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NEW GENERALIZATION OF THE EXISTENCE OF EQUILIBRIUM FOR GENERALIZED GAME IN ABSTRACT CONVEX SPACE

YAN-MEI DU

[1] Department of Mathematics, Tianjin Polytechnic University
Tianjin 300387, P.R. China
and
[2] Tianjin Key Laboratory of Optoelectronic Detection Technology and System
Tianjin 300387, P.R. China
E-mail: duyanmei@tjpu.edu.cn

Abstract. The purpose of this paper is to establish a general existence of equilibrium for generalized game in abstract convex space, where the preference correspondence has unionly open lower section and the constraint correspondence is transfer open valued. New notions of $\mathscr{U}_A$-mapping and $\mathscr{U}_A$-majorized mapping are introduced, in which the lower sections are unionly open. We first prove some new fixed point theorems for set-valued mapping in noncompact abstract convex space. Next, we obtain two existence theorems of maximal element for $\mathscr{U}_A$-mapping and $\mathscr{U}_A$-majorized mapping. Lastly, we establish new equilibrium existence theorems for qualitative game and generalized game. Besides, we can get more general results than that in the recent literature.
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1. Introduction

The existence of equilibrium plays an important part in the study of markets. Since the existence of equilibrium in a generalized game with compact strategy sets in R^n was proved in a seminal paper of Debreu [3], there are many generalizations of Debreu's theorem from two main respects. On one hand, the convexity structure of underlying space has lots of generalizations. Some existence theorems of generalized game were obtained by Yuan and Tarafdar [39], Ding and Yuan [9], Mehta et al. [20], Lin and Ansari [18], Yuan [37], Hou[15] and Yuan [38] in topological vector space. Since Horvath [13, 14] introduced H-space by replacing convex hull by contract subset, there are many generalizations of the concept of convex subsets of topological vector space, for example, G-convex space [30] and FC-space [4]. As a result, many authors established existence theorems of generalized game in H-space, G-convex space and FC-space, respectively, for example, Tan et al. [31], Wu [34], Chowdhury et al. [2], Ding and Xia [8], Ding [5], Ding and Wang [7], Ding and Feng [6]. In 2006, Park [22] introduced the abstract convex space, which include convex subsets in topological vector space, H-space, G-convex space and FC-space as special case. Abstract convex space will be the framework of this paper. On the other hand, the lower section of the preference correspondence or constrained correspondence is majorized by some characteristics of mapping. Since Borglin and Keiding [1] introduced the notion of KF-majorized mapping which was majorized by the correspondence with open lower section, there have appeared many majorized mappings with open, compactly open, transfer open or transfer compactly open lower section, for example, Yuan and Tarafdar [39], Ding and Yuan [9], Mehta et al.[20], Lin and Ansari [18], Yuan [37], Tan et al. [31], Wu [34], Chowdhury et al. [2], Ding and Xia [8], Ding [5], Ding and Wang [7], Ding and Feng [6], Du and Deng [10], Yang and Deng [35]. In 2010, Luc [19] introduced the notion of unionly open valued mapping, which includes open valued mapping and transfer open valued mapping as special case. In the paper, by using approximate technique for unionly open set-valued mapping in abstract convex space, We should introduce new notions of $\mathscr{U}_{A,\theta}$-mapping and $\mathscr{U}_{A,\theta}$-majorized mapping which are majorized by some mappings with unionly open lower section. Firstly, We prove new fixed point theorems in abstract convex space. Secondly, we obtain the existence theorems of maximal element for $\mathscr{U}_A$-mapping and $\mathscr{U}_A$-majorized mapping. Finally, we apply these results to establish general existence theorems of equilibrium for qualitative game and generalized game with infinite set of players, without compactness hypotheses in the abstract convex space. Our results unify the corresponding results in the existing literatures.

2. Preliminaries

Let X be a nonempty subset of topological space E. We shall denote by 2^X the family of all subsets of X, by $\langle X\rangle$ the family of all nonempty finite subsets of X, by $\text{int}_E(X)$ the interior of X in E, and by $\text{cl}_E(X)$ the closure of X in E.

If X and Y are topological space and $T, S : X \to 2^Y$ are two mappings, for any $D \subset X$ and $y \in Y$, let $S(D) = \cup_{x\in D} S(x)$ and $S^-(y) = \{x \in X : y \in S(x)\}$. The dom S denotes the domain of S, i.e., dom $S = \{x \in X : S(x) \neq \emptyset\}$, and $T \cap S : X \to 2^Y$ is a mapping defined by $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in X$. The graph of T is the set $\text{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ and the mapping $\bar{T} : X \to 2^Y$ is defined by $\bar{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X\times Y}(\text{Gr}(T))\}$. The mapping cl $T : X \to 2^Y$ is defined by $(\text{cl } T)(x) = \text{cl}_Y(T(x))$ for each $x \in X$.

Let X be a nonempty set and Y be a topological space. The mapping $F : X \to 2^Y$ is said to be transfer open valued on X if

$$\cup_{x\in Y} F(x) = \cup_{x\in X} \text{int}_Y(F(x))$$

The mapping $G : X \to 2^Y$ is said to be unionly open valued on X if

$$\text{int}_Y(\cup_{x\in X} G(x)) = \cup_{x\in X} \text{int}_Y(G(x)).$$

It is easy to prove that transfer open valued mapping must be unionly open valued. Moreover, we can obtain the following results.

