Fixed Point Theory, 17(2016), No. 2, 267-274
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

EXISTENCE THEOREM FOR DIRICHLET PROBLEM FOR DIFFERENTIAL INCLUSION DRIVEN BY THE $p(x)$-LAPLACIAN

GE BIN

Department of Applied Mathematics, Harbin Engineering University
Harbin, 150001, P.R. China
E-mail: gebin04523080261@163.com

Abstract. We study a class of second-order differential inclusion problem involving the $p(x)$-Laplacian. Using the techniques from multivalued analysis and the Leray-Schauder fixed point theorem, we establish the existence theorems under certain conditions.
Key Words and Phrases: Multifunction, maximal monotone operator, continuous selectors, differential inclusion, $p(x)$-Laplacian, Leray-Schauder fixed point theorem.
2010 Mathematics Subject Classification: 35R70, 35R20, 35J25, 35J70, 47H10.

1. Introduction

The differential equations and variational problems with nonstandard growth conditions have been studied in recent years. Some results on these problems have been obtained. For example, we refer to [6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 26, 27, 29] and references therein.

In the last decade, the second order differential systems driven by p-Laplacian operators (or p-Laplacian-like operators) have attracted increasing interest. Many works were carried out with various techniques employed, such as Pino-Elgueta-Manasevich [23], Manasevich-Mawhin [21], Zhang [28], Aizicovici-Papageorgiou-Staicu [1] with the Leray-Schauder degree, Kyritsi-Matzakos-Papageorgiou [19], Papalini [22], Zhang-Li [30] with fixed points of the multivalued maps, Bader-Papageorgiou [4], Papageorgiou-Staicu [25] with the method of upper-lower solutions etc.

The goal of this paper is to extend the works of Del Pino-Elgueta-Manasevich [23] and Zhang [28] to a larger class of differential inclusion problems, which involve the $p(x)$-Laplacian, that is

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in F(x,u(x)), & \text{in } \Omega, \\ u(x)=0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is bounded smooth domain in $\mathbb{R}^N$, $p(x)\in C(\overline{\Omega})$, $1<p^-\leq p(x)<+\infty$ and $F:\Omega\times\mathbb{R}\to 2^{\mathbb{R}}\backslash\emptyset$ is a multifunction.

Our method will be based on the techniques from multivalued analysis and nonlinear analysis. For the convenience of the readers, in the next section we recall the basic notions and results from these areas that we will need in the sequel. For further details we refer to the books of Barbu [2] and Hu-Papageorgiou [18].

2. Preliminary results

In this section we first review some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. We refer [12] for the details.

Let Ω and p be as in Section 1. Denote by $\mathbb{S}(\Omega)$ the set of all measurable real functions defined on Ω. Note that two measurable functions are considered as the same element of $\mathbb{S}(\Omega)$ when they are equal almost everywhere.

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u \in \mathbb{S}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty\},$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}$, and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm $\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $|\nabla u|_{p(x)}$ is an equivalent norm on $W_0^{1,p(x)}(\Omega)$. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are all separable and reflexive Banach space(see [12]).

Hereafter, let $p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \geq N. \end{cases}$

Lemma 2.1. [12]

(1) Poincaré inequality in $W_0^{1,p(x)}(\Omega)$ holds, that is, there exists a positive constant C such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \forall u \in W_0^{1,p(x)}(\Omega).$$

(2) The conjugates space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$,

$$\int_\Omega |uv| dx \leq \left(\frac{1}{p_-} + \frac{1}{q_-}\right) |u|_{p(x)} |v|_{q(x)}.$$

(3) If $q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W_0^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.

