

## FIXED POINT THEOREMS FOR MEIR-KEELER TYPE CONTRACTIONS IN METRIC SPACES

MORTAZA ABTAHI

School of Mathematics and Computer Sciences  
Damghan University  
Damghan, Iran  
E-mail: abtahi@du.ac.ir

**Abstract.** We establish a simple and powerful lemma that provides a criterion for sequences in metric spaces to be Cauchy. Using the lemma, it is then easily verified that the Picard iterates  $\{T^n x\}$ , where  $T$  is a contraction or asymptotic contraction of Meir-Keeler type, are Cauchy sequences. As an application, new and simple proofs for several known results on the existence of a fixed point for continuous and asymptotically regular self-maps of complete metric spaces satisfying a contractive condition of Meir-Keeler type are derived. These results include the remarkable fixed point theorem of Proinov in [Petko D. Proinov, Fixed point theorems in metric spaces, *Nonlinear Anal.* **46** (2006) 546–557], the fixed point theorem of Suzuki for asymptotic contractions in [Tomonari Suzuki, A definitive result on asymptotic contractions, *J. Math. Anal. Appl.* **335** (2007) 707–715], and others. We also prove some new fixed point theorems.

**Key Words and Phrases:** Meir-Keeler contractions, asymptotic contractions, fixed point theorems, complete metric spaces.

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### 1. INTRODUCTION

Metric fixed point theory is a very extensive area of analysis with various applications. Many of the most important nonlinear problems of applied mathematics reduce to finding solutions of nonlinear functional equations which can be formulated in terms of finding the fixed points of a given nonlinear operator of an infinite dimensional function space  $X$  into itself. There is a classical general existence theory of fixed points for mappings satisfying a variety of contractive conditions. The first basic result is the Banach contraction principle.

**Theorem 1.1** (Banach [2]). *Let  $X$  be a complete metric space and  $T : X \rightarrow X$  be a strict contraction, that is, there exists  $\alpha$ ,  $0 < \alpha < 1$ , such that*

$$\forall x, y \in X, \quad d(Tx, Ty) \leq \alpha d(x, y). \quad (1.1)$$

*Then  $T$  has a unique fixed point  $z$ , and  $T^n x \rightarrow z$ , for every  $x \in X$ .*

The Banach contraction principle is fundamental in fixed point theory. It has been extended to some larger classes of contractive mappings by replacing the strict contractive condition (1.1) by weaker conditions of various types; see, for example,

[4, 17, 30, 9, 3, 29, 8, 15, 10, 6, 13, 31, 11]. A comparative study of some of these results have been made by Rhoades [28].

There are thousands of theorems which assure the existence of a fixed point of a self-map  $T$  of a complete metric space  $X$ . These theorems can be categorized into different types, [34]. One type, and perhaps the most common one, is called the Leader-type [20]: the mapping  $T$  has a unique fixed point, and the fixed point can always be found by using Picard iterates  $\{T^n x\}$ , beginning with some initial choice  $x$  in  $X$ . Most of the theorems belong to Leader-type. For instance, Ćirić in [10] defined the class of *quasi-contractions* on a metric space  $X$  consisting of all mappings  $T$  for which there exists  $\alpha$ ,  $0 < \alpha < 1$ , such that  $d(Tx, Ty) \leq \alpha \mathbf{m}(x, y)$ , for every  $x, y \in X$ , where

$$\mathbf{m}(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

The results presented in [10] show that the condition of quasi-contraction implies all conclusions of Banach contraction principle. We remark that Ćirić's quasi-contraction is considered as the most general among contractions listed in [28].

Another interesting generalization of Banach contraction principle was given, in 1969, by Meir and Keeler [24]. They defined *weakly uniformly strict contraction* mappings and proved a fixed point theorem that generalized the fixed point theorem of Boyd and Wong [4] and extended the principle to wider classes of maps than those covered in [28].

