+\frac{h(2^{nj}x)}{2^{nj}}y+\frac{r(2^{nj}x,y)}{2^{nj}}\right)\\ &=xg(y)+d(x)y+\lim_{n\to\infty}\frac{h(0)}{2^{nj}}y.\end{aligned} \tag{2.21}$$

Set $x = 0$ in (2.21). Since $d(0) = 0$ and $0 \perp y$ for all $y \in \mathcal{A}$, we may conclude that

$$\lim_{n\to\infty} \frac{h(0)}{2^{nj}} y = 0.$$

Hence

$$d(xy) = xg(y) + d(x)y \tag{2.22}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$.

Now, let $x, y \in \mathcal{A}$ with $x \perp y$ and $n \in \mathbb{N}$ be fixed. Using (2.22) and orthogonal additivity of d, one can easily show that

$$\begin{aligned} xg(2^{nj}y) + 2^{nj}d(x)y &= xg(2^{nj}y) + d(x)2^{nj}y \\ &= d\big(x(2^{nj}y)\big) = d\big((2^{nj}x)y\big) \\ &= 2^{nj}xg(y) + d(2^{nj}x)y \\ &= 2^{nj}xg(y) + 2^{nj}d(x)y. \end{aligned}$$

If we compare the above relation with (2.22), we get

$$x\frac{g(2^{nj}y)}{2^{nj}} = xg(y) \tag{2.23}$$

and so

$$d(xy) = x\frac{g(2^{nj}y)}{2^{nj}} + d(x)y.$$

Taking the limit as $n \to \infty$, we see that

$$d(xy) = xd(y) + \lim_{n\to\infty} x\frac{g(0)}{2^{nj}} + d(x)y. \tag{2.24}$$

Letting $y = 0$ in (2.24), we may infer that $\lim\limits_{n\to\infty} x\frac{g(0)}{2^{nj}} = 0$.
Therefore, $d(xy) = xd(y) + d(x)y$. The proof of Theorem 2.1 is now complete.

In particular, given $\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p)$ and $\phi(x, y) = \theta\|x\|^q\|y\|^s$ for $\varepsilon, \theta \geq 0$ and some real numbers p, q, s in the main theorem, one gets the following corollary (as a consequence of Rassias theorem).

Corollary 2.2. *Let $j \in \{-1, 1\}$ and $f, g, h : \mathcal{A} \to \mathcal{B}$ be mappings satisfying*

$$\|f(x + y) - g(x) - h(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

$$\|f(xy) - xg(y) - h(x)y\| \leq \theta\|x\|^q\|y\|^s$$

for all $x, y \in \mathcal{A}$ with $x \perp y$, $\varepsilon, \theta \geq 0$ and real numbers p, q, s such that $p, q < 1$ for $j = 1$ and $p, q > 1$ for $j = -1$. If f is an odd mapping, then there exists a unique *orthogonally ring derivation $d : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - d(x)\| \leq \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \varepsilon \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right),$$

$$\|g(x) - g(0) - d(x)\| \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\}, \quad (2.25)$$

$$\|h(x) - h(0) - d(x)\| \leq \varepsilon \left\{ \frac{2^{\frac{j(1+j)(p-1)}{2}}}{1 - 2^{j(p-1)}} \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) + (\|x\|^p) \right\}$$

for any fixed $x \in \mathcal{A}$ and some $u_x \in \mathcal{A}$ with $x \perp u_x$.

Proof. Let $\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p)$ and $\phi(x, y) = \theta\|x\|^q\|y\|^s$.
Clearly, $\varphi(0, 0) = \phi(0, 0) = 0$. It follows from the hypotheses of the corollary that

$$\lim_{n\to\infty} \frac{\varphi(2^{nj}x, 2^{nj}y)}{2^{nj}} = \lim_{n\to\infty} \varepsilon 2^{nj(p-1)}(\|x\|^p + \|y\|^p) = 0,$$

$$\lim_{n\to\infty} \frac{\phi(2^{nj}x, y)}{2^{nj}} = \lim_{n\to\infty} \theta\ 2^{nj(q-1)}|x\|^q\|y\|^s = 0$$