Lemma 2.2. [12] *If we denote $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$, $\forall u \in W^{1,p(x)}$, then*

(1) for $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;

(2) $|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1)$;

(3) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$,

$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;

(4) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$, $|u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, u \in W_0^{1,p(x)}(\Omega).$$

We know that (see [5]), $J \in C^1(W_0^{1,p(x)}, \mathbb{R})$ and $p(x)-$Laplacian operator $-\Delta_{p(x)} u = -\mathrm{div}(|\nabla u|^{p(x)-2} \nabla u)$ is the derivative operator of J in the weak sense. We denote $A = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, then

$\langle A(u), v\rangle = \int_\Omega (|\nabla u(x)|^{p(x)-2} (\nabla u(x), \nabla v(x))_{\mathbb{R}^N} dx, \forall u, v \in W_0^{1,p(x)}(\Omega).$

Lemma 2.3. [13] *Set $X = W_0^{1,p(x)}(\Omega)$, A is as above, then*

(1) $A : X \to X^*$ is a continuous, bounded and strictly monotone operator;

(2) $A : X \to X^*$ is a mapping of type $(S)_+$, i.e., if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$, implies $u_n \to u$ in X;

(3) $A : X \to X^*$ is a homeomorphism.

Secondly, we give briefly some notions and results from multivalued analysis and the theory of nonlinear operators of monotone type.

Let E, E_1 be Banach spaces, and let

$\mathcal{P}_f = \{M \subseteq E : M$ is nonempty and closed$\}$,

$\mathcal{P}_k = \{M \subseteq E : M$ is nonempty and compact$\}$,

$\mathcal{P}_{(w)kc} = \{M \subseteq E : M$ is nonempty (weakly) compact and convex$\}$.

A multivalued map $T : E \to 2^{E_1} \setminus \emptyset$ is said to be upper semicontinuous (usc) if and only if the inverse image $T^{-1}(C) = \{x \in E : T(x \cap C) \neq \emptyset\}$ is closed for each closed subset C of E_1.

In this paper, we will need the multivalued generalization of the Leray-Schauder alternative theorem, due to Bader [3], that is:

Theorem 2.1. *If E, E_1 are Banach spaces, $T : E \to \mathcal{P}_{wkc}(E_1)$ is usc from E into E_1 endowed with weak topology, $\Psi : E_1 \to E$ is completely continuous and $\Phi = \Psi \circ T$ maps bounded sets into relatively compact sets, then one of the following statements holds:*

(1) the set $S = \{x \in E : x \in \mu\Phi(x), 0 < \mu < 1\}$ is unbounded, or

(2) Φ has a fixed point, i.e. there exists a $x \in E$, such that $x \in \Phi(x)$.

Remark 2.1. *We emphasize that the composition Φ need not have convex values. This makes Theorem 2.1 suitable for nonlinear problems (compare with a similar multivalued alternative theorem in Dugundji-Granas* [7]*, p. 98).*

3. Existence theorems

In this section we shall prove a sufficient condition for the existence of solutions for (P). We shall need the following conditions:

$\mathbf{H(F)}$: $F : \Omega \times \mathbb{R} \to \mathcal{P}_{kc}$ a multifunction with the following properties:

(i) $(x, t) \to F(x, t)$ is graph measurable;

(ii) for almost all $x \in \Omega$, $t \to F(x, t)$ has a closed graph;

(iii) there exist $\alpha \in C(\overline{\Omega})$ $(1 < \alpha^- \leq \alpha^+ < p^-)$, $a \in L^{\alpha'(x)}(\Omega)\left(\frac{1}{\alpha(x)} + \frac{1}{\alpha'(x)} = 1\right)$, and a positive constant c such that

$$|w| \leq a(x) + c|t|^{\alpha(x)-1} \text{ for all } w \in F(x, t);$$

In the following, for any $u \in X$, we will use the notations:

$$F(u)(x) = F(x, u(x)) \text{ and } N_F(u) = \{v \in V^* : v(x) \in F(x, u(x))\},$$

where $V = L^{\alpha(x)}(\Omega)$ and V^* its topological dual.

Now, we turn to study the multivalued map N_F. For the sake of completeness, we write their proofs in detail below.

