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STRONG CONVERGENCE THEOREMS BY HYBRID METHOD FOR SEMIGROUPS OF NOT NECESSARILY CONTINUOUS MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we establish a strong convergence theorem by the shrinking hybrid method for semigroups of not necessarily continuous mappings in Banach spaces. Using the result, we obtain well-known and new strong convergence theorems which are connected with results by the shrinking hybrid method in Hilbert spaces and Banach spaces.
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1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively and let C be a non-empty, closed and convex subset of H. Let T be a mapping of C into H. Then we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 2003, Nakajo and Takahashi [27] proved a strong convergence theorem for nonexpansive mappings in a Hilbert space by using the hybrid method in mathematical programming: Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let

$\{\alpha_n\}$ be a real sequence in $[0,1)$ such that $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$\begin{cases} u_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|u_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. Then $\{x_n\}$ converges strongly to $z \in F(T)$. Takahashi, Takeuchi and Kubota [34] proposed a hybrid method called the shrinking projection method which is different from Nakajo and Takahashi [27] and they obtained a strong convergence theorem for nonexpansive mappings in a Hilbert space. On the other hand, Kocourek, Takahashi and Yao [19] defined a class of nonlinear mappings called generalized hybrid mappings which contains nonexpansive mappings, nonspreading mappings [22, 23] and hybrid mappings [33] in a Hilbert space and they proved a fixed point theorem and a mean convergence theorem of Baillon's type iteration [5] for the mappings. They also extended generalized hybrid mappings to Banach spaces; see [20]. They called such mappings generalized nonspreading mappings and proved fixed point theorems and weak convergence theorems of Baillon's type iteration [5] and Mann's type iteration [25] for the mappings. Recently, using the concept of means and invariant means, Takahashi, Wong and Yao [35] introduced the concept of semigroups of not necessarily continuous mappings in a Hilbert space which contains discrete semigroups generated by generalized hybrid mappings and semigroups of nonexpansive mappings. They proved a fixed point theorem and a mean convergence theorem which generalized simultaneously the result of [19] for generalized hybrid mappings and the result of Baillon and Brezis [6] of one-parameter nonexpansive semigroups in a Hilbert space. They extended such results to Banach spaces; see [36]. Motivated by [35], Hussain and Takahashi [11] tried to prove a weak convergence theorem of Mann's type iteration and a strong convergence theorem of Halpern's type iteration [8] in a Hilbert space. To prove these two theorems, they introduced the concept of semigroups of not necessarily continuous mappings in a Hilbert space which is defined by using strongly asymptotically invariant nets. Alsulami, Hussain and Takahashi [3] extended such semigroups of mappings to Banach spaces and proved a weak convergence theorem of Mann's type iteration. However, they do not prove any strong convergence theorems in Banach spaces. It is natural to prove strong convergence theorems in Banch spaces for the semigroups defined by [3].

In this paper, we establish a strong convergence theorem by the shrinking hybrid method for semigroups of not necessarily continuous mappings in Banach spaces. Using the result, we obtain well-known and new strong convergence theorems which are connected with results by the shrinking hybrid method in Hilbert spaces and Banach spaces.

2. Preliminaries

Let E be a real Banach space with the norm $\|\cdot\|$ and let E^* be the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of the convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a non-empty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the *metric projection* of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called *smooth.* We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. The norm of E is said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. If the norm of E is Fréchet differentiable, then J is norm to norm continuous. For more details, see [31]. We also know the following result for duality mappings.

Lemma 2.1. *Let E be a smooth Banach space and let J be the duality mapping on E. Then $\langle x - y, Jx - Jy \rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let C be a non-empty subset of E and let T be a mapping of C into E. We denote the set of all fixed points of T by $F(T)$. A mapping $T\colon C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Let D be a subset of E and let P be a mapping of E onto D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

for $x \in E$ and $t \ge 0$. A mapping P of E onto C is said to be a *retraction* if $P^2 = P$. We denote the closure of the convex hull of D by $\overline{\mathrm{co}}D$.

