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A BANACH CONTRACTION PRINCIPLE FOR ABELIAN GROUPS

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Abstract. We apply the Banach Contraction Principle in order to obtain some information about fixed points of some maps defined on abelian groups.
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1. Introduction

There are examples in literature which show that some principles used in order to prove theorems for Banach spaces can be used in more abstract settings. For instance in [2] the authors prove that the main results about some almost additive maps (i.e. they can be approximate by additive maps) on Banach spaces can be extended to maps defined on semigroups. Conversely, some results proved for some very abstract structures can be used to derive classical results about Banach spaces. For instance, Baranga applied in [1] the Tarski-Kantorovitch Theorem stated on some partial ordered sets in order to prove the Banach Contraction Principle, even the standard proof for the Banach Contraction Principle uses standard properties of Banach spaces. In the present paper we apply this strategy in order to obtain some information about fixed points of some maps defined on abelian groups.

We will prove in Proposition 2.6 and Proposition 2.7 that the Banach Contraction Principle can be stated for torsion-free abelian groups which are complete and Hausdorff in the p-adic topology (where p is a fixed prime). These results cannot be extended to general abelian groups, cf. Remark 2.9.

In this paper all abelian groups are additive abelian groups, and the letter p will denote a fixed prime number. If G is an abelian group and $n \in \mathbb{N}$ then $nG = \{nx \mid x \in G\}$ is a subgroup of G. All unexplained notions and results can be consulted in [3] and [5].

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2. Banach contraction principle for abelian groups

Let G be an abelian group and let p be a prime. We recall that the p-adic topology defined on G has a basis of neighborhoods for 0 the set $\{p^nG \mid n \in \mathbb{N}\}$ and the addition map is additive. This topological space is Hausdorff if and only if $\bigcap_{n\in\mathbb{N}} p^nG = 0$, and every group which is complete and Hausdorff in the p-adic toplogy is a J_p-module, where J_p is the ring of p-adic integers.

Moreover, we can associate to every $x \in G$ its *p-height* which is defined as the positive integer $|x|_p$ with the property $x \in p^{|x|_p}G \setminus p^{|x|_p+1}G$, if such an integer exists. If $x \in \bigcap_{n\in\mathbb{N}} p^nG$ we define $|x|_p = \infty$. We recall that we use the convention $\infty + k = k + \infty = \infty$. Moreover, a torsion-free group is p-reduced if it has no non-zero elements of infinite p-height. Therefore, all torsion-free groups which are Hausdorff as p-adic spaces are p-reduced.

The basic properties of p-heights are the following:

Lemma 2.1. *Let G be an abelian group, and let p be a prime. The following are true:*

(1) *If $(q,p) = 1$ then $|qx|_p = |x|_p$;*
(2) *If $x \in G$ then $|px|_p \geq |x|_p + 1$;*
(3) *If $x, y \in G$ then $|x+y|_p \geq \min\{|x|_p, |y|_p\}$.*
(4) *If $x, y \in G$ and $|x|_p \neq |y|_p$ then $|x+y|_p = \min\{|x|_p, |y|_p\}$.*

Using a similar idea as that used in the definition of contraction maps defined on Banach spaces, see [5], we introduce a notion of contraction which is connected to the p-adic topology associated to a torsion-free abelian group.

Let G be a torsion free abelian group and $U \subseteq G$ a nonempty subset of G. We say that a map $f : G \to G$ is a *U-contraction* if $f(U) \subseteq U$ and

$$|f(x) - f(y)|_p \geq |x - y|_p + 1$$

for all $x, y \in U$.

The reader can verify without many efforts the following basic property:

Lemma 2.2. *Let G be a torsion-free abelian group and $U \subseteq G$. If f and g are U-contractions then $f \circ g$ is a U-contraction.*

Remark 2.3. Let G be a torsion-free abelian group, and let $f : G \to G$ be a U-contraction for a subset $U \subseteq G$. Fix an element $x \in U$.

If $|f(x)|_p < |x|_p$ then

$$|f^2(x) - f(x)|_p > |f(x)|_p + 1 > \min\{|f^2(x)|_p, |f(x)|\},$$

and it follows that $|f^2(x)|_p = |f(x)|_p$.