Lemma 3.1. *If hypothesis* $\mathbf{H(F)}$ *holds, then* $\forall u \in X$, $N_F(u)$ *is a nonempty, closed and convex subset of* V^*, *and* N_F *is husc (which means* N_F *is usc from* X *into* V^* *endowed with weak topology), and bounded on bounded sets.*

Proof. The closedness and convexity of the value of $N_F(\cdot)$ are clear. To prove the nonemptyness, let $u \in X$, and $\{u_n\}_{n\geq 1} \subseteq L^{p(x)}(\Omega)$ be a sequence of step function such that

$$u_n \to u \text{ in } L^{p(x)}(\Omega),$$

$$|u_n(x)| \leq |u(x)|, u_n(x) \to u(x) \text{ a.e. on } \Omega.$$

Then by virtue of hypothesis $\mathbf{H(F)}$(i), for every $n \geq 1$, $x \to F(x, u_n(x))$ is measurable from Ω into $\mathcal{P}_{kc}(\mathbb{R})$. So applying the Kuratowski and Ryll-nardzewski selection theorem (see [20]), we obtain a measurable

$$v_n : \Omega \to \mathbb{R} \text{ such that } v_n(x) \in F(x, u_n(x)) \text{ for } x \in \Omega.$$

From $\mathbf{H(F)}$(iii),

$$|v_n(x)| \leq a(x) + c|u_n(x)|^{\alpha(x)-1} \leq a(x) + c|u(x)|^{\alpha(x)-1}.$$

So, $\{v_n\}_{n\geq 1} \subseteq V^*$ is bounded and thus, we assume that $v_n \rightharpoonup v$ in V^*.

Then from Theorem 3.1 in [24] and $\mathbf{H(F)}$(ii) it follows that

$$v(x) \in \text{conv}\overline{\lim}\{v_n(x)\}_{n\geq 1} \subseteq \text{conv}\overline{\lim}F(x, u_n(x)) \subseteq F(x, u(x)) \text{ a.e. on } \Omega.$$

Since $v \in V^*, v \in N_F(u)$ and this proves that N_F has nonempty values.

Now we prove the upper semicontinuity of N_F from $X \to V^*_w$. For this we need to show that

$$N_F^-(C) = \{u \in X : N_F(u) \cap C \neq \emptyset\}$$

is closed for any weakly closed subset of V^*.

So let $\{u_n\}_{n\geq 1} \subseteq N_F^-(C)$ and assume that $u_n \to u$ in X. Because the embedding $X \hookrightarrow V$ is continuous, we can find $M > 0$ such that

$$|u_n|_{\alpha(x)} \leq M \text{ for all } n \geq 1.$$

Let $v_n \in N_F(u_n) \cap C$, then by $\mathbf{H(F)}$(iii) we have

$$|v_n(x)| \leq a(x) + c|u_n(x)|^{\alpha(x)-1} \text{ a.e. on } \Omega,$$

and evidently $\{v_n\}_{n\geq 1} \subseteq V^*$ is bounded. Hence, we can assume that $v_n \rightharpoonup v$ in V^*. As above we can easily check that $v \in N_F(u)$. Also $v \in C$ and so $v \in N_F(u) \cap C$, i.e., $u \in N_F^-(C)$, which proves the desired upper semicontinuity of N_F. Finally, from $\mathbf{H(F)}$(iii) it follows that N_F is bounded. □

Theorem 3.1. *If hypothesis* $\mathbf{H(F)}$ *holds, then problem* (P) *has at least one weak solution in* X.

Proof. From Lemma 3.1, we have $N_F(\cdot)$ has values in $P_{wkc}(V^*)$ and is usc into V^*_w. Then problem (P) is equivalent to the following abstract fixed point problem:

$$u \in (-\Delta_{p(x)})^{-1} N_F(u).$$

By virtue of $\mathbf{H}(\mathbf{F})$(iii) and $(-\Delta_{p(x)})^{-1} : V^* \to X$ is completely continuous, the multifunction $u \mapsto (-\Delta_{p(x)})^{-1} N_F(u)$ is compact.
Claim:

$$S = \{u \in X : u \in \lambda(-\Delta_{p(x)})^{-1} N_F(u) \text{ for some } \lambda \in (0,1)\}$$

is bounded.