Let E be a smooth Banach space. The function $\phi\colon E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see Alber [1] and Kamimura and Takahashi [18]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

for all $x, y, z \in E$. Furthermore, we have that

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.3}$$

for all $x, y, z, w \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

Assume that a Banach space E is smooth, strictly convex and reflexive. Let $\phi_* \colon E^* \times E^* \to \mathbb{R}$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.5}$$

for $x, y \in E$. The following results are well known; see [18].

Lemma 2.2 ([18])**.** *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq 1\}$.

Lemma 2.3 ([18])**.** *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a non-empty subset of a smooth Banach space E. A mapping $T : C \to E$ is called *generalized nonexpansive* [13] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y),\ \forall (x, y) \in C \times F(T).$$

If C is a non-empty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, then for all $x \in E$ there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x). \tag{2.6}$$

The mapping Π_C is called the *generalized projection* from E onto C; see [1], [2], and [18]. Kohsaka and Takahashi [21] proved the following results.

Lemma 2.4 ([21])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C_* be a non-empty, closed and convex subset of E^*. Let Π_{C_*} be the generalized projection of E^* onto C_*. Then a mapping R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C_*$.*

Lemma 2.5 ([21])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let D be a non-empty subset of E. Then, the following are equivalent:*

(1) *D is a sunny generalized nonexpansive retract of E;*
(2) *D is a generalized nonexpansive retract of E;*
(3) *JD is closed and convex.*

In this case, D is closed.

Lemma 2.6 ([21]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a non-empty and closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times D$. Then the following are equivalent:*

(1) *$z = Rx$;*
(2) *$\phi(x, z) = \min_{y \in D} \phi(x, y)$.*

Ibaraki and Takahashi [13] also proved the following results.

Lemma 2.7 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a non-empty and closed subset of E. Then, a sunny generalized nonexpansive retraction of E onto D is uniquely determined.*

Lemma 2.8 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a non-empty and closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction R of E onto D and let $(x, z) \in E \times D$. Then the following hold:*

(1) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0, \ \forall y \in D$;*
(2) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

For a sequence $\{C_n\}$ of non-empty, closed and convex subsets of a reflexive Banach space E, define s-$\mathrm{Li}_n C_n$ and w-$\mathrm{Ls}_n C_n$ as follows: $x \in$s-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-$\mathrm{Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies that

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.7}$$

we say that $\{C_n\}$ converges to C_0 in the sense of Mosco [26] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [26]. We know the following theorem which was proved by Ibaraki, Kimura and Takahashi [12].

Theorem 2.9 ([12]). *Let E be a smooth Banach space and let E^* have a Fréchet differentiable norm. Let $\{C_n\}$ be a sequence of non-empty, closed and convex subsets of E. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and non-empty, then for each $x \in E$, $\Pi_{C_n} x$ converges strongly to $\Pi_{C_0} x$, where Π_{C_n} and Π_{C_0} are the generalized projections of E onto C_n and C_0, respectively.*

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s \in S} f(s) \le \mu(f) \le \sup_{s \in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. If $S = \mathbb{N}$, an invariant mean on $C(S) = B(S)$ is a Banach limit on ℓ^∞. The following theorem is in [31, Theorem 1.4.5].

Theorem 2.10 ([31]). *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let E be a Banach space and let C be a non-empty subset of E. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of $T_s,\ s \in S$, i.e.,

$$F(\mathcal{S}) = \cap\{F(T_s) : s \in S\}.$$

The following definition [30] is crucial in the nonlinear ergodic theory of abstract semigroups; see also [9]. Let E be a reflexive Banach space and let E^* be the dual space of E. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s\langle u(s), y^*\rangle = \langle z_0, y^*\rangle, \quad \forall y^* \in E^*. \tag{2.8}$$

We call such z_0 the *mean vector* of u for μ. In particular, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Putting $u(s) = T_s x$ for all $s \in S$, we have that there exists $z_0 \in E$ such that

$$\mu_s\langle T_s x, y^*\rangle = \langle z_0, y^*\rangle, \quad \forall^* y \in E^*.$$

We denote such z_0 by $T_\mu x$. A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be *strongly asymptotically invariant* if for each $s \in S$,

$$\|\ell_s^*\mu_\alpha - \mu_\alpha\| \to 0 \quad \text{and} \quad \|r_s^*\mu_\alpha - \mu_\alpha\| \to 0,$$

where ℓ_s^* and r_s^* are the adjoint operators of ℓ_s and r_s, respectively. See [7] and [31] for more details.