for all $x, y \in \mathcal{A}$ with $x \perp y$, that is, the conditions (2.3) and (2.4) in the Theorem 2.1 are sharp here. Since the inequality

$$\begin{aligned} 2^{-j}\psi(2^j x, 2^j u_x) &= 2^{j(p-1)}\varepsilon \left(2\|x + u_x\|^p + 2\|x - u_x\|^p + 4\|x\|^p + 4\|u_x\|^p\right) \\ &\leq 2^{j(p-1)}\psi(x, u_x) \end{aligned}$$

holds for any fixed $x \in \mathcal{A}$, some $u_x \in \mathcal{A}$ with $x \perp u_x$, $\varepsilon \geq 0$ and real numbers p such that $p < 1$ for $j = 1$ and $p > 1$ for $j = -1$, we see that the inequality (2.6) in the Theorem 2.1 holds with $L = 2^{j(p-1)}$. Now, by (2.7) we conclude the assertion of this corollary.

Next, we are going to establish the hyperstability of the orthogonal Pexider ring derivation.

Corollary 2.3. *Assume that $f, g, h : \mathcal{A} \to \mathcal{B}$ are mappings satisfying the system*

$$\|f(x + y) - g(x) - h(y)\| \leq \varphi(x, y),$$

$$\|f(xy) - xg(y) - h(x)y\| \leq \phi(x, y),$$

where $\varphi, \phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ are mappings such that

$$\lim_{n\to\infty} \frac{\varphi(2^{nj}x, 2^{nj}y)}{2^{nj}} = 0,$$

$$\lim_{n\to\infty} \frac{\phi(2^{nj}x, y)}{2^{nj}} = \lim_{n\to\infty} \frac{\phi(x, 2^{nj}y)}{2^{nj}} = 0 \quad (2.26)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$, where $j \in \{-1, 1\}$. Let $g(0) = h(0) = 0$ and $\mathcal{B}$ be a Banach $\mathcal{A}$-bimodule without order, i.e. $\mathcal{A}x = 0$ or $x\mathcal{A} = 0$ implies that $x = 0$. If f is an odd mapping, $\varphi(0, 0) = \phi(0, 0) = 0$ and there exists $0 < L = L(j) < 1$ such that *for any fixed $x \in \mathcal{A}$ and some $u_x \in \mathcal{A}$ with $x \perp u_x$, the mapping ψ ((2.5) in Theorem 2.1) has the property*

$$\psi(x, u_x) \leq L2^j \psi(\frac{x}{2^j}, \frac{u_x}{2^j}),$$

then the mappings g, h are orthogonally ring derivations. Moreover, if either $\varphi(0, x) = 0$ or $\varphi(x, 0) = 0$ for all $x \in \mathcal{A}$, then f is orthogonally ring derivation.

Proof. According to Theorem 2.1, there exists an orthogonally ring derivation $d : \mathcal{A} \to \mathcal{B}$ such that

$$d(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty} \frac{g(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty} \frac{h(2^{nj}x)}{2^{nj}} \tag{2.27}$$

for all $x \in \mathcal{A}$, Since $g(0) = h(0) = 0$. By applying (2.27) in (2.23) we conclude that $x\big(d(y) - g(y)\big) = 0$ for all $x, y \in \mathcal{A}$. Therefore, $g = d$.

Let $x, y \in \mathcal{A}$ with $x \perp y$ and r be the mapping defined in Theorem 2.1. It follows from (2.26) that

$$\lim_{n\to\infty} \frac{r(x, 2^{nj}y)}{2^{nj}} = 0.$$

Using the above relation and (2.27), we obtain

$$d(xy) = xd(y) + h(x)y. \tag{2.28}$$

Similarly to the corresponding proof of Theorem 2.1, we have

$$\frac{h(2^{nj}x)}{2^{nj}} y = h(x)y.$$

By applying (2.27) in the previous relation we conclude that $h = d$.

Now, we only need to show that f is orthogonally ring derivation. Applying the last hypothesis of this corollary to the either relation (2.9) or relation (2.10), we get indeed the desired result.

3. The orthogonal Pexider Jordan ring derivation

In this section, we will apply the fixed point method for proving the Hyers–Ulam stability and hyperstability of the orthogonal Pexiderized Jordan ring derivation system of equations.