Let $u \in S$. We have

$$\begin{aligned} & -\Delta_{p(x)}\left(\frac{u}{\lambda}\right) \in N_F(u) \\ \Rightarrow & -\Delta_{p(x)}\left(\frac{u}{\lambda}\right) = v \ \text{ with } v \in N_F(u) \\ \Rightarrow & \left\langle -\Delta_{p(x)}\left(\frac{u}{\lambda}\right), \frac{u}{\lambda}\right\rangle_{X^*X} = \left\langle v, \frac{u}{\lambda}\right\rangle_{V^*V^*}, \end{aligned} \tag{3.1}$$

since embedding $X \hookrightarrow V$ is continuous. Thus,

$$\left\langle -\Delta_{p(x)}\left(\frac{u}{\lambda}\right), \frac{u}{\lambda}\right\rangle_{X^*X} = \int_\Omega \left|\nabla\left(\frac{u(x)}{\lambda}\right)\right|^{p(x)} dx \geq \frac{1}{\lambda^{p^-}} \int_\Omega |\nabla u(x)|^{p(x)} dx. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned} \int_\Omega |\nabla u|^{p(x)} dx \leq & \lambda^{p^-} \langle v, u\rangle_{X^*X} \\ \leq & \lambda^{p^-} |v|_{\alpha'(x)} |u|_{\alpha(x)} \\ \leq & \lambda^{p^-} c_0 |v|_{\alpha'(x)} \|u\| \\ \leq & c_0 |v|_{\alpha'(x)} \|u\|. \end{aligned} \tag{3.3}$$

On the other hand, from $\mathbf{H}(\mathbf{F})$(iii) we have

$$\begin{aligned} |v|_{\alpha'(x)} \leq & |a(x) + c|u|^{\alpha(x)-1}|_{\alpha'(x)} \\ \leq & |a|_{\alpha'(x)} + c||u|^{\alpha(x)-1}|_{\alpha'(x)}. \end{aligned} \tag{3.4}$$

Let us prove that

$$||u|^{\alpha(x)-1}|_{\alpha'(x)} \leq |u|^{\alpha^+-1}_{\alpha(x)} + 2. \tag{3.5}$$

Indeed, one has:

$$\text{(a) If } |u|_{\alpha(x)} \geq 1, \text{ then } ||u|^{\alpha(x)-1}|_{\alpha'(x)} \leq |u|^{\alpha^+-1}_{\alpha(x)}.$$

This is seen as follows: According to Lemma 2.2, to prove (a), it is equivalent to prove that $|u|_{\alpha(x)} \geq 1$ implies

$$\int_\Omega \frac{|u(x)|^{(\alpha(x)-1)\alpha'(x)}}{|u|^{(\alpha^+-1)\alpha'(x)}_{\alpha(x)}} = \int_\Omega \frac{|u(x)|^{\alpha(x)}}{|u|^{(\alpha^+-1)\alpha'(x)}_{\alpha(x)}} \leq 1.$$

This inequality is justified as follows. Since $|u|_{\alpha(x)} \geq 1$ and

$$\begin{aligned}(\alpha^+ - 1)\alpha'(x) - \alpha(x) &= \alpha^+\alpha'(x) - (\alpha(x) + \alpha'(x)) \\ &= \alpha^+\alpha'(x) - \alpha(x)\alpha'(x) \\ &= \alpha'(x)(\alpha^+ - \alpha(x)) \\ &\geq 0,\end{aligned}$$

we infer that

$$\frac{|u(x)|^{\alpha(x)}}{|u|_{\alpha(x)}^{(\alpha^+-1)\alpha'(x)}} = \frac{|u(x)|^{\alpha(x)}}{|u|_{\alpha(x)}^{\alpha(x)}} \frac{1}{|u|_{\alpha(x)}^{(\alpha^+-1)\alpha'(x)-\alpha(x)}} \leq \frac{|u(x)|^{\alpha(x)}}{|u|_{\alpha(x)}^{\alpha(x)}},$$

which implies that

$$\int_\Omega \frac{|u(x)|^{(\alpha(x)-1)\alpha'(x)}}{|u|_{\alpha(x)}^{(\alpha^+-1)\alpha'(x)}} \leq \int_\Omega \frac{|u(x)|^{\alpha(x)}}{|u|_{\alpha(x)}^{\alpha(x)}} = 1,$$

and the prove of (a) is complete.