**Definiton 1.2.** A mapping  $T$  on a metric space  $X$  is said to be a *Meir-Keeler contraction* (or a *weakly uniformly strict contraction* [24]) if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\forall x, y \in X, \quad \epsilon \leq d(x, y) < \epsilon + \delta \implies d(Tx, Ty) < \epsilon.$$

**Theorem 1.3** (Meir and Keeler [24]). *Let  $X$  be a complete metric space and let  $T$  be a Meir-Keeler contraction on  $X$ . Then  $T$  has a unique fixed point  $z$ , and  $T^n x \rightarrow z$ , for every  $x \in X$ .*

The Meir and Keeler's generalized version of Banach contraction principle initiated a lot of work in this direction and led to some important contribution in metric fixed point theory; see, for example, [21, 25, 27, 16, 7]. The following theorem of Ćirić [11] and Matkowski [19, Theorem 1.5.1] generalizes the above Meir-Keeler fixed point theorem.

**Definiton 1.4.** A mapping  $T$  on a metric space  $X$  is said to be a *Ćirić-Matkowski contraction* if  $d(Tx, Ty) < d(x, y)$  for every  $x, y \in X$  with  $x \neq y$ , and, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\forall x, y \in X, \quad \epsilon < d(x, y) < \epsilon + \delta \implies d(Tx, Ty) \leq \epsilon. \quad (1.2)$$

Obviously, the class of Ćirić-Matkowski contractions contains the class of Meir-Keeler contractions. As it is mentioned in [16, Proposition 1], it is easy to see that condition (1.2) in Definition 1.4 can be replaced by the following:

$$\forall x, y \in X, \quad d(x, y) < \epsilon + \delta \implies d(Tx, Ty) \leq \epsilon.$$

**Theorem 1.5** (Ćirić [11], Matkowski [19]). *Let  $X$  be a complete metric space and let  $T$  be a Ćirić-Matkowski contraction on  $X$ . Then  $T$  has a unique fixed point  $z$ , and  $T^n x \rightarrow z$ , for every  $x \in X$ .*

In 1995, Jachymski [16, Theorem 2] replaced the distance function  $d(x, y)$  in the Ćirić-Matkowski theorem by the following:

$$\mathbf{m}(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}.$$

As in [26], we refer to this result of Jachymski as the Jachymski-Matkowski theorem because it is equivalent to a result of Matkowski [23, Theorem 1].

In 2006, extending Ćirić's quasi-contraction to a very general setting, Proinov [26] obtained the following remarkable fixed point theorem generalizing Jachymski-Matkowski theorem.

**Definiton 1.6.** A self-map  $T$  of a metric space  $X$  is said to be *contractive* [12] if  $d(Tx, Ty) < d(x, y)$ , for all  $x, y \in X$  with  $x \neq y$ ; it is called *asymptotically regular* [5] if  $d(T^n x, T^{n+1} x) \rightarrow 0$ , for each  $x \in X$ .

**Theorem 1.7** (Proinov [26]). *Let  $T$  be a continuous and asymptotically regular self-map of a complete metric space  $X$ . Fix  $\gamma \geq 0$ , and define*

$$\mathbf{m}(x, y) = d(x, y) + \gamma[d(x, Tx) + d(y, Ty)]. \quad (1.3)$$

*Suppose  $d(Tx, Ty) < \mathbf{m}(x, y)$  for all  $x, y \in X$  with  $x \neq y$ , and, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\mathbf{m}(x, y) < \delta + \epsilon$  implies  $d(Tx, Ty) \leq \epsilon$ . Then  $T$  has a unique fixed point  $z$ , and the Picard iterates of  $T$  converge to  $z$ .*

After establishing a technical lemma in section 2, we present, in section 3, a fixed point theorem that generalizes the Proinov's Theorem 1.7. We shall also discuss asymptotic contractions of Meir-Keeler type in section 4. The significance of our results is their simple proofs despite their generality.

**Convention.** Since we are mainly concerned with Picard iterates  $\{T^n x\}_{n=0}^\infty$  of a given self-map  $T$ , it is more convenient to take  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  as the indexing set of all sequences in this paper.

## 2. A TECHNICAL LEMMA

The lemma we present in this section is fundamental in our discussion. It provides a criterion for sequences in metric spaces to be Cauchy. As a result, it can be easily verified that, if  $T$  is a contraction of Meir-Keeler type, then the Picard iterates of  $T$  are Cauchy sequences.