Proposition 2.1. *Let X be a nonempty subset of topological space E and Y be a topological space. If the mapping $S : X \to 2^Y$ is unionly open valued with $Y = S(X)$, then the mapping S is transfer open valued on X.*

Proof. Since S is unionly open valued on X, then $\text{int}_Y(\cup_{x\in X}S(x)) = \cup_{x\in X}\text{int}_Y(S(x))$. By $Y = S(X) = \cup_{x\in X}S(x)$, thus $\cup_{x\in X}S(x) = \cup_{x\in X}\text{int}_Y(S(x))$, S is transfer open valued on X. This completes the proof. $\square$

Proposition 2.2. *Let X be a nonempty subset of topological space E and Y be a topological space. For each $i \in I = \{1, 2, \cdots, n\}$, the mapping $S_i : X \to 2^Y$ is unionly open valued on X, then the mapping $\cap_{i=1}^n S_i$ is also unionly open valued on X.*

Proof. It is clear that $\cup_{x\in X}\text{int}_Y(\cap_{i=1}^n S_i(x)) \subset \text{int}_Y(\cup_{x\in X}(\cap_{i=1}^n S_i(x)))$, thus we only need to prove that $\text{int}_Y(\cup_{x\in X}(\cap_{i=1}^n S_i(x))) \subset \cup_{x\in X}\text{int}_Y(\cap_{i=1}^n S_i(x))$.
If $z \notin \cup_{x\in X}\text{int}_Y(\cap_{i=1}^n S_i(x))$, for each $x \in X$,

$$z \notin \text{int}_Y(\cap_{i=1}^n S_i(x)),\ z \in \text{cl}_Y(Y\backslash(\cap_{i=1}^n S_i(x))) = \text{cl}_Y(\cup_{i=1}^n(Y\backslash S_i(x))),$$

then for each neighborhood N_z of z, there exists a $i_0 \in I$, such that

$$N_z \cap (Y\backslash S_{i_0}(x)) \neq \emptyset.$$

That is $z \in \text{cl}_Y(Y\backslash S_{i_0}(x))$, i.e. $z \notin \text{int}_Y(S_{i_0}(x))$.
Since S_{i_0} is unionly open, then $z \notin \cup_{x\in X}\text{int}_Y(S_{i_0}(x)) = \text{int}_Y(\cup_{x\in X}(S_{i_0}(x)))$,
but $\cup_{x\in X}(\cap_{i=1}^n S_i(x)) \subset \cup_{x\in X}S_{i_0}(x)$, thus $z \notin \text{int}_Y(\cup_{x\in X}(\cap_{i=1}^n S_i(x)))$.
That is $\text{int}_Y(\cup_{x\in X}(\cap_{i=1}^n S_i(x))) \subset \cup_{x\in X}\text{int}_Y(\cap_{i=1}^n S_i(x))$. This completes the proof. $\square$

The following notions and lemmas were introduced by Park in [22, 23, 24].

Definition 2.1. *An abstract convex space $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D and a mapping $\Gamma : \langle D\rangle \to 2^E$ with nonempty value $\Gamma_A := \Gamma(A)$ for each $A \in \langle D\rangle$.*

For any $D' \subset D$, the Γ-convex hull of D' is denoted and defined by

$$\text{co}_\Gamma D' = \cup\{\Gamma_A | A \in \langle D'\rangle\} \subset E$$

A subset X of E is called a Γ-convex subset of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is $\text{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a Γ-convex subspace of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such a case, a subset X of E is said to be Γ-convex if $\text{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' = X \cap D$. If $E = D$, let $(E; \Gamma) = (E, E; \Gamma)$.

Definition 2.2. [23] *Let $(E, D; \Gamma)$ be an abstract convex space. If a mapping $G : D \to 2^E$ satisfies*

$$\Gamma_A \subset G(A) = \bigcup_{y\in A} G(y) \quad \textit{for all } A \in \langle D\rangle$$

then G is called a KKM mapping.

Definition 2.3. [23] *The partial KKM principle for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM mapping $G : D \to 2^E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property.*

Lemma 2.1. [24] *Let $(E, D; \Gamma)$ be an abstract convex space and $(X, D'; \Gamma')$ be an Γ-convex subspace. If $(E, D; \Gamma)$ satisfies the partial KKM principle, then so does $(X, D'; \Gamma')$.*

Lemma 2.2. [23] *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and $S : D \to 2^E$ be a mapping such that*
(i) for each $z \in X$, $S(z)$ is open;
(ii) $E = \bigcup_{z \in M} S(z)$ for some $M \in \langle D \rangle$.
Then there exists a finite subset $N \in \langle D \rangle$ such that $\Gamma_N \cap \bigcap_{z \in N} S(z) \neq \emptyset$.

By using the notion of unionly open mapping, we should introduce new notions of $\mathscr{U}_{A,\theta}$-mapping and $\mathscr{U}_{A,\theta}$-majorized mapping, that are majorized some mappings with unionly open lower section.

Definition 2.4. *Let X be a topological space and Y be a nonempty subset of an abstract convex space $(E; \Gamma)$. Let $\theta : X \to Y$ be a single valued mapping and $P : X \to 2^Y$ be a set-valued mapping. Then*
(i) P is said to be $\mathscr{U}_{A,\theta}$-mapping if there exists a mapping $\psi : X \to 2^Y$ such that
  (a) for each $x \in X$, $P(x) \subset \psi(x)$ and $\theta(x) \notin co\,_\Gamma(\psi(x)) \subset Y$;
  (b) the mapping $\psi^- : Y \to 2^X$ is unionly open valued on Y.
(ii) $(\psi_x; N_x)$ is said to be an $\mathscr{U}_{A,\theta}$-majorant of P at $x \in$dom P if N_x is an open neighborhood of x in X and the mapping $\psi_x : X \to 2^Y$ satisfies
  (a) for each $z \in N_x$, and $\theta(z) \notin co_\Gamma(\psi_x(z))$;
  (b) for each $z \in X$, $P(z) \subset \psi_x(z)$;
  (c) the mapping $\psi_x^- : Y \to 2^X$ is unionly open valued on Y.
(iii) P is said to be an $\mathscr{U}_{A,\theta}$-majorized mapping if for each $x \in$ dom P, there exists an $\mathscr{U}_{A,\theta}$-majorant $(\psi_x; N_x)$ of P at x.