(b) If $|u|_{\alpha(x)} < 1$, then $||u|^{\alpha(x)-1}|_{\alpha'(x)} < 2$.

Indeed, by $|u|_{\alpha(x)} < \int_\Omega |u(x)|^{\alpha(x)} dx + 1$ and Lemma 2.2 (3), one has:

$$||u|^{\alpha(x)-1}|_{\alpha'(x)} < \int_\Omega |u(x)|^{(\alpha(x)-1)\alpha'(x)} + 1 = \int_\Omega |u(x)|^{\alpha(x)} + 1 < 1 + 1 = 2.$$

Clearly, (3.5) is a consequence of (a) and (b).
Hence, by (3.4) and (3.5), we have

$$|v|_{\alpha'(x)} \leq c|u|_{\alpha(x)}^{\alpha^+-1} + 2c + |a|_{\alpha'(x)}. \tag{3.6}$$

From (3.3) and (3.6), we have

$$\int_\Omega |\nabla u(x)|^{p(x)} dx \leq c_0[c|u|_{\alpha(x)}^{\alpha^+-1} + 2c + |a|_{\alpha'(x)}]\|u\|. \tag{3.7}$$

Since $X \hookrightarrow V$ is a compact embedding, so there exists a $c_1 > 0$ such that $|u|_{\alpha(x)} \leq c_1\|u\|$. Therefore

$$\int_\Omega |\nabla u|^{p(x)} dx \leq c_0[c_2\|u\|^{\alpha^+-1} + c_3]\|u\|,$$

for positive constants c_2, c_3.
Without loss of generality, we may assume that $\|u\| = |\nabla u|_{p(x)} > 1$, otherwise, S is bounded set. Thus

$$\int_\Omega |\nabla u(x)|^{p(x)} dx \geq \|u\|^{p^-},$$

that is

$$\|u\|^{p^- -1} \leq c_0c_2\|u\|^{\alpha^+-1} + c_0c_3,$$

so $\|u\|$ is bounded (since $\alpha^+ < p^-$). This proves that $S \subseteq W_0^{1,p(x)}(\Omega)$ is bounded. Employing the Leray-Schauder alternative principle, we know that $u_0 \in W_0^{1,p(x)}(\Omega)$, such that

$$u_0 \in (-\Delta_{p(x)})^{-1} N_F(u_0),$$

that is, u_0 is a weak solution of problem (P). □

Acknowledgments. This paper is supported by the NNSF of China (No. 11201095), the Youth Scholar Backbone Supporting Plan Project of Harbin Engineering University, the Fundamental Research Funds for the Central Universities(No. 2016), Postdoctoral research startup foundation of Heilongjiang (No. LBH-Q14044).

References

[1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, *Periodic solution for second differential inclusions with the scalar p-Laplacian*, J. Math. Anal. Appl, **322**(2006), 913–929.

[2] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces,* Noordhoff, Leyden, 1976.

[3] R. Bader, *A topological fixed point index theory for evolution inclusions,* University of Munich, Preprint, 1999.

[4] R. Bader, N.S. Papageorgiou, *Nonlinear boundary value problems for second order differential inclusions,* Math. Nachr, **244**(2002), 5–25.

[5] K.C. Chang, *Critical Point Theory and Applications*, Shanghai Scientific and Technology Press, Shanghai, 1996.

[6] G. Dinca, *A fixed point method for the* $p(\cdot)$*-Laplacian,* C.R. Acad. Sci. Paris, Ser. I, **347**(2009), 757–762.

[7] N. Dugundji, A. Granas, *Fixed Point Theory,* Polish Sci. Publ., Warsaw, Poland, 1982.

[8] X.L. Fan, *Solutions for* $p(x)$*-Laplacian Dirichlet problems with singular coefficients,* J. Math. Anal. Appl, **312**(2005), 464–477.

[9] X.L. Fan, *Eigenvalues of the* $p(x)$*-Laplacian Neumann problems,* Nonlinear Anal., **67**(2007), 2982–2992.

