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KRASNOSELSKII ALGORITHM FOR FIXED POINTS OF MULTIVALUED QUASI-NONEXPANSIVE MAPPINGS IN CERTAIN BANACH SPACES

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Abstract. Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued quasi-nonexpansive mapping. A Krasnoselskii-type iteration sequence $\{x_n\}$ is constructed and shown to be an approximate fixed point sequence of T, that is, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$ holds. Convergence theorems are also proved under appropriate additional conditions.

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1. Introduction

Iterative processes for nonexpansive mappings are the key tools in signal processing and image restoration (see, *e.g.,* Byrne [5]). Let $T : K \to K$ be a nonexpansive mapping with nonempty fixed point set $F(T)$, K be a closed, convex nonempty subset of a normed linear space E. Krasnoselskii [24] proved that if E is a uniformly convex real Banach space, then for any $x_0 \in K$ fixed, the sequence $\{x_n\}$ generated by $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n,\ n \geq 0$ converges strongly to a fixed point of T. More generally, for any fixed element $x_0 \in K$, Schaefer [33] extended the result of Krasnoselskii [24] by considering the sequence $\{x_n\}$ generated by $x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\ n \geq 0,\ \lambda \in (0, 1)$. Edelstein [13] observed that the result of Krasnoselskii [24] holds even in a strictly convex real Banach space. The natural question of whether this result holds in any Banach space more general than strictly convex real Banach space remained open for many years. This question was answered in the affirmative by Edelstein and O'Brien [12] where they showed that the sequence $\{\|x_n - Tx_n\|\}$ converges to 0 uniformly in any normed linear space provided K is bounded.

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Another scheme studied for approximating fixed points of nonexpansive mappings is the Mann process: let $x_0 \in K$ be arbitrary but fixed,

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n, \; n \geq 0, \tag{1.1}$$

with $\lambda_n \in (0,1)$, $\lim\limits_{n\to\infty} \lambda_n = 0$, $\sum \lambda_n = \infty$. Ishikawa [19] proved that for the sequence $\{x_n\}$ generated by (1.1), with $0 \leq \lambda_n \leq b < 1$, and $\sum \lambda_n = \infty$, $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ holds true in any Banach space provided $\{x_n\}$ is bounded.

Important generalizations of nonexpansive mappings which have been studied by various authors include the following:

I. The class of Lipschitz pseudo-contractive mappings introduced by Browder. These mappings are intimately connected with the class of accretive operators (see Browder [4], Kato [20]) which are connected with evolution equations.
II. The class of quasi-nonexpansive mappings which was introduced (in general Banach spaces) by Diaz and Metcalf [9].

Let K be a nonempty subset of E. A mapping $T: K \to K$ satisfying $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$ is called a *nonexpansive* mapping. T is called *quasi-nonexpansive* if (i) $F(T) \neq \emptyset$ and (ii) $\|Tx - x^*\| \leq \|x - x^*\|$ for all $x \in K$, $x^* \in F(T)$. It is clear from this definition that every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive. Suppose T satisfies the condition

$$\begin{aligned} \|Tx - Ty\| \leq\ & a_1\|x-y\| + a_2\|x - Tx\| + a_3\|y - Ty\| \\ & + a_4\|x - Ty\| + a_5\|y - Tx\| \quad \forall x, y \in K, \end{aligned} \tag{1.2}$$

with $a_i \geq 0$ and $\sum\limits_{i=1}^{5} a_i < 1$, then T is qusi-nonexpansive (see, e.g., (Hardy and Rogers [17])). The following example [10] shows that the class of quasi-nonepansive mappings contains *properly* the class of nonexpansive mappings with nonempty fixed point sets.

Example 1.1. ([10]) Let $T: \mathbb{R} \to \mathbb{R}$ be defined by $Tx = \frac{1}{2}\sin x, x \neq 0,$ and $T0 = 0$.

For several years, the study of fixed point theory for *multi-valued mappings* has attracted, and continues to attract, the interest of several mathematicians (see, for example, Brouwer [3], Kakutani [16], Nash [28, 29], Geanakoplos [15], Nadler [27], Downing and Kirk [11]). Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in *Game Theory and Market Economy* and in other areas of mathematics, such as in *Non-Smooth Differential Equations*. For further details, see Chidume *et al.* [6].