Theorem 3.1. *Suppose that $f, g, h : \mathcal{A} \to \mathcal{B}$ are mappings satisfying the following system of functional inequalities*

$$\|f(x + y) - g(x) - h(y)\| \leq \varphi(x, y), \tag{3.1}$$

$$\|f(xy + yx) - xg(y) - h(x)y - yg(x) - h(y)x\| \leq \phi(x, y), \tag{3.2}$$

where $\varphi, \phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ are mappings such that

$$\lim_{n\to\infty} \frac{\varphi(2^{nj}x, 2^{nj}y)}{2^{nj}} = 0,$$

$$\lim_{n\to\infty} \frac{\phi(2^{nj}x, y)}{2^{nj}} = \lim_{n\to\infty} \frac{\phi(x, 2^{nj}y)}{2^{nj}} = 0 \tag{3.3}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$, where $j \in \{-1, 1\}$. If f is an odd mapping, $\varphi(0,0) = \phi(0,0) = 0$ and there exists $0 < L = L(j) < 1$ such that for any fixed $x \in \mathcal{A}$ and some $u_x \in \mathcal{A}$ with $x \perp u_x$, the mapping ψ ((2.5) in Theorem 2.1) has the property

$$\psi(x, u_x) \leq L2^j \psi(\frac{x}{2^j}, \frac{u_x}{2^j}),$$

then there exists a unique orthogonally Jordan ring derivation $d : \mathcal{A} \to \mathcal{B}$ such that

$$\begin{aligned}
\|f(x) - d(x)\| &\leq \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x, u_x), \\
\|g(x) - g(0) - d(x)\| &\leq \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x, u_x) + \varphi(x, 0), \\
\|h(x) - h(0) - d(x)\| &\leq \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x, u_x) + \varphi(0, x).
\end{aligned}$$

Proof. Letting $x, y = 0$ in (3.1) and (3.2), we get

$$f(0) = 0 \quad , \quad g(0) + h(0) = 0.$$

Applying the similar argument to the corresponding part of Theorem 2.1, we conclude that there exists a unique orthogonally additive mapping $d : \mathcal{A} \to \mathcal{B}$ which is the fixed point of T and satisfies

$$\|f(x) - d(x)\| \leq \frac{L^{\frac{1+j}{2}}}{1-L}\psi(x, u_x).$$

Moreover,

$$d(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{nj}} = \lim_{n\to\infty} \frac{g(2^{nj}x) - g(0)}{2^{nj}} = \lim_{n\to\infty} \frac{h(2^{nj}x) - h(0)}{2^{nj}}. \tag{3.4}$$

Now, we are going to show that the mapping d satisfies the functional equation (1.2). Define $r : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ by $r(x, y) = f(xy + yx) - xg(y) - h(x)y - yg(x) - h(y)x$ for all $x, y \in \mathcal{A}$ with $x \perp y$. It follows from (3.3) that

$$\lim_{n\to\infty} \frac{r(2^{nj}x, y)}{2^{nj}} = 0. \tag{3.5}$$

Making use of (3.4) and (3.5), we get

$$\begin{aligned}
d(xy + yx) &= \lim_{n\to\infty} \frac{f\big(2^{nj}(xy + yx)\big)}{2^{nj}} = \lim_{n\to\infty} \frac{f\big((2^{nj}x)y + y(2^{nj}x)\big)}{2^{nj}} \\
&= \lim_{n\to\infty} \frac{2^{nj}xg(y) + h(2^{nj}x)y + yg(2^{nj}x) + h(y)2^{nj}x + r(2^{nj}x, y)}{2^{nj}} \\
&= \lim_{n\to\infty} \left(xg(y) + \frac{h(2^{nj}x)}{2^{nj}}y + y\frac{g(2^{nj}x)}{2^{nj}} + h(y)x + \frac{r(2^{nj}x, y)}{2^{nj}} \right) \\
&= xg(y) + d(x)y + yd(x) + h(y)x + \lim_{n\to\infty} \left(\frac{h(0)}{2^{nj}}y + y\frac{g(0)}{2^{nj}} \right). \qquad (3.6)
\end{aligned}$$

Let $x = 0$ in (3.6). Employing the orthogonal additivity of d and the fact that $0 \perp y$ for all $y \in \mathcal{A}$, one proves that $\lim_{n\to\infty}\left(\frac{h(0)}{2^{nj}}y + y\frac{g(0)}{2^{nj}}\right) = 0$. Hence,

$$d(xy + yx) = xg(y) + d(x)y + yd(x) + h(y)x \tag{3.7}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$.