Let E be a smooth Banach space and let C be a non-empty subset of E. For a mapping T from C into C, we denote by $A(T)$ the set of *attractive points* [34, 24] of T, that is,

$$A(T) = \{u \in E : \phi(u, Tx) \leq \phi(u, x),\ \forall x \in C\}.$$

We know from Lin and Takahashi [24] that $A(T)$ is always closed and convex. Let S be a commutative semitopological semigroup with identity. For a continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings of C into itself, we denote the set $A(\mathcal{S})$ of *common attractive points* [4, 36] of $\mathcal{S} = \{T_s : s \in S\}$ by

$$A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}.$$

It is obvious from Lin and Takahashi [24] that $A(\mathcal{S})$ is closed and convex. Using the technique developed by Takahashi [30], Takahashi, Wong and Yao [36] proved the following attractive point theorem for a family of mappings in a Banach space.

Theorem 2.11 ([36])**.** *Let E be a smooth and reflexive Banach space with the duality mapping J and let C be a non-empty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y)$$

for all $y \in C$ and $t \in S$. Then $A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}$ is non-empty. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is non-empty.

Let E be a smooth Banach space and let C be a non-empty subset of E. Let T be a mapping from C into C. We denote by $B(T)$ the set of *skew-attractive points* [24] of T, i.e.,

$$B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z),\ \forall x \in C\}.$$

Lin and Takahashi [24] proved that $B(T)$ is always closed. Using the duality theory of nonlinear mappings [37] and [10], they also proved that $JB(T)$ is closed and convex. We can also define by $B(\mathcal{S})$ the set of all *common skew-attractive points* of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself, i.e., $B(\mathcal{S}) = \cap\{B(T_s) : s \in S\}$. Takahashi, Wong and Yao [36] obtained the following skew-attractive point theorem.

Theorem 2.12 ([36])**.** *Let E be a strictly convex and reflexive Banach space with a Fréchet differentiable norm and let C be a non-empty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \phi(T_t y, T_s x) \leq \mu_s \phi(y, T_s x)$$

for all $y \in C$ and $t \in S$. Then, $B(\mathcal{S}) = \cap\{B(T_t) : t \in S\}$ is non-empty. In particular, if C is closed and and JC is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is non-empty.

3. Strong convergence theorem

In this section, using the shrinking hybrid method due to [34], we prove a strong convergence theorem for semigroups of not necessarily continuous mappings in a Banach space. For proving the result, we need the following lemma which was obtained by Alsulami, Hussain and Takahashi [3].

Lemma 3.1 ([3]). *Let E be a smooth and reflexive Banach space and let C be a non-empty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $B(\mathcal{S}) \neq \emptyset$. Let μ be a mean on $C(S)$. Then*

$$\phi(T_\mu x, m) \leq \phi(x, m), \quad \forall x \in C,\ m \in B(\mathcal{S}),$$

where $T_\mu x$ is a mean vector of $\{T_s x : s \in S\}$ for μ.

Theorem 3.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a non-empty, closed and convex subset of E such that JC is closed and convex. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$. Suppose that*

$$\limsup_{\nu} \sup_{x,y \in D} (\mu_\nu)_s(\phi(T_s x, T_t y) - \phi(T_s x, y)) \leq 0, \quad \forall t \in S \tag{3.1}$$

for every strongly asymptotically invariant net $\{\mu_\nu\}$ of means on $C(S)$ and every bounded subset D of C. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}, and $\{\alpha_n\} \subset [0, 1)$ is a sequence such that $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 = R_{F(\mathcal{S})} x$, where $R_{F(\mathcal{S})}$ is the sunny generalized nonexpansive retraction of E onto $F(\mathcal{S})$.