Remark 2.1. The notion of $\mathscr{U}_{A,\theta}$-mapping (resp., $\mathscr{U}_{A,\theta}$-majorized mapping), which includes the notion of a mapping being of class $\mathscr{L}$ (resp., $\mathscr{L}$-majorized mapping) introduced by Yuan [38], and the notion of a mapping being of class $\mathscr{L}^*_{\theta,F_c}$ (resp., $\mathscr{L}^*_{\theta,F_c}$-majorized mapping) introduced by Ding and Xia [8] as special case. These notions also generalize the corresponding notions in Tan et al. [31], Hou [16], Yang and Deng [35], Ding and Yuan [9].

In this paper, we shall deal mainly with either the case (I) $X = Y$ and X is an abstract convex space and $\theta = I_X$, which is the identity mapping on X, or the case (II) $X = \Pi_{i \in I} X_i$ and $\theta = \pi_i : X \to X_i$ is the projection of X onto X_i and X_i is an abstract convex space. In both case (I) and (II), we shall write $\mathscr{U}_A$ in place of $\mathscr{U}_{A,\theta}$.

3. Fixed point theorems

Let X be a topological space and $T : X \to 2^X$ be a mapping. A point $\hat{x} \in X$ is called a fixed point of T if there exists a point $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

In the section, we should prove some new fixed point theorems in noncompact abstract convex space.

Theorem 3.1. *Let $(X; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and the mappings $S, T : X \to 2^X$ be such that*

(i) for each $x \in X$, $S(x) \subset T(x)$*;*
(ii) the mapping $S^- : X \to 2^X$ *is transfer open valued and for each* $x \in K$, $S(x) \neq \emptyset$*;*
(iii) there exists a nonempty compact subset K *of* X *such that either (a) or (b) hold. (a) and (b) are expressed as follows.*

(a) $X \backslash K \subset \bigcup\{int_X(T^-(z)), z \in M\}$ *for some* $M \in \langle X \rangle$*;*

(b) *for each* $N \in \langle X \rangle$*, there exists a compact* Γ*-convex subset* L_N *of* X *containing* N *such that*

$$L_N \backslash K \subset \bigcup\{int_X(T^-(x)) : x \in L_N\}.$$

then there exists a point $\hat{x} \in X$ *such that* $\hat{x} \in co_\Gamma(T(\hat{x}))$*.*

Proof. By (ii) and (i), $K \subset \cup_{z \in X}(S^-(z)) = \cup_{z \in X}\text{int}_X(S^-(z)) \subset \cup_{z \in X}\text{int}_X(T^-(z))$, then there exists a finite set $N \in \langle X \rangle$ such that

$$K \subset \cup\{\text{int}_X(T^-(z)) : z \in N\}. \tag{3.1}$$

By (iii), if (a) holds, then

$$X = \cup\{\text{int}_X(T^-(z)) : z \in M \cup N\}.$$

By Lemma 2.2, there exists a set $A \in \langle X \rangle$, such that $\Gamma_A \cap \bigcap_{z \in A} \text{int}_X(T^-(z)) \neq \emptyset$. Put $\hat{x} \in \Gamma_A \cap \bigcap_{z \in A} \text{int}_X(T^-(z))$, then for each $z \in A$, $\hat{x} \in \text{int}_X(T^-(z)) \subset T^-(z)$, that is $A \subset T(\hat{x})$, thus $\Gamma_A \subset \text{co}_\Gamma(T(\hat{x}))$. Since $\hat{x} \in \Gamma_A$, then $\hat{x} \in \text{co}_\Gamma(T(\hat{x}))$.

If (b) holds, there exists a compact Γ-convex subset L_N of X containing N such that

$$L_N \backslash K \subset \cup\{\text{int}_X(T^-(z)) : z \in L_N\}.$$

By (3.1),

$$L_N \cap K \subset \cup\{\text{int}_X(T^-(z)) : z \in N\} \subset \cup\{\text{int}_X(T^-(z)) : z \in L_N\}$$

then

$$L_N \subset \cup\{\text{int}_X(T^-(z)) : z \in L_N\}.$$

Since L_N is compact, thus there exists a finite set $A \in \langle L_N \rangle$ such that

$$L_N = \cup\{\text{int}_X(T^-(z)) \cap L_N : z \in A\}.$$

Define $\Gamma' : \langle L_N \rangle \to 2^{L_N}$, by $\Gamma'_C = \Gamma_C \cap L_N$ for each $C \in \langle L_N \rangle$, then $(L_N; \Gamma')$ is an Γ-convex subspace of $(X; \Gamma)$ and satisfies the partial KKM principle by Lemma 2.1. Let $T' : L_N \to 2^{L_N}$ by $T'(z) = \text{int}_X(T^-(z)) \cap L_N$ for each $z \in L_N$, It is easy to prove that the all the hypotheses of Lemma 2.2 are satisfied. By Lemma 2.2, there exists a finite set $B \in \langle L_N \rangle$ such that $\Gamma'_B \cap \bigcap_{z \in B} T'(z) \neq \emptyset$. Let $\hat{x} \in \Gamma'_B \cap \bigcap_{z \in B} T'(z)$, then for each $z \in B$, $\hat{x} \in T'(z) = \text{int}_X(T^-(z)) \cap L_N \subset T^-(z)$, that is $B \subset T(\hat{x})$ and $\Gamma_B \subset \text{co}_\Gamma(T(\hat{x}))$. Since $\hat{x} \in \Gamma'_B = \Gamma_B \cap L_N \subset \Gamma_B$, thus $\hat{x} \in \text{co}_\Gamma(T(\hat{x}))$. This completes the proof. □

Remark 3.1. Theorem 3.1 extends Theorem 3.1 of Ding and Wang [7] from *FC*-space to abstract convex space. It is easy to check that Theorem 3.1 also generalizes Theorem 2.3 of Chowdhury et al. [2], Theorem 4.1 of Park [25] and Corollary 12.1 of Park [26] with more weaker hypotheses.