If $|f(x)|_p = |x|_p$, suppose $|f^2(x)|_p \neq |f(x)|_p$. Then

$$\begin{aligned} |f(x)|_p \geq \min\{|f^2(x)|_p, |f(x)|_p\} &= |f^2(x) - f(x)|_p \geq |f(x) - x|_p + 1 \\ &\geq \min\{|f(x)|_p, |x|_p\} + 1 \geq |f(x)|_p + 1, \end{aligned}$$

a contradiction. It follows that $|f^2(x)|_p = |f(x)|_p$.

Therefore, we can assume w.l.o.g. that the restriction $f_{|U}$ does not decrease the p-heights, i.e. $|x|_p \leq |f(x)|_p$ for all $x \in U$.

Lemma 2.4. *Let G be a p-reduced torsion free abelian group and $U \subseteq G$ a nonempty subset of G. If $f : G \to G$ is a U-contraction such that $f_{|U}$ does not decrease the p-heights. Then*

$$|f^n(x) - f^m(y)|_p \geq \min\{|x|_p, |y|_p\} + m$$

for all $x, y \in U$ and all $0 < m < n \in \mathbb{N}$.

Proof. In order to prove this we will use the induction on m.

For $m = 1$ we have

$$\begin{aligned} |f^n(x) - f(y)|_p &\geq |f^{n-1}(x) - y|_p + 1 \\ &\geq \min\{|f^{n-1}(x)|_p, |y|_p\} + 1. \end{aligned}$$

Since f do not decrease the p-heights, it follows that

$$|f^n(x) - f(y)|_p \geq \min\{|x|_p, |y|_p\} + 1.$$

Suppose that $|f^n(x) - f^m(y)|_p \geq \min\{|x|_p, |y|_p\} + m$ for a fixed m and all $n > m$. If $n > m + 1$ we obtain, in the same way as before,

$$\begin{aligned} |f^n(x) - f^{m+1}(y)|_p = |f(f^{n-1}(x)) - f(f^m(y))|_p &\geq |f^{n-1}(x) - f^m(y)|_p + 1 \\ &\geq \min\{|x|_p, |y|_p\} + m + 1, \end{aligned}$$

and the induction step is complete. □

Using this we can find the set of fixed points of f which lie in U.

Remark 2.5. Before we state the main result of the paper, let us note that in the hypotheses of the previous lemma, if $0 \in U$ then $|f(x)|_p \geq |x|_p + 1$ for all $x \in U$, so 0 is the only fixed point of f which belongs to U.

Proposition 2.6. *Let G be a p-reduced torsion free abelian group and $U \subseteq G$ a nonempty subset of G. If $f : G \to G$ is a U-contraction such that $f_{|U}$ does not decrease the p-heights, then*

$$\bigcap_{n\in\mathbb{N}} f^n(U) = \{x \in U \mid f(x) = x\},$$

and this set has at most one element.

Proof. Suppose that $x, y \in U$ are fixed points for f. Since f is a U-contraction we obtain $|x - y|_p \geq |x - y|_p + 1$, and this is possible only if the p-height of $x - y$ is infinite. But G is p-reduced and torsion-free, and this implies $x - y = 0$.

Let $z \in \bigcap_{n\in\mathbb{N}} f^n(U)$. Then for every $n \in \mathbb{N}$ there exists $x_n \in U$ such that $f^n(x_n) = z$. If we choose $n \in \mathbb{N}$ and we evaluate the p-height of $f(z) - z$ we obtain, using Lemma 2.4,

$$|f(z) - z|_p = |f^{n+1}(x_n) - f^n(x_n)|_p \geq |x_n|_p + n.$$

Then $|f(z) - z|_p = \infty$. But 0 is the only element of infinite p-height, since G is p-reduced. It follows that $f(z) = z$.

Then $\bigcap_{n\in\mathbb{N}} f^n(U) \subseteq \{x \in U \mid f(x) = x\}$. Since the converse inclusion is obvious, the proof is complete. □

This result suggests us that it can be useful to study the sequences $(f^n(x_n))_{n\in\mathbb{N}}$, where $x_n \in U$.