Proof. From $B(\mathcal{S}) \neq \emptyset$ we have that $\{T_s x\}$ is bounded for all $x \in C$. Since S is commutative, we have from Theorem 2.10 that there exists an invariant mean μ on $C(S)$. Put $\mu_\nu = \mu$ in (3.1). Then we have that

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y), \quad \forall x, y \in C,\ t \in S. \tag{3.2}$$

We have from Theorem 2.11 that $F(\mathcal{S})$ is non-empty. We also have from (3.2) and $A(\mathcal{S}) = B(\mathcal{S})$ that

$$F(\mathcal{S}) = C \cap A(\mathcal{S}) \subset B(\mathcal{S}). \tag{3.3}$$

Using that JC is closed and convex, we get that $F(\mathcal{S})$ is closed and $JF(\mathcal{S})$ is closed and convex. In fact, we first show that $JF(\mathcal{S})$ is closed. Let $\{x_n^*\} \subset JF(\mathcal{S})$ such that

$x_n^* \to x^*$ for some $x^* \in E^*$. Since JC is closed, we have $x^* \in JC$. Since E is smooth, strictly convex and reflexive, $J : E \to E^*$ is one-to-one and onto. Then there exist $x \in C$ and $\{x_n\} \subset F(\mathcal{S})$ such that $x^* = Jx$ and $x_n^* = Jx_n$ for all $n \in \mathbb{N}$. From (3.3) we have that for any $t \in S$

$$\begin{aligned}
\phi(T_t x, x) &= \|T_t x\|^2 - 2\langle T_t x, Jx\rangle + \|Jx\|^2 \\
&= \|T_t x\|^2 - 2\langle T_t x, x^*\rangle + \|x^*\|^2 \\
&= \lim_{n\to\infty} (\|T_t x\|^2 - 2\langle T_t x, x_n^*\rangle + \|x_n^*\|^2) \\
&= \lim_{n\to\infty} (\|T_t x\|^2 - 2\langle T_t x, Jx_n\rangle + \|Jx_n\|^2) \\
&= \lim_{n\to\infty} \phi(T_t x, x_n) \\
&\le \lim_{n\to\infty} \phi(x, x_n) \\
&= \lim_{n\to\infty} (\|x\|^2 - 2\langle x, x_n^*\rangle + \|x_n^*\|^2) \\
&= \lim_{n\to\infty} (\|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2) \\
&= \|x\|^2 - 2\langle x, Jx\rangle + \|x\|^2 \\
&= \phi(x, x) = 0.
\end{aligned}$$

Thus we have $\phi(T_t x, x) = 0$. Since E is strictly convex, we have $x = T_t x$. This implies that $x^* = Jx \in JF(\mathcal{S})$. We next show that $JF(\mathcal{S})$ is convex. Let $x^*, y^* \in JF(\mathcal{S})$ and let $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$. Then we have $x, y \in F(\mathcal{S})$ such that $x^* = Jx$ and $y^* = Jy$. From $x, y \in F(\mathcal{S})$, we have that for any $t \in S$

$$\begin{aligned}
&\phi(T_t J^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
&\quad = \|T_t J^{-1}(\alpha Jx + \beta Jy)\|^2 - 2\langle T_t J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&\qquad + \|J^{-1}(\alpha Jx + \beta Jy)\|^2 + \alpha\|x\|^2 + \beta\|y\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\quad = \alpha\phi(T_t J^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(T_t J^{-1}(\alpha Jx + \beta Jy), y) \\
&\qquad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\quad \le \alpha\phi(J^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(J^{-1}(\alpha Jx + \beta Jy), y) \\
&\qquad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\quad = \alpha\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jx\rangle + \|x\|^2\big\} \\
&\qquad + \beta\big\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jy\rangle + \|y\|^2\big\} \\
&\qquad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\quad = 2\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&\quad = 2\|\alpha Jx + \beta Jy\|^2 - 2\|\alpha Jx + \beta Jy\|^2 = 0.
\end{aligned}$$

Then we have $T_t J^{-1}(\alpha Jx + \beta Jy) = J^{-1}(\alpha Jx + \beta Jy)$ and hence $J^{-1}(\alpha Jx + \beta Jy) \in F(\mathcal{S})$. This implies that $\alpha Jx + \beta Jy \in JF(\mathcal{S})$. Therefore $JF(\mathcal{S})$ is convex. Since $JF(\mathcal{S})$ is closed and convex, $JF(\mathcal{S})$ is weakly closed. Furthermore, since J is norm to weak continuous, $F(\mathcal{S})$ is closed. From Lemma 2.5 we get that $F(\mathcal{S})$ is a sunny generalized nonexpansive retract of E.