For a topological space (X, τ), the compactly generated extension of the topology τ is the new topology consisting of all compactly closed [resp., open] subsets. In this way, we have the following modified form of Theorem 3.1.

Theorem 3.2. *Let $(X;\Gamma)$ be an abstract convex space satisfying the partial KKM principle and the mappings $S, T : X \to 2^X$ be such that*
(i) for each $x \in X$, $S(x) \subset T(x)$;
(ii) the mapping $S^- : X \to 2^X$ is transfer compactly open valued on X and for $x \in K$, $S(x) \neq \emptyset$;
(iii) there exists a nonempty compact subset K of X such that either (a) or (b). (a) and (b) are expressed as follows.
(a) *$X \backslash K \subset \bigcup\{int_X(T^-(z)), z \in M\}$ for some $M \in \langle X\rangle$;*
(b) *for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$L_N\backslash K \subset \bigcup\{int_X(T^-(x)) : x \in L_N\}.$$

then there exists a point $\hat{x} \in X$ such that $\hat{x} \in co_\Gamma(T(\hat{x}))$.

Proof. Replace the topology of X by its compactly generated extension, then $(X;\Gamma)$ with this new topology is another abstract convex space. All the hypotheses of Theorem 3.1 are satisfied. By Theorem 3.1, there exists a point $\hat{x} \in X$ such that $\hat{x} \in \mathrm{co}_\Gamma(T(\hat{x}))$. □

Remark 3.2. Theorem 3.2 generalizes Theorem 3.1 of Ding and Feng [6] and Theorem 2.4 of Chowdhury [2].

By using Proposition 2.1 and Theorem 3.1, we can derive the following fixed pointed theorem.

Corollary 3.1. *Let $(X;\Gamma)$ be an abstract convex space satisfying the partial KKM principle and the mappings $S, T : X \to 2^X$ be such that*
(i) for each $x \in X$, $S(x) \neq \emptyset$ and $S(x) \subset T(x)$;
(ii) the mapping $S^- : X \to 2^X$ is unionly open valued on X and $X = S(X)$;
(iii) there exists a nonempty compact subset K of X such that either (a) or (b). (a) and (b) are expressed as follows.
(a) *$X \backslash K \subset \bigcup\{int_X(T^-(z)), z \in M\}$ for some $M \in \langle X\rangle$;*
(b) *for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$L_N\backslash K \subset \bigcup\{int_X(T^-(x)) : x \in L_N\}.$$

then there exists a point $\hat{x} \in X$ such that $\hat{x} \in co_\Gamma(T(\hat{x}))$.

Remark 3.3. Corollary 3.1 generalizes Theorem 3.3 of Park [27] and Theorem 5.4 of Park [28] under weaken assumptions.

4. Existence of maximal element

Let X be a topological space and $T : X \to 2^X$ be a mapping. A point $\hat{x} \in X$ is called a maximal element of T if $T(\hat{x}) = \emptyset$.

In the section, we shall establish some new existence theorems of maximal element for $\mathscr{U}_A$-mapping and $\mathscr{U}_A$-majorized mapping defined on noncompact abstract convex space.

Theorem 4.1. *Let $(X;\Gamma)$ be an abstract convex space satisfying the partial KKM principle and K be a nonempty compact subset of X. Suppose that $P : X \to 2^X$ is an $\mathscr{U}_A$-mapping such that*

(i) for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that

$$L_N \backslash K \subset \bigcup\{int_X(P^-(y)) : y \in L_N\}.$$

Then there exists a point $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$.

Proof. Since P is an $\mathscr{U}_A$-mapping, then there exists a mapping $\psi : X \to 2^X$ such that

(a) for each $x \in X$, $P(x) \subset \psi(x)$ and $x \notin$ co $_\Gamma(\psi(x)) \subset X$;

(b) the mapping $\psi^- : X \to 2^X$ is unionly open valued on X.

Suppose for each $x \in X$, $P(x) \neq \emptyset$. By (a), $\psi(x) \neq \emptyset$ for each $x \in X$, then $X = \psi(X)$. By (i) and (a), for each $N \in \langle X\rangle$, there exists a compact abstract convex subset L_N of X containing N such that $L_N\backslash K \subset \cup\{\text{int}_X(P^-(y)) : y \in L_N\} \subset \cup\{\text{int}_X(\psi^-(y)) : y \in L_N\}$. Therefore, P and ψ satisfy all the hypotheses of Corollary 3.1. By Corollary 3.1, there exists a point $\bar{x} \in K$ such that $\bar{x} \in\text{co}_\Gamma(\psi(\bar{x}))$, that contradicts with condition (a). Hence there exists a point $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$. This completes the proof. □

Remark 4.1. The condition (i) of Theorem 4.1 can be replaced by the following statement: (i)′ for each $N \in \langle D\rangle$, there exists a compact abstract convex subset L_N of X containing N such that for each $x \in L_N\backslash K$, there exists a point $\bar{y} \in L_N$ such that $x \in \text{int}_X(P^-(\bar{y}))$.
Indeed, for each $N \in \langle D\rangle$, there exists a compact abstract convex subset L_N of X containing N such that for each $x \in L_N\backslash K$, there exists $\bar{y} \in L_N$ such that $x \in\text{int}_X(P^-(\bar{y}))$, thus $x \in \cup\{\text{int}_X(P^-(y)) : y \in L_N\}$, that is $L_N\backslash K \subset \cup\{\text{int}_X(P^-(y)) : y \in L_N\}$. Thus Theorem 4.1 includes Theorem 3.1 of Yang and Deng [35] as special case. Moreover, Theorem 4.1 generalizes Theorem 1 of Kim [17], Theorem 5.1 of Ding and Wang [7] and Theorem 3.1 of Chowdhury et al. [2].