Let D be a nonempty subset of a normed space E. The set D is called *proximinal* (see *e.g.*, [32, 30, 35]) if for each $x \in E$ there exists $u \in D$ such that

$$\|x - u\| = \inf\{\|x - y\| : y \in K\} := \text{dist}(x, D).$$

Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $P(D)$ and $K(D)$ denote the families of nonempty, closed and bounded subsets of D, nonempty, proximinal and bounded subsets of D and nonempty compact subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by:

$$H(A,B) = \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\}$$

for all $A, B \in CB(D)$. Let $T : D(T) \subseteq E \to CB(E)$ be a *multi-valued mapping* on E. A point $x \in D(T)$ is called a *fixed point of* T if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\| \ \ \forall x, y \in D(T). \tag{1.3}$$

T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\| \ \ \forall x \in D(T),\ p \in F(T). \tag{1.4}$$

Let (X, ρ) be a metric space. A map $T : X \to K(X)$ is said to be $*$-*nonexpansive* [18] if for all $x, y \in X$ and $u_x \in Tx$, with $\rho(x, u_x) = \operatorname{dist}(x, Tx)$ there exists $u_y \in Ty$ with $\rho(y, u_y) = \operatorname{dist}(y, Ty)$ such that

$$\rho(u_x, u_y) \leq \rho(x, y).$$

It is obvious that this notion reduces to the notion of nonexpansiveness for single-valued mappings. However, for multi-valued mappings, $*$-nonexpansive mappings are not comparable to nonexpansive mappings in general (see, e.g., [36]). Associated with a map $T : X \to K(X)$ is the map $P_T : X \to K(X)$ defined by $P_T(x) := \{u \in Tx : \rho(u, x) = \operatorname{dist}(x, Tx)\}$. It is clear that from the definition of P_T, $P_T(x^*) = \{x^*\}$ for every fixed point x^* of P_T. It is also known that T is $*$-nonexpansive if and only if P_T is nonexpansive [2].

Several papers deal with the problem of approximating fixed points of *multi-valued nonexpansive* mappings (see, for example [1, 21, 22, 32, 30, 35] and the references therein) and their generalizations (see *e.g.*, [8, 14]). Chidume *et al.* [6] proved strong convergence theorems for strictly pseudo-contractive mappings, a proper superclass of multi-valued nonexpansive mappings. For iterative approximation schemes of fixed points for multi-valued maps on metric spaces, one can see Petruşel and Rus [31].

Kuhfittig [25] proved strong convergence result for a multi-valued mapping T which is nonexpansive around a known fixed point. He generated a Krasnoselskii sequence *using the known fixed point* and obtained strong convergence to additional fixed point. More recently, Shahzad and Zegeye [34] proved strong convergence of the sequence of *Ishikawa-type iterates* to a fixed point of a quasi-nonexpansive mapping on uniformly convex Banach space. They extended and improved the results of Sastry and Babu [32], Panyanak [30] and Son and Wang [35]. Furthermore, they proved the convergence of Ishikawa-type sequence: $x_0 \in K$ fixed arbitrarily, $\alpha_n, \beta_n \in [0, 1]$,

$$y_n = (1 - \beta_n)x_n + \beta_n z_n,\ \ n \geq 0,\ \ z_n \in Tx_n,$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z'_n,\ \ n \geq 0,\ \ z'_n \in Ty_n,$$

to a fixed point of $*$-nonexpansive mapping.

Dotson [10] proved that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ where T is a single-valued quasi-nonexpansive mapping in the setting of a uniformly convex Banach space. Unlike in the case of nonexpansive mappings, the following example of Chidume [7] shows that

$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ does not hold for quasi-nonexpansive mappings in arbitrary Banach space even when K is bounded, where $\{x_n\}$ is a Krasnoselskii sequence.

Example 1.2. (Chidume [7]) Let $E := l^\infty$ and $B(0,1) := \{x \in l^\infty : \|x\| \le 1\}$. Define

$$T : B(0,1) \to B(0,1)$$

by

$$Tx := \{0, x_1^2, x_2^2, x_3^2, \cdots\}$$

where $x := \{x_1, x_2, x_3\} \in l^\infty$. Then (i) T is continuous, (ii) $Tp = p$ if and only if $p = 0$, (iii) $\|Tx - p\| \le \|x - p\|$ for all $x \in B(0,1)$ and fixed point p and (iv) for all $n \ge 0$, there exists $x \in B(0,1)$ such that $\lambda\|Tx_n - x_n\| = \|T_\lambda^{n+1}x - T_\lambda^n x\| > \lambda^2(1-\lambda)^2$, for any $\lambda \in (0,1)$, where $T_\lambda := (1-\lambda)I + \lambda T$ with I being the identity map on $B(0,1)$.
Let K be a closed convex nonempty subset of a uniformly convex real Banach space E and let $T : K \to CB(K)$ be a multi-valued quasi-nonexpansive mapping. It is our purpose in this paper to prove that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$ holds for quasi-nonexpansive mappings T for which all its fixed points are strict fixed points (see, e.g., [31]), i.e., $Tx^* = \{x^*\}$ for each fixed point x^* of T, where $\{x_n\}$ is a Krasnoselskii sequence.

