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SOME REMARKS ON THE PAPER "ON THE SET OF SOLUTIONS FOR THE DARBOUX PROBLEM FOR FRACTIONAL ORDER PARTIAL HYPERBOLIC FUNCTIONAL DIFFERENTIAL INCLUSIONS"

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Abstract. In this note we point out some errors in the paper "On the set of solutions for the Darboux problem for fractional order partial hyperbolic functional differential inclusions" by S. Abbas and M. Benchohra published in Fixed Point Theory, vol. 14, no. 2, 2013, 253-262.
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1. Introduction

In [1] it is considered the following problem

$$D_c^r u(x,y) \in F(x,y,u(x,y),G(x,y,u(x,y))) \quad a.e.\ (x,y)\in \Pi, \tag{1.1}$$

$$u(x,0)=\varphi(x), \quad u(0,y)=\psi(y) \quad (x,y)\in\Pi, \tag{1.2}$$

where $\Pi=[0,a]\times[0,b]$, $\varphi(.):[0,a]\to\mathbb{R}^n$, $\psi(.):[0,a]\to\mathbb{R}^n$ are given absolutely continuous functions with $\varphi(0)=\psi(0)$, $F(.,.):\Pi\times\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$, $G(.,.):\Pi\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ are given set-valued maps and D_c^r is the Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$.

The authors state that they prove the arcwise connectedness of the solution set of this problem. As we can see afterwards, in general, this is not true.

Since a continuous function $u(.,.):\Pi\to\mathbb{R}^n$ is a solution of problem (1.1)-(1.2) if and only if $u(.,.)$ is a solution of the problem

$$u(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\xi(s,t)dsdt, \tag{1.3}$$

$$\xi(x,y)\in F(x,y,u(x,y),G(x,y,u(x,y))) \quad a.e.\ (\Pi), \tag{1.4}$$

where $\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0)$ and $\Gamma(.)$ is Euler's Gamma function, it is enough to obtain the desired properties for the solution set of problem (1.3)-(1.4).

The same authors consider in [2] the similar problem

$$u(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b(x-s)^{r_1-1}(y-t)^{r_2-1}f(x,y,s,t,\xi(s,t))dsdt, \tag{1.5}$$

$$\xi(x,y)\in F(x,y,u(x,y),G(x,y,u(x,y)))\quad a.e.\ (\Pi). \tag{1.6}$$

Since the crossing from problem (1.3)-(1.4) to problem (1.5)-(1.6) is obvious and since the result in [2] contains the same errors as in [1], in what follows we are concerned only with problem (1.3)-(1.4).

The paper is organized as follows: in Section 2 we present some definitions and preliminary results needed for our considerations and in Section 3 we present a discussion of the result in [1].

2. Preliminaries

Let (X,d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\ \ d^*(A,B)=\sup\{d(a,B);\ a\in A\},$$

where $d(x,B)=\inf\{d(x,y);y\in B\}$. Let Z be a metric space, let X be a nonempty set and let $F:X\to\mathcal{P}(Z)$ be a set-valued map. The multifunction F is called Hausdorff continuous if for any $x_0\in X$ and every $\epsilon>0$ there exists $\delta>0$ such that $x\in X, d_X(x,x_0)<\delta$ implies $d_H(F(x),F(x_0))<\epsilon$. A nonempty set $K\subset L^1(T,X)$ is called decomposable if, for every $u,v\in K$ and every $A\in\mathcal{F}$, one has $\chi_A.u+\chi_{T\setminus A}.v\in K$ where $\chi_B, B\in\mathcal{F}$ indicates the characteristic function of B.

We denote by $C(\Pi,\mathbb{R}^n)$ the Banach space of all continuous functions $u:\Pi\to\mathbb{R}^n$ endowed with the norm

$$|u|_C=\sup_{(x,y)\in\Pi}||u(x,y)||.$$

Given a continuous strictly positive function $d:\Pi\to\mathbb{R}$ we denote by $L^1(\Pi,\mathbb{R}^n)$ the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma:\Pi\to\mathbb{R}^n$, endowed with the norm

$$|\sigma|_1=\int\int_\Pi d(x,y)||\sigma(x,y)||dxdy.$$

By $\mathcal{M}$ we mean the linear subspace of $C(\Pi,\mathbb{R}^n)$ consisting of all $\mu\in C(\Pi,\mathbb{R}^n)$ such that there exist continuous functions $\varphi(.):[0,a]\to\mathbb{R}^n$, $\psi(.):[0,a]\to\mathbb{R}^n$ with $\varphi(0)=\psi(0)$ satisfying $\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0)$. $\mathcal{M}$, equipped with the norm of $C(\Pi,\mathbb{R}^n)$, is a separable Banach space.

Definition 2.1. a) The left-sided mixed Riemann-Liouville integral of order $r=(r_1,r_2)\in(0,1]\times(0,1]$ of $f(.,.)\in L^1(\Pi,\mathbb{R}^n)$ is defined by

$$(I_0^rf)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dsdt,$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by

$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\mathrm{d}t,\ \alpha>0.$$

b) The Caputo fractional-order derivative of order r of $f(.,.) \in L^1(\Pi, \mathbb{R}^n)$ is defined by

$$(D_c^r f)(x,y) = \left(I_0^{1-r} \frac{\partial^2 f}{\partial x \partial y} \right)(x,y).$$

In the definition above by $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1] \times (0,1]$.