**Lemma 2.1.** *Let  $\{x_n\}$  be a sequence in a metric space. If  $d(x_n, x_{n+1}) \rightarrow 0$ , then the following condition implies that  $\{x_n\}$  is a Cauchy sequence.*

- *For every  $\epsilon > 0$ , there exists a sequence  $\{\nu_n\}$  of nonnegative integers such that, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup d(x_{p_n}, x_{q_n}) \leq \epsilon$ , then, for some  $N$ ,*

$$d(x_{p_n+\nu_n}, x_{q_n+\nu_n}) \leq \epsilon, \quad (n \geq N). \quad (2.1)$$

It should be mentioned that the following proof of the lemma actually stems from the work of Geraghty in [14].

*Proof.* Assume, towards a contradiction, that  $\{x_n\}$  is not a Cauchy sequence. Then, there exists  $\epsilon > 0$  such that

$$\forall k \in \mathbb{N}, \exists p, q \geq k, \quad d(x_p, x_q) > \epsilon. \quad (2.2)$$

For this  $\epsilon$ , let  $\{\nu_n\}$  be the sequence of nonnegative integers given by the assumption. Since  $d(x_n, x_{n+1}) \rightarrow 0$ , there exist positive integers  $k_1 < k_2 < \dots$  such that

$$d(x_\ell, x_{\ell+1}) < \frac{1}{n(\nu_n + 1)}, \quad (\ell \geq k_n).$$

For each  $k_n + \nu_n$ , by (2.2), there exist integers  $s_n$  and  $t_n$  such that  $t_n > s_n \geq k_n + \nu_n$  and  $d(x_{s_n}, x_{t_n}) > \epsilon$ . We let  $t_n$  be the smallest such integer so that  $d(x_{s_n}, x_{t_n-1}) \leq \epsilon$ . Take  $p_n = s_n - \nu_n$  and  $q_n = t_n - \nu_n$ . Then  $q_n > p_n \geq k_n$ , and

$$d(x_{p_n+\nu_n}, x_{q_n+\nu_n}) > \epsilon \quad \text{and} \quad d(x_{p_n+\nu_n}, x_{q_n+\nu_n-1}) \leq \epsilon.$$

Using triangle inequality, we have, for every  $n$ ,

$$d(x_{p_n}, x_{q_n}) \leq \frac{\nu_n}{n(\nu_n + 1)} + d(x_{p_n+\nu_n}, x_{q_n+\nu_n-1}) + \frac{\nu_n + 1}{n(\nu_n + 1)}.$$

This implies that  $\limsup d(x_{p_n}, x_{q_n}) \leq \epsilon$ . Since  $d(x_{p_n+\nu_n}, x_{q_n+\nu_n}) > \epsilon$ , for every  $n$ , we get a contradiction.  $\square$

*Remark.* In the above proof of the lemma, if we apply the triangle inequality for the second time, we get

$$\epsilon < d(x_{p_n+\nu_n}, x_{q_n+\nu_n}) \leq \frac{\nu_n}{n(\nu_n + 1)} + d(x_{p_n}, x_{q_n}) + \frac{\nu_n}{n(\nu_n + 1)}.$$

This implies that  $\epsilon \leq \liminf d(x_{p_n}, x_{q_n})$ . Hence we have  $d(x_{p_n}, x_{q_n}) \rightarrow \epsilon$ .

Using the following theorem, and its successive corollary, we will be able to give very simple proofs for theorems mentioned in section 1.

**Theorem 2.2.** *Let  $X$  be a metric space and let  $\{x_n\}$  be a sequence in  $X$ . Suppose  $\mathbf{m}$  is a nonnegative function on  $X \times X$  such that, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ ,*

$$\limsup_{n \rightarrow \infty} \mathbf{m}(x_{p_n}, x_{q_n}) \leq \limsup_{n \rightarrow \infty} d(x_{p_n}, x_{q_n}). \quad (2.3)$$

*If  $d(x_n, x_{n+1}) \rightarrow 0$ , then the following condition implies that  $\{x_n\}$  is Cauchy.*