2. Preliminaries

Lemma 2.1. *(Xu [37]) Let E be a uniformly convex real Banach space and $R > 0$. Then there exists a continuous, convex, strictly increasing function*

$$g : [0,\infty) \to [0,\infty), \quad g(0) = 0,$$

such that for all $x, y \in B(0,R) := \{u \in E : \|u\| < R\}$ and $\lambda \in (0,1)$,

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|).$$

Lemma 2.2. *(Xu [2]) Let X be a metric space and $T : X \to K(X)$ be a multi-valued map. Then T is $*$-nonexpansive if and only if P_T is nonexpansive.*

Lemma 2.3. *(Xu [2]) Let E be a uniformly convex real Banach space and let D be a closed convex and bounded subset of E. Suppose $T : D \to K(D)$ is $*$-nonexpansive multi-valued mapping. Then T has a fixed point.*

3. Main results

We start this section by showing that the result of Edelstein [13] is easily extended to the class of quasi-nonexpansive continuous mappings and for general $\lambda \in (0,1)$. We use the method of Edelstein [13].

Theorem 3.1. *Let K be a closed convex subset of a strictly convex real Banach space E and $T : K \to K$ be a continuous quasi-nonexpansive mapping such that $T(K)$ is contained in a compact subset K_1 of K. Then for each $\lambda \in (0,1)$ and for each $x_0 \in K$, the sequence $\{T_\lambda^n x_0\}$, where $T_\lambda : K \to K$ is defined by $T_\lambda x = ((1-\lambda)I + \lambda T)\,x$, converges strongly to a fixed point of T.*

Proof. Clearly, the quasi-nonexpansiveness of T implies that of T_λ. Therefore, for each $x \in K \setminus F(T)$, and $x^* \in F(T)$,

$$\|T_\lambda x - x^*\| = \|(1-\lambda)(x - x^*) + \lambda(Tx - x^*)\| \le \|x - x^*\|.$$

This implies

$$\left\|(1-\lambda)\frac{(x - x^*)}{\|x - x^*\|} + \lambda\frac{(Tx - x^*)}{\|x - x^*\|}\right\| \le 1.$$

Strict convexity of E and quasi-nonexpansiveness of T give

$$\left\|(1-\lambda)\frac{(x - x^*)}{\|x - x^*\|} + \lambda\frac{(Tx - x^*)}{\|x - x^*\|}\right\| < 1.$$

Hence, for all $x \in K \setminus F(T),\ x^* \in F(T)$ we have,

$$\|T_\lambda x - x^*\| < \|x - x^*\|. \tag{3.1}$$

We note that $\{T_\lambda^n x_0\}$ is contained in $\bar{co}\,(K_1 \cup \{x_0\})$ which, by Mazur Theorem [26], is compact. Therefore, $\{T_\lambda^n x_0\}$ has a convergent subsequence $\{T_\lambda^{n_j} x_0\}$ with limit $p \in K$. By the quasi-nonexpansiveness of T_λ, we have that for any $q \in F(T)$ and for any $n \ge 0$,

$$\|T_\lambda^{n+1} x_0 - q\| \le \|T_\lambda^n x_0 - q\|.$$

Hence, $\lim_{n\to\infty} \|T_\lambda^n x_0 - q\|$ exists. Therefore, convergence of a subsequence of $\{T_\lambda^n x_0\}$ to an element of $F(T)$ implies the convergence of the whole sequence to the same element. So, to prove the theorem, it suffices to show that the limit p of $\{T_\lambda^{n_j} x_0\}$ belongs to $F(T)$. We do this by contradiction. Suppose $T(p) \ne p$. Then no term of the sequence $\{T_\lambda^n x_0\}$ is a fixed point of T; for if there exists $N \ge 1$, such that $T_\lambda^N x_0 \in F(T)$, then $T_\lambda^n x_0 = T_\lambda^N x_0, \quad \forall n \ge N$ and so the whole sequence $\{T_\lambda^n x_0\}$ converges to $T_\lambda^N x_0 \in F(T)$ which gives $p \in F(T)$, a contradiction.
Hence, from (3.1), for any member $q \in F(T)$ we have

$$\|T_\lambda^{n+1} x - q\| < \|T_\lambda^n x - q\|, \quad \forall n \ge 1. \tag{3.2}$$