Definition 2.2. A function $u(.,.) \in C(\Pi, \mathbb{R}^n)$ is said to be a solution of problem (1.1)-(1.2) if there exists $f(.,.) \in L^1(\Pi, \mathbb{R}^n)$ such that

$$f(x,y) \in F(x,y,u(x,y), G(x,y,u(x,y))) \quad a.e. \ (\Pi), \tag{2.1}$$

$$D_c^r u(x,y) = f(x,y) \quad (x,y) \in \Pi, \tag{2.2}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y) \quad (x,y) \in \Pi, \tag{2.3}$$

where $F(x,y,u,G(x,y,u)) = \cup_{v \in G(x,y,u)} F(x,y,u,v)$.

Lemma 2.3. ([1]) *$u(.,.) \in C(\Pi, \mathbb{R}^n)$ is a solution of problem (2.2)-(2.3) if and only if $u(.,.)$ satisfies*

$$u(x,y) = \mu(x,y) + (I_0^r f)(x,y), \quad (x,y) \in \Pi,$$

where $\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0)$.

We denote by $S(\mu)$ the solution set of problem (1.3)-(1.4).

The key tool in the attempt of proving the result in [1] is the next result of Marano and Staicu, concerning the arcwise connectedness of the fixed point set of nonclosed nonconvex contractions.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space, S a separable Banach space, let $(X, |.|_X)$ be a real Banach space and $E = L^1(T, X)$.

Lemma 2.4. ([5]) *Assume that $\phi : S \times E \to \mathcal{P}(E)$ and $\psi : S \times E \times E \to \mathcal{P}(E)$ are Hausdorff continuous multifunctions with nonempty, closed, decomposable values, satisfying the following conditions*

a) There exists $L \in [0,1)$ such that, for every $s \in S$ and every $u, u' \in E$,

$$d_H(\phi(s,u), \phi(s,u')) \le L|u - u'|_E.$$

b) There exists $M \in [0,1)$ such that $L + M < 1$ and for every $s \in S$ and every $(u,v), (u',v') \in E \times E$,

$$d_H(\psi(s,u,v), \psi(s,u',v')) \le M(|u-u'|_E + |v - v'|_E).$$

Set $Fix(\Gamma(s,.)) = \{u \in E; u \in \Gamma(s,u)\}$, where $\Gamma(s,u) = \psi(s,u,\phi(s,u)), (s,u) \in S \times E$. Then

1) For every $s \in S$ the set $Fix(\Gamma(s,.))$ is nonempty and arcwise connected.

2) For any $s_i \in S$, and any $u_i \in Fix(\Gamma(s,.)), i = 1, ..., p$ there exists a continuous function $\gamma : S \to E$ such that $\gamma(s) \in Fix(\Gamma(s,.))$ for all $s \in S$ and $\gamma(s_i) = u_i, i = 1, ..., p$.

The next technical result is due to De Blasi, Pianigiani and Staicu [4].

Lemma 2.5. *Let $\xi \in (0,1)$ and let $N : \Pi \to \mathbb{R}$ be a positive integrable function. Then there exists a continuous strictly positive function $d : \Pi \to \mathbb{R}$ which, for every $(x,y) \in \Pi$, satisfies*

$$\int\int_{R(x,y)} N(\xi,\eta) d(\xi,\eta) d\xi d\eta = \xi(d(x,y) - 1), \tag{2.4}$$

where $R(x,y) = [x,a] \times [y,b]$.

3. A discussion of the result in [1]

In this section we discuss the unique result in [1]; namely, Theorem 4.2, whose statement follows in our "Theorem 3.2".

Hypothesis 3.1. Let $F : \Pi \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ and $G : \Pi \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be two set-valued maps with nonempty closed values, satisfying the following assumptions

i) The set-valued maps $(x,y) \to F(x,y,u,v)$ and $(x,y) \to G(x,y,u)$ are measurable for all $u,v \in \mathbb{R}^n$.

ii) There exists $l \in L^1(\Pi,\mathbb{R})$ such that, for every $u,u' \in \mathbb{R}^n$,

$$d_H(G(x,y,u),G(x,y,u')) \le l(x,y)||u-u'|| \quad a.e.\ (\Pi).$$

iii) There exist $m \in L^1(\Pi,\mathbb{R})$ and $\eta \in [0,1)$ such that, for every $u,v,u',v' \in \mathbb{R}^n$,

$$d_H(F(x,y,u,v),F(x,y,u',v')) \le m(x,y)||u-u'|| + \eta||v-v'|| \quad a.e.\ (\Pi).$$

iv) There exist $f_1, f_2 \in L^1(\Pi,\mathbb{R})$ such that

$$d(\{0\},F(x,y,\{0\},\{0\})) \le f_1(x,y), \quad d(\{0\},G(x,y,\{0\})) \le f_2(x,y) \quad a.e.\ (\Pi).$$

Theorem 3.2. *Assume that Hypothesis 3.1 is satisfied. Then,*

1) For every $\mu \in \mathcal{M}$, the solution set $S(\mu)$ of (1.1)-(1.2) is nonempty and arcwise connected in the space $C(\Pi,\mathbb{R}^n)$.