Theorem 4.2. *Let $(X;\Gamma)$ be a Housdorff and paracompact abstract convex space satisfying the partial KKM principle and K be a nonempty compact subset of X. Let $P : X \to 2^X$ be an $\mathscr{U}_A$-majorized mapping such that*

(i) for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that

$$L_N \backslash K \subset \bigcup\{int_X(P^-(y)) : y \in L_N\}.$$

then there exists a point $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$.

Proof. Suppose $P(x) \neq \emptyset$ for each $x \in X$, that is dom $P = X$. Since P is an $\mathscr{U}_A$-majorized mapping, for each $x \in X$, let N_x be an open neighborhood of x in X and $\psi_x : X \to 2^X$ be mapping such that

(1) for each $z \in N_x$, $z \notin \text{co}_\Gamma(\psi_x(z))$;

(2) for each $z \in X$, $P(z) \subset \psi_x(z)$;
(3) the mapping $\psi_x^- : X \to 2^X$ is unionly open valued on X.

Since X is Housdorff and paracompact, then X is regular. For each $x \in X$, there exists an open neighborhood G_x of x in X such that $\mathrm{cl}_X G_x \subset N_x$. By Theorem VIII.1.4 of Dugundji [11], the open covering $\{G_x\}$ of X has an open precise neighborhood finite refinement $\{G'_x\}$. Given any $x \in X$, we define the mapping $\psi'_x : X \to 2^X$ by

$$\psi'_x(z) = \begin{cases} \psi_x(z) & \text{if } z \in \mathrm{cl}_X G'_x; \\ X & \text{if } z \in X \backslash \mathrm{cl}_X G'_x. \end{cases}$$

then we have
(i) by (3.1), $P(z) \subset \psi'_x(z)$ for each $z \in X$;
(ii) for each $y \in X$, the set

$$\begin{aligned} (\psi'_x)^-(y) &= \{z \in \mathrm{cl}_X G'_x : y \in \psi_x(z)\} \cup \{z \in X \backslash \mathrm{cl}_X G'_x : y \in X\} \\ &= [(\mathrm{cl}_X G'_x) \cap \psi_x^-(y)] \cup (X \backslash \mathrm{cl}_X G'_x) \\ &= \psi_x^-(y) \cup (X \backslash \mathrm{cl}_X G'_x) \end{aligned}$$

By dom $P = X$ and (2), then dom $\psi_x = X$, thus ψ_x^- is transfer open valued by Proposition 2.1. It is easy to check that $(\psi'_x)^-$ is transfer open valued. Now define $\psi : X \to 2^X$ by

$$\psi(z) = \bigcap_{x \in X} \psi'_x(z) \quad \text{for each} \quad z \in X$$

(a) For each $z \in X$, then there exists a point $x \in X$ such that $z \in \mathrm{cl}_X G'_x$, so that $\psi'_x(z) = \psi_x(z)$ and hence $\psi(z) \subset \psi_x(z)$. By (3.1), we have $z \notin \mathrm{co}_\Gamma(\psi(z))$.
(b) Now we prove that $P(z) \subset \psi(z)$ for each $z \in X$. Indeed, for each $x \in X$, if $z \in X \backslash \mathrm{cl}_X G'_x$, then $\psi'_x(z) = X$, so $P(z) \subset \psi'_x(z)$; if $z \in \mathrm{cl}_X G'_x$, we have $z \in \mathrm{cl}_X G'_x \subset \mathrm{cl}_X G_x \subset N_x$, by (3.1), $P(z) \subset \psi_x(z) \subset \psi'_x(z)$. It follows that $P(z) \subset \psi'_x(z)$ for all $x \in X$, thus $P(z) \subset \cap_{x \in X} \psi'_x(z) = \psi(z)$.
(c) Finally we show that the mapping $\psi^- : X \to 2^X$ is transfer open valued on X. Indeed, let $y_0 \in X$ be such that $\psi^-(y_0) \neq \emptyset$. Given a point u such that

$$u \in \psi^-(y_0) = \{z \in X : y_0 \in \psi(z)\}.$$

Since $\{G'_x\}$ is a neighborhood finite refinement, there exists an open neighborhood M_u of u in X such that $\{x \in X : M_u \cap G'_x \neq \emptyset\} = \{x_1, x_2, \cdots, x_n\}$. Note that for each $x \in X$ with $x \notin \{x_1, x_2, \cdots, x_n\}$, $\emptyset = M_u \cap G'_x = M_u \cap \mathrm{cl}_X G'_x$, so $\psi'_x(z) = X$ for $z \in M_u$. Then we have $\psi(z) = \cap_{x \in X} \psi'_x(z) = \cap_{i=1}^n \psi'_{x_i}(z)$ for all $z \in M_u$. It follows that

$$\begin{aligned} \psi^-(y_0) &= \{z \in X : y_0 \in \psi(z)\} = \{z \in X : y_0 \in \bigcap_{x \in X} \psi'_x(z)\} \\ &\supset \{z \in M_u : y_0 \in \bigcap_{x \in X} \psi'_x(z)\} \\ &= \{z \in M_u : y_0 \in \bigcap_{i=1}^n \psi'_{x_i}(z)\} \\ &= M_u \cap \{z \in X : y_0 \in \bigcap_{i=1}^n \psi'_{x_i}(z)\} \\ &= M_u \cap [\bigcap_{i=1}^n (\psi'_{x_i})^-(y_0)] \end{aligned}$$