By continuity of T_λ at p, setting $r := \frac{1}{2}\Big(\|p - q\| - \|T_\lambda p - q\|\Big) > 0$, and $B := \{w \in K : \|w - T_\lambda(p)\| < r\}$, we obtain an open ball B' centered at p such that $T_\lambda(B') \subset B$. Convergence of $\{T_\lambda^{n_j} x_0\}$ to p guarantees the existence of $k \ge 1$ such that $T_\lambda^k x_0 \in B'$. So $T_\lambda^{k+1} x_0 \in B$. Using (3.2) we have for each $i \ge 1$,

$$\|T_\lambda^{k+i} x_0 - q\| < \|T_\lambda^{k+(i-1)} x_0 - q\| < \cdots < \|T_\lambda^{k+1} x_0 - q\| \le \|T_\lambda^{k+1} x_0 - T_\lambda p\| + \|T_\lambda p - q\|.$$

Therefore,

$$\|T_\lambda^{k+i} x_0 - q\| < r + \|T_\lambda p - q\| = \frac{1}{2}\Big(\|p - q\| + \|T_\lambda p - q\|\Big).$$

This now implies that for each $i \geq 1$,

$$\begin{aligned}\|T_\lambda^{k+i}x_0 - p\| &= \|T_\lambda^{k+i}x_0 - q + q - p\| \\ &\geq \|q-p\| - \|T_\lambda^{k+i}x_0 - q\| \\ &> \|q-p\| - \frac{1}{2}\Big(\|T_\lambda p - q\| + \|p-q\|\Big) \\ &= \frac{1}{2}\Big(\|p-q\| - \|T_\lambda p - q\|\Big) = r,\end{aligned}$$

which contradicts the fact that $\{T_\lambda^{n_j}x_0\}$ converges to p. Thus $p \in F(T)$ and the proof is complete.

□

We now prove the main theorem of this paper.

Theorem 3.2. *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. Suppose that $T : K \to CB(K)$ is a multi-valued quasi-nonexpansive mapping such that $Tp = \{p\}$ for some $p \in F(T)$. Then for any fixed $x_0 \in K$ and arbitrary $\lambda \in (0,1)$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \quad n \geq 0, \tag{3.3}$$

where $y_n \in Tx_n$. Then, $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$.

Proof. We first note that for any $x, y, z \in K$ such that $Ty = \{z\}$, we have

$$\begin{aligned}H(Tx,Ty) &= \max\left\{\sup_{u\in Tx}\text{dist}(u,Ty), \sup_{u\in Ty}\text{dist}(u,Tx)\right\} \\ &= \max\{\sup_{u\in Tx}\|u-z\|, \text{dist}(z,Tx)\} \\ &= \sup_{u\in Tx}\|u-z\| \\ &\geq \|u-z\| \quad \forall u \in Tx.\end{aligned} \tag{3.4}$$

We next show that $\{x_n\}$ is bounded. Let $p \in F(T)$ such that $Tp = \{p\}$. Then using inequality (3.4) and the assumption on T we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\lambda)(x_n - p) + \lambda(y_n - p)\| \\ &\leq (1-\lambda)\|x_n - p\| + \lambda\|y_n - p\| \\ &\leq (1-\lambda)\|x_n - p\| + \lambda H(Tx_n, Ty_n) \\ &\leq (1-\lambda)\|x_n - p\| + \lambda\|x_n - p\| \\ &= \|x_n - p\|, \quad \forall n \geq 0.\end{aligned}$$

This implies that $\{x_n\}$ is bounded and, so $\{y_n\}$ is bounded. We also have that the limit $\lim_{n\to\infty}\|x_n - p\|$ exists.