We show that JC_n is closed and convex and $F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from the assumption that $JC_1 = JC$ is closed and convex and $F(\mathcal{S}) \subset C_1$. Suppose that JC_k is closed and convex and $F(\mathcal{S}) \subset C_k$ for some $k \in \mathbb{N}$. By the definition of ϕ, we know that for $z \in C_k$

$$\begin{aligned} &\phi(y_k, z) \le \phi(x_k, z) \\ \iff &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, Jz\rangle \le 0. \end{aligned}$$

Then JC_{k+1} is closed and convex. If $z \in F(\mathcal{S}) \subset C_k$, then we have from (3.3) and Lemma 3.1 that

$$\phi(T_{\mu_k} y, z) \le \phi(y, z), \quad \forall y \in C. \tag{3.4}$$

Using (3.4), we have that

$$\begin{aligned} \phi(y_k, z) &= \phi(\alpha_k x_k + (1-\alpha_k) T_{\mu_k} x_k, z) \\ &\le \alpha_k \phi(x_k, z) + (1-\alpha_k)\phi(T_{\mu_k} x_k, z) \\ &\le \alpha_k \phi(x_k, z) + (1-\alpha_k)\phi(x_k, z) \\ &= \phi(x_k, z). \end{aligned} \tag{3.5}$$

Hence we have $z \in C_{k+1}$. By induction, we have that JC_n are closed and convex, and $F(\mathcal{S}) \subset C_n$ for all $n \in \mathbb{N}$. Since JC_n is closed and convex, from Lemma 2.5 there exists a unique sunny generalized nonexpansive retraction R_{C_n} of E onto C_n. We also know from Lemma 2.4 that such R_{C_n} is denoted by $J^{-1}\Pi_{JC_n}J$, where J is the duality mapping and Π_{JC_n} is the generalized projection of E onto JC_n. Thus $\{x_n\}$ is well-defined.

Since $\{JC_n\}$ is a nonincreasing sequence of non-empty, closed and convex subsets of E^* with respect to inclusion, it follows that

$$\emptyset \ne JF(\mathcal{S}) \subset \text{M-}\lim_{n\to\infty} JC_n = \cap_{n=1}^{\infty} JC_n. \tag{3.6}$$

Put $C_0^* = \cap_{n=1}^{\infty} JC_n$. Then by Theorem 2.9 we have that $\{\Pi_{JC_{n+1}}Jx\}$ converges strongly to $x_0^* = \Pi_{C_0^*}Jx$. Since E^* has a Fréchet differentiable norm, J^{-1} is continuous. Then we have

$$x_{n+1} = J^{-1}\Pi_{JC_{n+1}}Jx \to J^{-1}x_0^*.$$

To complete the proof, it is sufficient to show that $J^{-1}x_0^* = R_{F(\mathcal{S})}x$. Since $x_n = R_{C_n}x$ and $x_{n+1} = R_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have from Lemma 2.8 and (2.3) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, Jx_n - Jx_{n+1}\rangle \\ &= \phi(x, x_{n+1}) - \phi(x, x_n) - \phi(x_n, x_{n+1}) \\ &\le \phi(x, x_{n+1}) - \phi(x, x_n). \end{aligned}$$

Thus we get that

$$\phi(x, x_n) \le \phi(x, x_{n+1}). \tag{3.7}$$

Furthermore, since $x_n = R_{C_n} x$ and $z \in F(\mathcal{S}) \subset C_n$, from Lemma 2.6 we have

$$\phi(x, x_n) \leq \phi(x, z). \tag{3.8}$$

Thus we have that $\lim_{n\to\infty} \phi(x, x_n)$ exists. This implies that $\{x_n\}$ is bounded. Hence $\{y_n\}$ and $\{T_{\mu_n} x_n\}$ are also bounded. Since, from Lemma 2.8,

$$\begin{aligned}\phi(x_n, x_{n+1}) &= \phi(R_{Q_n} x, x_{n+1})\\ &= \phi(x, x_{n+1}) - \phi(x, R_{Q_n} x)\\ &= \phi(x, x_{n+1}) - \phi(x, x_n) \to 0,\end{aligned}$$

we have that

$$\phi(x_n, x_{n+1}) \to 0. \tag{3.9}$$

By $x_{n+1} \in C_{n+1}$, we also have that $\phi(y_n, x_{n+1}) \leq \phi(x_n, x_{n+1})$. This implies that $\phi(y_n, x_{n+1}) \to 0$. Using Lemma 2.3, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0,$$

from which it follows that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.10}$$