then $M_u \cap (\cup_{y\in X}\text{int}_X[\bigcap_{i=1}^{n}(\psi'_{x_i})^-(y)]) \subset \cup_{y\in X}\text{int}_X(\psi^-(y))$. Since $(\psi'_{x_i})^-$ is transfer open valued on X by (iii), then $\cap_{i=1}^n(\psi'_{x_i})^-$ is transfer open by Proposition 2.2, that is $\cup_{y\in X}\text{int}_X\ [\bigcap_{i=1}^{n}(\psi'_{x_i})^-(y)] = \cup_{y\in X}[\bigcap_{i=1}^{n}(\psi'_{x_i})^-(y)]$. Note $y_0 \in \psi(u) = \cap_{i=1}^n(\psi'_{x_i})(u)$, $u \in \cap_{i=1}^n(\psi'_{x_i})^-(y_0)$, then $u \in M_u \cap (\cup_{y\in X}[\bigcap_{i=1}^{n}(\psi'_{x_i})^-(y)])$, thus $u \in \cup_{y\in X}\text{int}_X(\psi^-(y))$. Hence $\cup_{y\in X}\psi^-(y) = \cup_{y\in X}\text{int}_X(\psi^-(y))$. This shows that $\psi^- : Y \to 2^X$ is transfer open valued on X. Therefore, all the hypotheses of Theorem 3.1 are satisfied. By Theorem 3.1, there exists a point $\bar{x} \in X$ such that $\bar{x} \in \text{co}_\Gamma(\psi(\bar{x}))$, that contradicts with condition (a). Thus there exists a point $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$. This completes the proof. □

Remark 4.2. Theorem 4.2 improves and generalizes Theorem 3.3 of Ding and Xia [8] from G-convex space to abstract convex space. Theorem 4.2 also generalizes Theorem 3.3 of Yang and Deng [35], Theorem 2.3 of Ding and Yuan [9].

5. Equilibrium of abstract economy

Let I be a finite or infinite set of players. For each $i \in I$, let its strategy set X_i be nonempty subset of an abstract convex space with $X = \prod_{i\in I} X_i$. $P_i : X \to 2^{X_i}$ be the preference correspondence of i-th player. Following the notion of Gale and Mas-Colell [12], the collection $\Lambda = (X_i; P_i)_{i\in I}$ will be called a qualitative game. A point $\hat{x} \in X$ is said to be an equilibrium of the qualitative game, if $P_i(\hat{x}) = \emptyset$ for each $i \in I$.

A generalized game (=abstract economy) is a quadruples family

$$\Lambda = (X_i; A_i; B_i; P_i)_{i\in I}$$

where I is a finite or infinite set of players such that for each $i \in I$, X_i is a nonempty subset of an abstract convex space with $X = \prod_{i\in I} X_i$. $A_i, B_i : X \to 2^{X_i}$ are the constraint correspondences and $P_i : X \to 2^{X_i}$ is the preference correspondence. When $I = \{1, 2, \cdots, N\}$, where N is a positive integer, $\Lambda = (X_i; A_i; B_i; P_i)_{i\in I}$ is also called an N-person game. Particularly, if $I = \{1\}$, $\Lambda = (X; A; B; P)$ is said to be one person game. An equilibrium of the generalized game Λ is a point $\hat{x} \in X$ such that for each $i \in I, \hat{x}_i = \pi_i(\hat{x}) \in \bar{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

The notion of abstract economy in abstract convex space is the natural extension of the abstract economy introduced by Nash [21], Tarafdar [33], Yang and Deng [35], Borglin and Keiding [1], Yannelis and Prabhakar[36].

The following Lemma shows the product of a family of abstract convex space is also an abstract convex space.

Lemma 5.1. [23, 29] *Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces. Let $X := \Pi_{i\in I}X_i$ be equipped with the product topology and $D := \Pi_{i\in I}D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D\rangle$, define $\Gamma(A) := \Pi_{i\in I}\Gamma_i(\pi_i(A))$. Then $(X, D; \Gamma)$ is an abstract convex space.*

We now apply Theorem 4.1 to establish a new existence theorem of equilibrium for one person game in abstract convex space.

Theorem 5.1. *Let $(X;\Gamma)$ be an abstract convex space satisfying the partial KKM principle and K be a compact subset of X. Suppose the mappings $A, B, P : X \to 2^X$ satisfy*
(i) the mapping $A^- : X \to 2^X$ is transfer open valued on X;
(ii) dom $P = X$ and P is an $\mathscr{U}_A$-mapping;
(iii) for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that

$$L_N \backslash K \subset \bigcup\{int_X((A\cap P)^-(y)) : y \in L_N\}.$$

Then there exists a point $\hat{x} \in K$ such that $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. Let $W = \{x \in X : x \notin \overline{B}(x)\}$, then W is open in X. Define $Q : X \to 2^X$ by

$$Q(x) = \begin{cases} P(x) & \text{if } x \in W; \\ A(x) \cap P(x) & \text{if } x \notin W. \end{cases}$$

By (ii), P is an $\mathscr{U}_A$-mapping, then there exists a mapping $\psi : X \to 2^X$ such that
(a) for each $x \in X$,$P(x) \subset \psi(x)$ and $x \notin \mathrm{co}_\Gamma(\psi(x))$;
(b) the mapping $\psi^- : X \to 2^X$ is unionly open valued on X.
Define $\Psi : X \to 2^X$ by

$$\Psi(x) = \begin{cases} \psi(x) & \text{if } x \in W; \\ A(x) \cap \psi(x) & \text{if } x \notin W. \end{cases}$$

Then we have
(a') by (a), for each $x \in X$, $Q(x) \subset \Psi(x)$, if $x \in W$, $x \notin \mathrm{co}_\Gamma(\psi(x))=\mathrm{co}_\Gamma(\Psi(x))$; if $x \notin W$, by $A(x) \cap \psi(x) \subset \psi(x)$, $x \notin \mathrm{co}_\Gamma(\psi(x) \cap A(x)) = \mathrm{co}_\Gamma(\Psi(x))$, thus $x \notin \mathrm{co}_\Gamma(\Psi(x))$ for each $x \in X$;
(b') for each $y \in X$,

$$\begin{aligned}\Psi^-(y) &= \{x \in X : y \in \Psi(x)\} = \{x \in W : y \in \psi(x)\} \cup \{x \in X\backslash W : y \in A(x) \cap \psi(x)\} \\ &= [\psi^-(y) \cap W] \cup [(X\backslash W) \cap \psi^-(y) \cap A^-(y)] = [W \cup A^-(y)] \cap \psi^-(y).\end{aligned}$$