To prove the assertion of the theorem, let $R > 0$ such that $\{x_n\}, \{y_n\}$ are contained in $B(0,R)$. Then by Lemma 2.1, there exists a continuous, convex, and strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that for all $n \geq 0$ we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1-\lambda)(x_n - p) + \lambda(y_n - p)\|^2 \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)g\left(\|x_n - y_n\|\right) \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda H\left(Tx_n, Tp\right)^2 - \lambda(1-\lambda)g\left(\|x_n - y_n\|\right) \\ &\leq (1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 - \lambda(1-\lambda)g\left(\|x_n - y_n\|\right) \\ &= \|x_n - p\|^2 - \lambda(1-\lambda)g\left(\|x_n - y_n\|\right). \end{aligned}$$

It then follows that

$$\lambda(1-\lambda)g\left(\|x_n - y_n\|\right) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2, \quad \forall n \geq 0.$$

Since $\lim\limits_{n\to\infty} \|x_n - p\|$ exists and $\lambda \in (0, 1)$, we have $\lim\limits_{n\to\infty} g\left(\|x_n - y_n\|\right) = 0$. The fact that g is strictly increasing and $g(0) = 0$, imply that $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$. Since $y_n \in Tx_n$, we have that $\lim\limits_{n\to\infty} \mathrm{dist}(x_n, Tx_n) = 0$. This completes the proof. □

The following corollary follows.

Corollary 3.3. *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E. Suppose that $T : K \to K$ is a quasi-nonexpansive mapping. Then for any fixed $x_0 \in K$ and arbitrary $\lambda \in (0, 1)$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n. \tag{3.5}$$

Then, $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0.$

A mapping $T : K \to CB(K)$ is called *demicompact* if, for any sequence $\{x_n\}$ in K, $\{x_n\}$ bounded and $\mathrm{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$ imply the existence of a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. We note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is demicompact.

Corollary 3.4. *Let K be a nonempty, closed and convex subset of a uniformly convex real Banach space E and let $T : K \to CB(K)$ be a multi-valued quasi-nonexpansive mapping such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is demicompact and continuous with respect to the Hausdorff metric. Let $\{x_n\}$ be a sequence generated by* (3.3)*. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 3.2, we have that $\lim\limits_{n\to\infty} \mathrm{dist}(x_n, Tx_n) = 0$. Since $\{x_n\}$ is bounded and T is demicompact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ as $k \to \infty$ for some $q \in K$. Since T is continuous, we also have $\mathrm{dist}(x_{n_k}, Tx_{n_k}) \to \mathrm{dist}(q, Tq)$ as $k \to \infty$. Therefore, $\mathrm{dist}(q, Tq) = 0$ and so, by closedness of Tq, $q \in F(T)$. Setting $p = q$ in the proof of Theorem 3.2, it follows from (3.5) that $\lim\limits_{n\to\infty} \|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. □

Corollary 3.5. *Let K be a nonempty, compact and convex subset of uniformly convex real Banach space E and $T : K \to CB(K)$ be a multi-valued mapping such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous with respect to the Hausdorff* *metric. Let* $\{x_n\}$ *be a sequence generated by* (3.3)*. Then, the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Proof. The proof follows from Corollary 3.4 and the fact that compactness of K implies demicompactness of T. □

Corollary 3.6. *Let* K *be a nonempty, closed and convex subset of a uniformly convex real Banach space* E *and* $T : K \to CB(K)$ *be a multi-valued nonexpansive mapping with a nonempty fixed point set* $F(T)$ *such that* $Tp = \{p\}$ *for all* $p \in F(T)$*. Suppose that* T *is demicompact. Let* $\{x_n\}$ *be a sequence generated by* (3.3)*. Then, the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Proof. Since T is nonexpansive with nonempty fixed point set, it is quasi-nonexpansive and continuous with respect to the Hausdorff metric. So, the proof follows from Corollary 3.4. □

A mapping $T : K \to CB(K)$ is said to satisfy *Condition (I)* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\text{dist}(x, T(x)) \geq f(\text{dist}(x, F(T)) \ \forall x \in K.$$

Corollary 3.7. *Let* K *be a nonempty, closed and convex subset of a uniformly convex real Banach space* E *and* $T : K \to P(K)$ *be a multi-valued quasi-nonexpansive mapping such that* $Tp = \{p\}$ *for all* $p \in F(T)$*. Suppose that* T *satisfies condition (I). Let* $\{x_n\}$ *be a sequence generated by* (3.3)*. Then, the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Proof. From Theorem 3.2, we have that $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$. Using the fact that T satisfies condition (I), it follows that $\lim_{n\to\infty} f(\text{dist}(x_n, F(T))) = 0$ which in turn, using the nondecreasing property of f, gives $\lim_{n\to\infty} \text{dist}(x_n, F(T)) = 0$. Thus there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F(T)$ such that

$$\|x_{n_k} - p_k\| < \frac{1}{2^k} \ \forall k.$$

By setting $p = p_k$ and following the same arguments as in the proof of Theorem 3.2, we obtain from inequality (3.5) that