Since $\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1 - \alpha_n) T_{\mu_n} x_n\| = (1 - \alpha_n)\|x_n - T_{\mu_n} x_n\|$ and $0 \leq \alpha_n \leq a < 1$, we also have that

$$\|T_{\mu_n} x_n - x_n\| \to 0. \tag{3.11}$$

Since $x_n \to J^{-1} x_0^*$, we have that $T_{\mu_n} x_n \to J^{-1} x_0^*$. We have from (2.3) that for $y \in C$ and $s, t \in S$

$$2\langle T_s x_n - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = \phi(T_s x_n, T_t y) - \phi(T_s x_n, y).$$

Applying μ_n to both sides of the above equality, we have that

$$\begin{aligned}2(\mu_n)_s\langle T_s x_n - T_t y, &Jy - JT_t y\rangle - \phi(T_t y, y)\\ &= (\mu_n)_s\big(\phi(T_s x_n, T_t y) - \phi(T_s x_n, y)\big)\end{aligned}$$

and hence

$$\begin{aligned}2\langle T_{\mu_n} x_n - T_t y, &Jy - JT_t y\rangle - \phi(T_t y, y)\\ &= (\mu_n)_s\big(\phi(T_s x_n, T_t y) - \phi(T_s x_n, y)\big).\end{aligned}$$

Since $T_{\mu_n} x_n \to J^{-1} x_0^*$ and $\limsup_{n\to\infty} (\mu_n)_s\big(\phi(T_s x_n, T_t y) - \phi(T_s x_n, y)\big) \leq 0$, we get that

$$2\langle J^{-1} x_0^* - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) \leq 0.$$

Since $2\langle J^{-1} x_0^* - T_t y, Jy - JT_t y\rangle - \phi(T_t y, y) = \phi(J^{-1} x_0^*, T_t y) - \phi(J^{-1} x_0^*, y)$, we have that

$$\phi(J^{-1} x_0^*, T_t y) \leq \phi(J^{-1} x_0^*, y), \quad y \in C,\ t \in S. \tag{3.12}$$

Putting $y = J^{-1} x_0^*$, we have $J^{-1} x_0^* \in F(T_t)$. Therefore $J^{-1} x_0^* \in F(\mathcal{S})$.
Put $z_0 = R_{F(\mathcal{S})} x$. Since $z_0 = R_{F(\mathcal{S})} x \subset C_{n+1}$ and $x_{n+1} = R_{C_{n+1}} x$, we have that

$$\phi(x, x_{n+1}) \leq \phi(x, z_0). \tag{3.13}$$

Since $x_n \to J^{-1}x_0^*$ and $Jx_n \to x_0^*$, we have that

$$\begin{aligned}\phi(x, J^{-1}x_0^*) &= \|x\|^2 - 2\langle x, x_0^*\rangle + \|J^{-1}x_0^*\|^2 \\ &= \lim_{n\to\infty} \left(\|x\|^2 - 2\langle x, Jx_n\rangle + \|x_n\|^2\right) \\ &= \lim_{n\to\infty} \phi(x, x_n) \\ &\le \phi(x, z_0).\end{aligned}$$

Consequently, we get $R_{F(\mathcal{S})}x = z_0 = J^{-1}x_0^*$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

Remark. We do not know whether a strong convergence theorem under the Nakajo and Takahashi hybrid method [27] for our semigroups of mappings holds or not; see Introduction.

4. Applications

In this section, using Theorem 3.2, we obtain well-known and new strong convergence theorems in Hilbert spaces and Banach spaces.

Let H be a real Hilbert space and let C be a non-empty, closed and convex subset of H. A family $\mathcal{S} = \{T_s : 0 \le s < \infty\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C if it satisfies the following:

(1) $T_0x = x$ for all $x \in C$;
(2) $T_{s+t} = T_sT_t$ for every $s, t \ge 0$;
(3) $\|T_sx - T_sy\| \le \|x - y\|$ for each $s \ge 0$ and $x, y \in C$;
(4) for all $x \in C$, $s \longmapsto T_sx$ is continuous.