[10] X.L. Fan, X.Y. Han, *Existence and multiplicity of solutions for* $p(x)$*-Laplacian equations in* $\mathbb{R}^N$. Nonlinear Anal., **59**(2004), 173–188.

[11] X.L. Fan, D. Zhao, *On the generalize Orlicz-Sobolev space* $W^{k,p(x)}(\Omega)$*,* J. Gansu Educ. College, **12**(1998), 1–6.

[12] X.L. Fan, D. Zhao, *On the space* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$*,* J. Math. Anal. Appl, **263**(2001), 424–446.

[13] X.L. Fan, Q.H. Zhang, *Existence of solutions for* $p(x)$*-Laplacian Dirichlet problem,* Nonlinear Anal., **52**(2003), 843–1852.

[14] Y.Q. Fu, X. Zhang, *A multiplicity result for* $p(x)$*-Laplacian problem in* $\mathbb{R}^N$*,* Nonlinear Anal., **70**(2009), 2261–2269.

[15] B. Ge, Q.M. Zhou, X.P. Xue, *Infinitely many solutions for a differential inclusion problem in RN involving* $p(x)$*-Laplacian and oscillatory terms,* Z. Angew. Math. Phys., **63**(2012), 691–711.

[16] B. Ge, X.P. Xue, Q.M. Zhou, *The existence of radial solutions for differential inclusion problems in* $\mathbb{R}^N$ *involving the* $p(x)$*-Laplacian,* Nonlinear Anal., **73**(2010), 622–633.

[17] B. Ge, X.P. Xue, *Multiple solutions for inequality Dirichlet problems by the* $p(x)$*-Laplacian,* Nonlinear Anal. Real World Appl., **11**(2010), 3198–3210.

[18] S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis. Volume I: Theory,* Kluwer, Dordrecht, The Netherlands, 1997.

[19] S. Kyritsi, N. Matzakos, N.S. Papageorgiou, *Periodic problems for strongly nonlinear second-order differential inclusions,* J. Diff. Eq., **182**(2002), 279–302.

[20] K. Kuratowski, C. Ryll Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci, **13**(1965), 397–403.

[21] R. Mansevich, J. Mawhin, *Periodic solutions for nonlinear systems with p-Laplacian-like operators,* J. Diff. Eq., **145**(1998), 367–393.

[22] F. Papalini, *Solvability of strongly nonlinear boundary value problems for second order differential inclusions,* Nonlinear Anal., **66**(2007), 2166–2189.

[23] M.D. Pino, M. Elgueta, R. Manasevich, *A homotopic deformation along p of a leray-schauder degree result and existence for* $(|u'|^{p-2})' + f(x,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, J. Differential Equations, **80**(1989), 1–13.

[24] N.S. Papageorgiou, *Convergence theorem for Banach space valued integrable multifunctions,* Intern. J. Math. and Math. Sci, **10**(1987), 433–422.

[25] N.S. Papageorgiou, V. Staicu, *The method of upper-lower solutions for second order differential inclusions,* Nonlinear Anal., **67**(2007), 708–726.

[26] C.Y. Qian, Z.F. Shen, *Existence and multiplicity of solutions for* $p(x)$*-Laplacian equation with nonsmooth potential,* Nonlinear Anal. Real World Appl., **11**(2010), 106–116.

[27] Z.F. Shen, C.Y. Qian, M.B. Yang, *Existence of solutions for* $p(x)$*-Laplacian nonhomogeneous Neumann problems with indefinite weight,* Nonlinear Anal. Real World Appl., **11**(2010), 446–458.

[28] M.R. Zhang, *Nonuniform nonresonance of semilinear differential equations,* J. Diff. Eq., **166**(2000), 33–50.

[29] X. Zhang, Y.Q. Fu, *Bifurcation results for a class of* $p(x)$*-Laplacian equations,* Nonlinear Anal., **73**(2010), 3641–3650.

[30] Q. Zhang, G. Li, *Nonlinear boundary value problems for second order differential inclusions,* Nonlinear Anal., **70**(2009), 3390–3406.

Received: January 17, 2013; Accepted: May 19, 2014.