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

We now show that $\{p_k\}$ is a Cauchy sequence in K. Notice that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in K and thus converges strongly to some $q \in K$. Using the fact that T is quasi-nonexpansive and $p_k \to q$, we have

$$\begin{aligned} \text{dist}(p_k, Tq) &\leq H(Tp_k, Tq) \\ &\leq \|p_k - q\|, \end{aligned}$$

so that $\text{dist}(q, Tq) = 0$ and thus $q \in Tq$. Therefore, $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Setting $p = q$ in the proof of Theorem 3.2, it follows from inequality (3.5) that $\lim_{n\to\infty} \|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to q. This completes the proof. □

Corollary 3.8. *Let D be a nonempty, closed and convex subset of a uniformly convex real Banach space E and let $T : K \to K(D)$ be a multi-valued $*$-nonexpansive mapping with nonempty fixed point set $F(T)$. Suppose $\lambda \in (0, 1)$ and let $\{x_n\}$ be a sequence defined by $x_0 \in D$ fixed and*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \quad n \geq 0, \tag{3.6}$$

where $y_n \in P_T x_n$. Then, $\lim_{n\to\infty} dist(x_n, P_T x_n) = 0$. Moreover, if P_T is demicompact or it satisfies condition (I), then $\{x_n\}$ converges to a fixed point of T.

Proof. Using the fact that T is $*$-nonexpansive, we obtain by virtue of Lemma 2.2 that P_T is nonexpansive. Since $F(T) \neq \emptyset$ and $F(T) = F(P_T)$, it follows that P_T is quasi-nonexpansive. The results then follow from Theorem 3.2, Corollaries 3.4 and 3.7. □

Corollary 3.9. *Let D be a nonempty, closed, convex and bounded subset of a uniformly convex real Banach space E and let $T : K \to K(D)$ be a multi-valued $*$-nonexpansive mapping. Suppose $\lambda \in (0, 1)$ and let $\{x_n\}$ be a sequence defined by $x_0 \in D$ fixed and*

$$x_{n+1} = (1 - \lambda)x_n + \lambda y_n, \quad n \geq 0, \tag{3.7}$$

where $y_n \in P_T x_n$. Then, $\lim_{n\to\infty} dist(x_n, P_T x_n) = 0$. Moreover, if P_T is demicompact or it satisfies condition (I), then $\{x_n\}$ converges to a fixed point of T.

Proof. Using Lemma 2.3, the proof follows from Corollary 3.8. □

Ishikawa [19] proved that the if the Mann sequence $\{x_n\}$ *is bounded*, then it is an approximate fixed point sequence. Observe that in this result of Ishikawa, one can choose $\lambda_n = \lambda \in (0, b], 0 < b < 1, \ \forall n \geq 0$ (which will not be the case in the Mann process where $\lim \lambda_n = 0$ is required). If a nonexpansive mapping has a fixed point, x^* say, it is trivial to see that the sequence $\{\|x_n - x^*\|\}$ is monotone decreasing and so $\{x_n\}$ is bounded. Consequently, to approximate a fixed point of a nonexpansive mapping (using the Mann-type sequence) when existence is known, it can be assumed by the above result of Ishikawa that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Also, Edelstein and O'Brien [12] proved that for a nonexpansive map $T : K \to K$, where K is a bounded convex subset of *an arbitrary normed linear space*, the Krasnoselskii sequence always yields $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ *uniformly.* Example 2 shows that the above result does not hold if T is a quasi-nonepansive mapping. We have noted that the results holds for single-valued quasi-nonexpansive mappings in uniformly convex real Banach spaces (Dotson [10]). This brings us to the following open question.

Question. Let K be nonempty closed convex subset of a real Banach space E and let $T : K \to K$ be a quasi-nonexpansive continuous mapping. Let $\{x_n\}$ be defined by $x_0 \in K$,

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n \geq 0.$$

Is $\{x_n\}$ an approximate fixed point sequence in any Banach space E more general than uniformly convex Banach spaces?

Remark 1. Theorem 3.2 extends this result of Dotson to the multi-valued quasi-nonexpansive mappings on uniformly convex real Banach spaces.

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