Now let $x, y \in \mathcal{A}$ with $x \perp y$ and $n \in \mathbb{N}$ be fixed. By (3.7) and orthogonal additivity of d, it can be shown that

$$\begin{aligned} xg(2^{nj}y) + 2^{nj}d(x)y + 2^{nj}yd(x) + h(2^{nj}y)x \\ &= xg(2^{nj}y) + d(x)2^{nj}y + 2^{nj}yd(x) + h(2^{nj}y)x \\ &= d\big(x(2^{nj}y) + (2^{nj}y)x\big) = d\big((2^{nj}x)y + y(2^{nj}x)\big) \\ &= 2^{nj}xg(y) + d(2^{nj}x)y + yd(2^{nj}x) + h(y)2^{nj}x \\ &= 2^{nj}xg(y) + 2^{nj}d(x)y + 2^{nj}yd(x) + h(y)2^{nj}x \end{aligned}$$

and then

$$x\frac{g(2^{nj}y)}{2^{nj}} + \frac{h(2^{nj}y)}{2^{nj}}x = xg(y) + h(y)x.$$

Comparing the above relation with (3.7), we get

$$d(xy + yx) = x\frac{g(2^{nj}y)}{2^{nj}} + d(x)y + yd(x) + \frac{h(2^{nj}y)}{2^{nj}}x.$$

Sending n to infinity, we obtain

$$d(xy + yx) = xd(y) + \lim_{n\to\infty} x\frac{g(0)}{2^{nj}} + d(x)y + yd(x) + d(y)x + \lim_{n\to\infty}\frac{h(0)}{2^{nj}}x. \tag{3.8}$$

Putting $y = 0$ in (3.8), one gets

$$\lim_{n\to\infty} x\frac{g(0)}{2^{nj}} + \lim_{n\to\infty}\frac{h(0)}{2^{nj}}x = 0.$$

Whence, $d(xy + yx) = xd(y) + d(x)y + yd(x) + d(y)x$. This completes the proof of the theorem.

As a special case, if one takes $\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p)$ and $\phi(x, y) = \theta\|x\|^q\|y\|^s$ for $\varepsilon, \theta \geq 0$ and some real numbers p, q, s in Theorem 3.1, then one has the following corollary (as a consequence of Rassias theorem).

Corollary 3.2. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be mappings satisfying*

$$\|f(x + y) - g(x) - h(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

$$\|f(xy + yx) - xg(y) - h(x)y - yg(x) - h(y)x\| \leq \theta\|x\|^q\|y\|^s$$

for all $x, y \in \mathcal{A}$ with $x \perp y$, $\varepsilon, \theta \geq 0$ and real numbers p, q, s such that $p, q < 1$ for $j = 1$ and $p, q > 1$ for $j = -1$. If f is an odd mapping, then there exists a unique conditional Jordan ring derivation $d : \mathcal{A} \to \mathcal{B}$ such that (2.25) *in Corollary* 2.2 *is sharp here for any fixed $x \in \mathcal{A}$ and some $u_x \in \mathcal{A}$ with $x \perp u_x$, where $j \in \{-1, 1\}$.*

Proof. The proof of this corollary is omitted as similar to the proof of Corollary 2.2.

We now present the hyperstability result concerning the orthogonal Pexider Jordan ring derivation. The proof is similar to that of Corollary 2.3 and we omit it.