2) For any $\mu_i \in \mathcal{M}$ and any $u_i \in S(\mu_i), i = 1,...,p$, there exists a continuous function $s : \mathcal{M} \to C(\Pi,\mathbb{R}^n)$ such that $s(\mu) \in S(\mu)$ for any $\mu \in \mathcal{M}$ and $s(\mu_i) = u_i, i = 1,...,p$.

3) The set $S = \cup_{\mu\in\mathcal{M}} S(\mu)$ is arcwise connected in $C(\Pi,\mathbb{R}^n)$.

Several remarks are in order.

Remark 3.3. If $r_1 = r_2 = 1$ then problem (1.1)-(1.2) reduces to the "classical" Darboux problem for hyperbolic differential inclusions of the form

$$u_{xy}(x,y) \in F(x,y,u(x,y),G(x,y,u(x,y))) \quad a.e.\ (x,y) \in \Pi, \tag{3.1}$$

$$u(x,0) = \varphi(x), \quad u(0,y) = \psi(y) \quad (x,y) \in \Pi, \tag{3.2}$$

The arcwise connectedness of the solution set of problem (3.1)-(3.2) was obtained in [3] under Hypothesis 3.1 **and** under the hypothesis that $\xi \in (0,1)$ is fixed such that $2\xi + \eta < 1$ and the mapping $d(.,.)$ which defines the norm on $L^1(\Pi,\mathbb{R}^n)$ is provided by Lemma 2.5 and, obviously, depends on ξ.

Even if it is not specified in [1], the wish of the authors was to extend the result in [3] to problem (1.1)-(1.2). Unfortunately, because of a superficial understanding of the proof in [3], this extension fails.

Remark 3.4. At the beginning of the proof of Theorem 4.2 in [1], following [3], there are taken $\xi \in (0,1)$ such that $2\xi + \eta < 1$ and the corresponding mapping $d(.,.)$ given by Lemma 2.5.

If in [3] such a choice is needed in order to obtain that certain Lipschitz constant is less than 1, the choice in [1] is unjustified since it is not used, in any way, in the next computations of the proof.

Remark 3.5. Consider the set-valued maps $\alpha : \mathcal{M} \times L^1(\Pi, \mathbb{R}^n) \to \mathcal{P}(L^1(\Pi, \mathbb{R}^n))$ and $\beta : \mathcal{M} \times L^1(\Pi, \mathbb{R}^n) \times L^1(\Pi, \mathbb{R}^n) \to \mathcal{P}(L^1(\Pi, \mathbb{R}^n))$ given by

$$\alpha(\mu, u) = \{v \in L^1(\Pi, \mathbb{R}^n); \quad v(x, y) \in G(x, y, u_\mu(x, y)) \quad a.e.\ (\Pi)\},$$

$$\beta(\mu, u, v) = \{w \in L^1(\Pi, \mathbb{R}^n); \quad w(x, y) \in F(x, y, u_\mu(x, y), v(x, y)) \quad a.e.\ (\Pi)\},$$

where $u_\mu(x, y) = \mu(x, y) + (I_0^r u)(x, y)$.

The crucial step of the proof consists in showing that these set-valued maps verify the hypothesis of Lemma 2.4, **especially** the fact that the sum between the Lipschitz constant (with respect to the second variable) of α and the Lipschitz constant (with respect to the second and third variable) of β is less than 1.

In [1] is given an estimate of the Lipschitz constant of α. More exactly, this Lipschitz constant is estimated as

$$L(r_3) := \frac{\xi^{r_3} N^* a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1 + 1)^{(1-r_3)} (\omega_2 + 1)^{(1-r_3)} \Gamma(r_1)\Gamma(r_2)}, \tag{3.1}$$

with $0 < r_3 < \min\{r_1, r_2\}$,

$$\omega_1 = \frac{r_1 - 1}{1 - r_3},\ \omega_2 = \frac{r_2 - 1}{1 - r_3},$$

$$N^* = \sup_{(x,y)\in\Pi} (N(x, y))^{\frac{1-r_3}{r_3}},$$

$$N(x, y) = \max\{l(x, y), m(x, y)\}, (x, y) \in \Pi.$$

Surprisingly, the Lipschitz constant of β is not computed!

If the estimates (3.1) are correct, then the Lipschitz constant of β follows to be $L(r_3) + \eta$. Therefore, Lemma 2.4 may be applied and Theorem 4.2 in [1] is true if, in addition, $L(r_3) + L(r_3) + \eta = 2L(r_3) + \eta < 1$.

Consequently, an open problem is: if (or when) one may found r_3 such that $0 < r_3 < \min\{r_1, r_2\}$ and $2L(r_3) + \eta < 1$?

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