Using Theorem 3.2, we first obtain the following result by Takahashi, Takeuchi and Kubota [34].

Theorem 4.1 ([34]). *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H. Let $\mathcal{S} = \{T_s : 0 \le s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T_s x_n \, ds\,, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, $0 \le \alpha_n \le a < 1$ and $\{t_n\} \subset (0,\infty)$ satisfies $t_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Proof. We first have that $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. Put $S = [0, s)$ in Theorem 3.2. Since $\mathcal{S} = \{T_s : 0 \le s < \infty\}$ is a one-parameter nonexpansive semigroup on C, we have that

$$\|T_{t+s}x - T_ty\|^2 \le \|T_sx - y\|^2$$

for all $x, y \in C$ and $s, t \in S$. Furthermore, from the definitions of $A(\mathcal{S})$ and $B(\mathcal{S})$, we have that $A(\mathcal{S}) = B(\mathcal{S}) = F(\mathcal{S}) \neq \emptyset$. Let D be a bounded subset of C. Then we have that for all $x, y \in D$ and $s, t \in S$

$$\begin{aligned}\|T_s x - T_t y\|^2 - \|T_s x - y\|^2 &= \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2 + \|T_{s+t} x - T_t y\|^2 - \|T_s x - y\|^2 \\ &\leq \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2 + \|T_s x - y\|^2 - \|T_s x - y\|^2 \\ &= \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2.\end{aligned}$$

If $\{\mu_\nu\}$ is a strongly asymptotically invariant net of means on $C(S)$, then

$$\begin{aligned}(\mu_\nu)_s(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) &\leq (\mu_\nu)_s(\|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2) \\ &= (\mu_\nu)_s\|T_s x - T_t y\|^2 - (\ell_t^* \mu_\nu)_s\|T_s x - T_t y\|^2 \\ &\leq \|\mu_\alpha - \ell_t^* \mu_\alpha\| \sup_s \|T_s x - T_t y\|^2\end{aligned}$$

and hence

$$\limsup_\nu \sup_{x,y \in D} (\mu_\nu)_s(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) \leq 0$$

for all $t \in S$. Finally, define

$$\mu_n(f) = \frac{1}{t_n} \int_0^{t_n} f(t) dt$$

for all $t_n > 0$ with $t_n \to \infty$ and $f \in C(S)$. As in the proof of [9, Theorem 6], we have that $\{\mu_n\}$ is a strongly asymptotically invariant sequence of means on $C(S)$. Furthermore, we have from [31, Theorem 3.5.2] that for any $u \in C$ and $n \in \mathbb{N}$

$$T_{\mu_n} u = \frac{1}{t_n} \int_0^{t_n} T_s u \, ds.$$

Therefore, we obtain Theorem 4.1 by using Theorem 3.2. □

Let E be a smooth Banach space and let C be a non-empty, closed and convex subset of E. A mapping $T : C \to E$ is called *generalized nonspreading* [20] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Tx, Ty) + (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ \leq \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}\end{aligned} \tag{4.1}$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. We call such a mapping an $(\alpha, \beta, \gamma, \delta)$-*generalized nonspreading* mapping. Let T be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping. Observe that if $F(T) \neq \emptyset$, then $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. Indeed, putting $x = u \in F(T)$ in (4.1), we obtain that

$$\phi(u, Ty) + \gamma\{\phi(Ty, u) - \phi(Ty, u)\} \leq \phi(u, y) + \delta\{\phi(y, u) - \phi(y, u)\}.$$

Then we have that $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$.

Theorem 4.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on ℓ^∞, i.e., a sequence of means on ℓ^∞ such that*

$$\|\mu_n - \ell_1^* \mu_n\| \to 0.$$

Assume that $\{\alpha_n\} \subset [0, 1)$ satisfies $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}. Then $\{x_n\}$ converges strongly to $z_0 = R_{F(T)} x$.