By (i), we can obtain that $W \cup A^-(y)$ is transfer open, then the mapping $\Psi^- : X \to 2^X$ is unionly open valued on X by (b) and Proposition 2.2. This shows that Q is an $\mathscr{U}_A$-mapping. By the definition of Q and condition (ii), for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that

$$L_N\backslash K \subset \cup\{\mathrm{int}_X((A \cap P)^-(y)) : y \in L_N\} \subset \cup\{\mathrm{int}_X(Q^-(y)) : y \in L_N\}.$$

Hence all the hypotheses of Theorem 4.1 are satisfied. By Theorem 4.1, there exists a point $\hat{x} \in X$ such that $Q(\hat{x}) = \emptyset$. By the definition of Q and condition (iii), $\hat{x} \notin W$, that is $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. This completes the proof. □

Remark 5.1. Theorem 5.1 improves Theorem 5.1 of Ding and Feng [6] from two aspects.
(i) The underlying space changes from FC-space to abstract convex space;
(ii) The preference correspondence P is an $\mathscr{U}_A$-mapping mapping instead of the $\mathscr{L}_F^*$ class mapping.
Moreover, Theorem 5.1 generalizes Theorem 3.2 of Ding and Yuan [9] from topological vector space to abstract convex space under weaken assumptions.

From Theorem 4.2, we can derive the following existence of equilibrium for qualitative game.

Theorem 5.2. *Let $\Lambda = (X_i; P_i)_{i\in I}$ be a qualitative game, For each $i \in I$, suppose the following conditions are satisfied*
(i) $(X_i; \Gamma_i)_{i\in I}$ is a family of paracompact abstract convex space such that $(X;\Gamma)$ satisfies the partial KKM principle and K is a nonempty compact subset of X;
(ii) $P_i : X \to 2^{X_i}$ is an $\mathscr{U}_A$-majorized mapping;
(iii) $W_i = \{x \in X : P_i(x) \neq \emptyset\}$ is open in X;
(iv) for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that

$$L_N\backslash K \subset \bigcup\{int_X(P_i^-(\pi_i(y))) : y \in L_N\}.$$

Then Λ has an equilibrium point in K.

Proof. For each $x \in X$, let $I(x) = \{i \in I : P_i(x) \neq \emptyset\}$. Define a mapping $P'_i : X \to 2^X$ by $P'_i(x) = \pi_i^-(P_i(x)) = \Pi_{j\in I, j\neq i} X_i \otimes P_i(x)$ for each $x \in X$, where the mapping $\pi_i : X \to X_i$ is projection of X onto X_i. Furthermore, define the mapping $P : X \to 2^X$ by

$$P(x) = \begin{cases} \bigcap\limits_{i\in I(x)} P'_i(x) & \text{if } I(x) \neq \emptyset; \\ \emptyset & \text{if } I(x) = \emptyset. \end{cases}$$

Then for each $x \in X$, $P(x) \neq \emptyset$ if and only if $I(x) \neq \emptyset$. We shall show that P is an $\mathscr{U}_A$-majorized mapping. For each $x \in X$ with $P(x) \neq \emptyset$, let $i \in I(x)$ with $P_i(x) \neq \emptyset$, by (ii) , let N_x be an open neighborhood of x in X and $\psi_{i,x} : X \to 2^{X_i}$ be mapping such that
(a) for each $z \in N_x$, $z_i \notin \text{co}_\Gamma(\psi_{i,x}(z))$;
(b) for each $z \in X$, $P_i(z) \subset \psi_{i,x}(z)$;
(c) the mapping $\psi_{i,x}^- : X_i \to 2^X$ is unionly open valued on X_i.
By (iii), we may assume that $N_x \subset W_i$, hence $P_i(z) \neq \emptyset$ and $i \in I(z)$ for all $z \in N_x$. Now define the mapping $\psi_x : X \to 2^X$ by

$$\psi_x(z) = \pi_i^-(\psi_{i,x}(z)) \quad \text{for each } z \in X.$$

Then we have
(a') for each $z \in N_x$, by (a), $z \notin \text{co}_\Gamma(\psi_x(z))$;
(b') for each $z \in X$, by (b), $P(z) = \cap_{i\in I(z)} P'_i(z) \subset P'_i(z) = \pi_i^-(P_i(z)) \subset \pi_i^-(\psi_{i,x}(z)) = \phi_x(z)$;
(c') for each $y \in X$, $\psi_x^-(y) = \{z \in X : y \in \psi_x(z)\} = \{z \in X : y_i \in \psi_{i,x}(z)\} = \psi_{i,x}^-(y_i)$ is unionly open in X by (b).
(a), (b) and (c) show that P is an $\mathscr{U}_A$-majorized mapping. By $P^-(y) = P_i^-(\pi_i(y))$ and condition (iv), for each $N \in \langle X\rangle$, there exists a compact abstract convex subset L_N of X containing N such that

$$L_N\backslash K \subset \cup\{\text{int}_X(P_i^-(\pi_i(y)) : y \in L_N\} = \cup\{\text{int}_X(P^-(y)) : y \in L_N\}.$$

Hence all the hypotheses of Theorem 4.2 are satisfied. By Theorem 4.2, there exists a point $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$. That implies that $I(\hat{x}) = \emptyset$, thus $P_i(\hat{x}) = \emptyset$ for each $i \in I$. This completes the proof. □

Remark 5.2. Theorem 5.2 generalizes Theorem 6.2 of Ding and Wang [7] in which the $\mathscr{L}_F$-majorized mapping is replaced by $\mathscr{U}_A$-mapping mapping.