Proposition 2.7. *Let G be a p-reduced torsion free abelian group and $U \subseteq G$ a nonempty subset of G. If $f : G \to G$ is a U-contraction such that $f_{|U}$ does not decrease the p-heights, the following statements are true:*

(1) *For every $x \in U$ the sequence $(f^n(x))_{n>0}$ is Cauchy in the p-adic topology.*
(2) *If G is reduced, $x \in U$ and x^* is the limit in the p-adic topology for the sequence $(f^n(x))_{n>0}$ then $f(x^*) = x^*$.*
(3) *If G is reduced, $U = G \setminus \{0\}$ and x^* is the limit in the p-adic topology for the sequence $(f^n(x))_{n>0}$ then $x^* = 0$.*

Proof. (1) Using Lemma 2.4 we obtain that

$$|f^n(x) - f^m(x)|_p \geq |x|_p + m$$

for all $0 < m < n \in \mathbb{N}$, and the conclusion is now obvious.

(2) In order to prove that x^* is a fixed point for f, it is enough to prove that $f(x^*)$ is also a limit for the sequence $(f^n(x))_n$.

Let us fix a positive integer k. It follows that there exists an integer $u > 0$ such that $f^v(x) - x^* \in p^k G$ for all integers $v \geq u$. Using the hypothesis we obtain

$$|f^{v+1}(x) - f(x^*)|_p \geq |f^v(x) - x^*|_p + 1 \geq k + 1,$$

hence $f(x^*)$ is also a limit for $(f^n(x))_n$. Since our topological space is Hausdorff, it follows that $f(x^*) = x^*$.

(3) Suppose that $0 \neq x^* \in G$. Note that the p-height of x^* is finite, and let $y^* \in G$ be the (unique) element of G such that $p^{|x|_p} y = x^*$. It is not hard to see that $y \in G \setminus pG$, so $|y|_p = 0$.

Applying the hypothesis we obtain that the inequality

$$|f(pz) - f(z)|_p \geq |pz - z|_p + 1 = |z|_p + 1$$

is valid for all $z \in U$ such that $pz \in U$.

For the case $z = y$ we obtain $|f(py) - f(y)|_p \geq 1$. If $|f(py)|_p > |f(y)|_p$ then we obtain $|f(y)|_p \geq 1$, and it follows that $|f(x^*)|_p > |x^*|_p$, a contradiction. Therefore $|f(py)|_p = |f(y)|_p$. We can repeat the same steps for $z = py$, and we obtain

$$|f(p^2 y)|_p = |f(py)|_p = |f(y)|_p.$$

If we continue in this way, we obtain $|f(x)|_p = |f(p^k y)|_p$ for all k. But this implies the f decrease some heights, a contradiction. □

Example 2.8. Let G be a p-reduced torsion-free group. Fix a nonzero element $g \in G \setminus pG$, and we consider the set $U = g + pG$. It is easy to prove that the map

$$f : G \to G,\ f(x) = g + px$$

is a U-contraction such that $f_{|U}$ does not decrease the p-heights. If the sum $\sum_{n>0} p^n g$ exists in G (for instance, in the case G is algebraically compact) then

$$g^* = g + \sum_{n>0} p^n g$$

is a fixed point of f.

Remark 2.9. If in the above example $p > 2$ and we suppose that G has elements of order $p - 1$ then every solution of the equation $(1 - p)x = g$ is a fixed point for f.

Remark 2.10. The proof of Proposition 2.7 suggests us that we can also consider a theory of (weakly) Picard operators associated to maps defined on abelian groups. Recall that an operator $f : X \to X$ defined on a metric space (X, d) is called *weakly Picard* if for every $x \in X$ the sequence $(f^n(x))_n$ is convergent and the limit is a fixed point of f, [4]. It follows that the maps studied in Proposition 2.7 have a similar property. Let $f : G \to G$ be a map which does not decrease the heights. We say that $x \in G$ is a *a weakly Picard point* associated to f if the sequence $(f^n(x))_n$ is a Cauchy sequence with respect the p-adic topology. It would be nice to have more information about the set of all weakly Picard points associated to f, but standard computations show that this is not possible without some additional hypotheses.

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