Corollary 3.3. *Assume that* $f, g, h : \mathcal{A} \to \mathcal{B}$ *are mappings satisfying the system*

$$\|f(x+y) - g(x) - h(y)\| \leq \varphi(x, y),$$

$$\|f(xy + yx) - xg(y) - h(x)y - yg(x) - h(y)x\| \leq \phi(x, y),$$

where $\varphi, \phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ *are mappings such that*

$$\lim_{n\to\infty} \frac{\varphi(2^{nj}x, 2^{nj}y)}{2^{nj}} = 0,$$

$$\lim_{n\to\infty} \frac{\phi(2^{nj}x, y)}{2^{nj}} = \lim_{n\to\infty} \frac{\phi(x, 2^{nj}y)}{2^{nj}} = 0$$

for all $x, y \in \mathcal{A}$ *with* $x \perp y$*, where* $j \in \{-1, 1\}$*. Let* $g(0) = h(0) = 0$ *and* $\mathcal{B}$ *be a Banach* $\mathcal{A}$*-bimodule without order, i.e.* $\mathcal{A}x = 0$ *or* $x\mathcal{A} = 0$ *implies that* $x = 0$*. If* f *is an odd mapping,* $\varphi(0, 0) = \phi(0, 0) = 0$ *and there exists* $0 < L = L(j) < 1$ *such that for any fixed* $x \in \mathcal{A}$ *and some* $u_x \in \mathcal{A}$ *with* $x \perp u_x$*, the mapping* ψ *((2.5) in Theorem 2.1) has the property*

$$\psi(x, u_x) \leq L 2^j \psi\left(\frac{x}{2^j}, \frac{u_x}{2^j}\right),$$

then the mappings g, h *are orthogonally Jordan ring derivations. Moreover, if either* $\varphi(0, x) = 0$ *or* $\varphi(x, 0) = 0$ *for all* $x \in \mathcal{A}$*, then* f *is orthogonally Jordan ring derivation.*

References

[1] J. Alonso, C. Benítez, *Carlos orthogonality in normed linear spaces: A survey. II. Relations between main orthogonalities*, Extracta Math., **4**(1989), 121-131.

[2] J. Alonso, C. Benítez, *Orthogonality in normed linear spaces: A survey. I. Main properties*, Extracta Math., **3**(1988), 1-15.

[3] R. Badora, *On approximate derivations*, Math. Inequal. Appl., **9**(2006), 167-173.

[4] J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc., **80**(1980), 411–416.

[5] J. Baker, J. Lawrence, F. Zorzitto, *The stability of the equation* $f(x + y) = f(x)f(y)$, Proc. Amer. Math. Soc., **74**(1979), 242–246.

[6] J.A. Baker, *The stability of certain functional equations*, Proc. Amer. Math. Soc., **112**(1991), 729–732.

[7] M. Eshaghi Gordji, A. Najati, *Approximately* J^**-homomorphisms: A fixed point approach*, J. Geom. Phys., **60**(2010), 809–814.

[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431-436.

[9] R. Ger, *Superstability is not natural*, Rocznik Nauk.-Dydakt. Prace Mat., **159**(1993), 109–123.

[10] R. Ger, P. Šemrl, *The stability of the exponential equation*, Proc. Amer. Math. Soc., **124**(1996), 779–787.

[11] S. Gudder, D. Strawther, *Orthogonally additive and orthogonally increasing functions on vector spaces*, Pacific J. Math., **58**(1975), 427–436.

[12] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA., **27**(1941), 222-224.

[13] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.

[14] K.W. Jun, D.S. Shin, B.D. Kim, *On the Hyers-Ulam-Rassias stability of the Pexider equation*, J. Math. Anal. Appl., **239**(1999), 20-29.

[15] Y.-H. Lee, K.-W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Pexider equation*, J. Math. Anal. Appl., **246**(2000), 627-638.

[16] B. Margolis, J. Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74**(1968), 305-309.

[17] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4**(2003), 91-96.

[18] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251**(2000), 264–284.

[19] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[20] J. Rätz, *On orthogonally additive mappings*, Aequat. Math., **28**(1985), 35-49.

[21] P. Šemrl, *The functional equation of multiplicative derivation is superstable on standard operator algebras*, Inegr. Equat. Oper. Th., **18**(1994), 118-122.

[22] L. Székelyhidi, *On a theorem of Baker, Lawrence and Zorzitto*, Proc. Amer. Math. Soc., **84**(1982), 95–96.

[23] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Publ., New York, 1960.

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