By using Theorem 5.2, we can obtain the following equilibrium existence theorem for a noncompact generalized game.

Theorem 5.3. *Let $\Lambda = (X_i; A_i; B_i; P_i)_{i\in I}$ be a generalized game. Let K be a compact subset of X. Suppose that for each $i \in I$,*
(i) $(X_i;\Gamma_i)$ is a paracompact abstract convex space such that $(X;\Gamma)$ satisfies the partial KKM principle;
(ii) for each $y \in X_i$, $A_i^-(y)$ is transfer open in X;
(iii) $W_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in X;
(iv) dom $P_i = X$ and $P_i : X \to 2^{X_i}$ is an $\mathscr{U}_A$-majorized mapping;
(v) for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that

$$L_N\backslash K \subset \bigcup\{int_X((A_i \cap P_i)^-(\pi_i(y))) : y \in L_N\}.$$

Then Λ has an equilibrium point $\hat{x}$ in K.

Proof. For each $i \in I$, let $F_i = \{x \in X : x_i \notin \overline{B}_i(x)\}$, then F_i is open in X. Define $Q_i : X \to 2^{X_i}$ by

$$Q_i(x) = \begin{cases} P_i(x) & \text{if } x \in F_i; \\ A_i(x) \cap P_i(x) & \text{if } x \notin F_i. \end{cases}$$

We shall prove that the qualitative game $\Lambda' = (X_i, Q_i)_{i\in I}$ satisfies all the hypotheses of Theorem 5.2. For each $i \in I$, we have that the set

$$\begin{aligned} \{x \in X : Q_i(x) \neq \emptyset\} &= \{x \in F_i : P_i(x) \neq \emptyset\} \cup \{x \in X\backslash F_i : (A_i \cap P_i)(x) \neq \emptyset\} \\ &= F_i \cup [(X\backslash F_i) \cap W_i] = F_i \cup W_i \end{aligned}$$

is open in X and hence the condition (iii) of Theorem 5.2 is satisfied. By (v), for each $x \in X$, there exist an open neighborhood N_x of x in X and a mapping $\psi_{i,x} : X \to 2^{X_i}$ such that
(a) for each $z \in N_x$, $z_i \notin \text{co}_\Gamma(\psi_{i,x}(z))$;
(b) for each $z \in X$, $P_i(z) \subset \psi_{i,x}(z)$;
(c) the mapping $\psi_{i,x}^- : X_i \to 2^X$ is unionly open valued on X_i.
Define $\Psi_{i,x} : X \to 2^{X_i}$ by

$$\Psi_{i,x}(z) = \begin{cases} \psi_{i,x}(z) & \text{if } z \in F_i; \\ A_i(z) \cap \psi_{i,x}(z) & \text{if } z \notin F_i. \end{cases}$$

Now for each $x \in X$ with $Q_i(x) \neq \emptyset$, the set $U_x = N_x$ is open in X. Then
(a') by (a), for each $z \in U_x$, if $z \in U_x \cap F_i$, $z \notin \text{co}_\Gamma(\psi_{i,x}(z)) = \text{co}_\Gamma(\Psi_{i,x}(z))$; if $z \in U_x \cap (X\backslash F_i)$, by $A_i(z) \cap \psi_{i,x}(z) \subset \psi_{i,x}(z)$, $z \notin \text{co}_\Gamma(A_i(z) \cap \psi_{i,x}(z)) = \text{co}_\Gamma(\Psi_{i,x}(z))$, that is $z \notin \text{co}\ _\Gamma(\Psi_{i,x}(z))$ for each $z \in U_x$;
(b') by (b), for each $z \in X$, $Q_i(z) \subset \Psi_{i,x}(z)$;
(c') for each $y \in X_i$,

$$\begin{aligned} \Psi_{i,x}^-(y) &= \{z \in X : y \in \Psi_{i,x}(z)\} = \{z \in F_i : y \in \psi_{i,x}(x)\} \\ &\cup \{z \in X\backslash F_i : y \in \psi_{i,x}(z) \cap A_i(z)\} \\ &= \psi_{i,x}^-(y) \cap [F_i \cup A_i^-(y)] \end{aligned}$$

By (i), it is easy to check that $W_i \cup A_i^-(y)$ is transfer open, then the mapping $\Psi_{i,x}^- : X \to 2^X$ is unionly open valued on X by (c) and Proposition 2.2. Thus Q_i is an $\mathscr{U}_A$-majorized mapping. By the definition of Q and condition (vi), for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that

$$L_N \backslash K \subset \cup \{\mathrm{int}_X((A_i \cap P_i)^-(\pi_i(y)) : y \in L_N\} \subset \cup \{\mathrm{int}_X(Q_i^-(\pi_i(y)) : y \in L_N\}.$$

Hence, all the hypotheses of Theorem 5.2 are satisfied. By Theorem 5.2, there exists a point $\hat{x} \in X$ such that $Q_i(\hat{x}) = \emptyset (i \in I)$. By the definition of Q_i, $\hat{x} \in X \backslash F_i$, that is for all $i \in I$, $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. This completes the proof. □

Remark 5.3. Theorem 5.3 greatly improves Theorem 2.2 of Hou [16] from three aspects.
(i) The underlying space changes from topological vector space to abstract convex space;
(ii) The preference correspondence P_i is an $\mathscr{U}_A$-majorized mapping instead of the $\mathscr{L}^*$-majorized mapping;
(iii) The constraint correspondence A_i has transfer open lower section instead of open.
Moreover, Theorem 5.3 also generalizes Theorem 3.1 of Yuan [38], Theorem 6.1 of Yannelis and Prabhakar [36] under weaken hypotheses.

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