Proof. Put $S = \{0\} \cup \mathbb{N}$ and consider $\mathcal{S} = \{T^k : k \in S\}$. Putting $y = u \in F(T)$ in (4.1), we obtain that for $x \in E$

$$\begin{aligned} &\alpha\phi(Tx, u) + (1 - \alpha)\phi(x, u) + \gamma\{\phi(u, Tx) - \phi(u, x)\} \\ &\qquad \leq \beta\phi(Tx, u) + (1 - \beta)\phi(x, u) + \delta\{\phi(u, Tx) - \phi(u, x)\} \end{aligned}$$

and hence

$$(\alpha - \beta)\{\phi(Tx, u) - \phi(x, u)\} + (\gamma - \delta)\{\phi(u, Tx) - \phi(u, x)\} \leq 0.$$

Therefore, we have that $\alpha > \beta$ together with $\gamma \leq \delta$ implies that

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. We have that $A(T) = A(T) \cap E = F(T)$ and $B(T) = B(T) \cap E = F(T)$. Then we have that $A(T) = B(T) = F(T) \neq \emptyset$. As in the proof of [3, Theorem 5.2], we have that

$$\limsup_{\nu} \sup_{x,y \in D} (\mu_\nu)_k (\phi(T^k x, Ty) - \phi(T^k x, y)) \leq 0$$

for every strongly asymptotically invariant net $\{\mu_\nu\}$ of means on ℓ^∞ and every bounded subset D of E. Therefore we have the desired result from Theorem 3.2 □

Let E be a smooth Banach space and let C be a non-empty subset of E. Let S be a commutative semitopological semigroup with identity. A continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings on C is a *ϕ-nonexpansive semigroup* on C if each T_s is ϕ-nonexpansive, i.e., $\phi(T_s x, T_s y) \leq \phi(x, y)$ for all $x, y \in C$. In the case when E is a Hilbert space, a ϕ-nonexpansive semigroup on C is a nonexpansive semigroup on C; see Atsushiba and Takakashi [4].

Theorem 4.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a non-empty, closed and convex subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$. Let $\{\mu_n\}$ be a strongly* *asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that*

$$\|\mu_n - \ell_s^*\mu_n\| \to 0, \quad \forall s \in S.$$

Assume that $\{\alpha_n\} \subset [0,1)$ satisfies $0 \le \alpha_n \le a < 1$ for some $a \in \mathbb{R}$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}. Then $\{x_n\}$ converges strongly to $z_0 = R_{F(\mathcal{S})}x$, where $R_{F(\mathcal{S})}$ is the sunny generalized non-expansive retraction of E onto $F(\mathcal{S})$.

Proof. Since $\mathcal{S} = \{T_t : t \in S\}$ is a ϕ-nonexpansive semigroup on C, we have that for all $x, y \in C$ and $s, t \in S$

$$\begin{aligned} \phi(T_s x, &T_t y) - \phi(T_s x, y) \\ &= \phi(T_s x, T_t y) - \phi(T_{s+t}x, T_t y) + \phi(T_{s+t}x, T_t y) - \phi(T_s x, y) \\ &\le \phi(T_s x, T_t y) - \phi(T_{s+t}x, T_t y) + \phi(T_s x, y) - \phi(T_s x, y) \\ &= \phi(T_s x, T_t y) - \phi(T_{s+t}x, T_t y). \end{aligned}$$

If $\{\mu_\alpha\}$ is a strongly asymptotically invariant net of means on $C(S)$ and D is a bounded subset of C, then

$$\begin{aligned} (\mu_\alpha)_s&(\phi(T_s x, T_t y) - \phi(T_s x, y)) \\ &\le (\mu_\alpha)_s(\phi(T_s x, T_t y) - \phi(T_{s+t}x, T_t y)) \\ &= (\mu_\alpha)_s\phi(T_s x, T_t y) - (\ell_t^*\mu_\alpha)_s\phi(T_s x, T_t y)) \\ &\le \|\mu_\alpha - \ell_t^*\mu_\alpha\| \sup_s \phi(T_s x, T_t y) \end{aligned}$$

and hence

$$\limsup_\alpha \sup_{x,y \in D} (\mu_\alpha)_s(\phi(T_s x, T_t y) - \phi(T_s x, y)) \le 0$$

for all $t \in S$. Therefore, we have the desired result from Theorem 3.2. □

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