- *For every  $\epsilon > 0$ , there exists a sequence  $\{\nu_n\}$  of nonnegative integers such that, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ , then, for some  $N \in \mathbb{N}$ ,*

$$d(x_{p_n+\nu_n}, x_{q_n+\nu_n}) \leq \epsilon, \quad (n \geq N).$$

*Proof.* Let  $\epsilon > 0$  and let  $\{\nu_n\}$  be the sequence of nonnegative integers given by the assumption. Let  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$  be two subsequences with  $\limsup d(x_{p_n}, x_{q_n}) \leq \epsilon$ . Then  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ , and thus (2.1) holds. All conditions in Lemma 2.1 are fulfilled and so the sequence is Cauchy.  $\square$

If  $\{\nu_n\}$  is a constant sequence, e.g.  $\nu_n = \nu$  for all  $n$ , then we get the following result which is of particular importance.

**Corollary 2.3.** *Suppose the function  $\mathbf{m}$  satisfies (2.3) and  $d(x_n, x_{n+1}) \rightarrow 0$ . Then each of the following conditions implies that  $\{x_n\}$  is Cauchy.*

- (i) *For every  $\epsilon > 0$  and for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ , then, for some  $N$ ,*

$$d(x_{p_n}, x_{q_n}) \leq \epsilon, \quad (n \geq N).$$

- (ii) *For every  $\epsilon > 0$  and for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ , then, for some  $N$ ,*

$$d(x_{p_n+1}, x_{q_n+1}) \leq \epsilon, \quad (n \geq N).$$

- (iii) *For every  $\epsilon > 0$ , there exists  $\nu \in \mathbb{N}$  such that, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ , then, for some  $N \in \mathbb{N}$ ,*

$$d(x_{p_n+\nu}, x_{q_n+\nu}) \leq \epsilon, \quad (n \geq N).$$

### 3. FIXED POINT THEOREMS

In this section, using Theorem 2.2, we present our fixed point theorem.

**Lemma 3.1.** *If  $\{x_n\}$  is a sequence in a metric space  $X$  and  $\mathbf{m}$  is a nonnegative function on  $X \times X$ , then the following statements are equivalent:*

- (i) *for every  $\epsilon > 0$ , there exists  $\delta > 0$  and  $N \in \mathbb{N}_0$  such that*

$$\forall p, q \geq N, \quad \mathbf{m}(x_p, x_q) < \epsilon + \delta \implies d(x_{p+1}, x_{q+1}) \leq \epsilon. \quad (3.1)$$

- (ii) *for every  $\epsilon > 0$ , and for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$  then, for some  $N$ ,*

$$d(x_{p_n+1}, x_{q_n+1}) \leq \epsilon, \quad (n \geq N).$$

*Proof.* (i)  $\implies$  (ii): Let  $\epsilon > 0$  and assume, for subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , we have  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ . There exists, by (i), some  $\delta > 0$  and  $N \in \mathbb{N}_0$  such that (3.1) holds. Take  $N_1 \in \mathbb{N}_0$  such that  $\mathbf{m}(x_{p_n}, x_{q_n}) < \epsilon + \delta$  for  $n \geq N_1$ . Therefore, we have  $d(x_{p_n+1}, x_{q_n+1}) \leq \epsilon$ , for  $n > \max\{N, N_1\}$ .

(ii)  $\implies$  (i): Assume, to get a contradiction, that (i) fails to hold. Then there exist  $\epsilon > 0$  and subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$  such that

$$\mathbf{m}(x_{p_n}, x_{q_n}) < \epsilon + \frac{1}{n} \quad \text{and} \quad \epsilon < d(x_{p_n+1}, x_{q_n+1}).$$

This contradicts (ii) because  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ . □

**Definiton 3.2.** Let  $X$  be a metric space and let  $\mathbf{m}$  be a nonnegative function on  $X \times X$ . A sequence  $\{x_n\}$  in  $X$  is said to be  $\mathbf{m}$ -contractive if it satisfies one (and hence all) of the conditions in Lemma 3.1.

The following is a direct consequence of Corollary 2.3 and the above lemma.

**Theorem 3.3.** *Let  $X$  be a metric space,  $\{x_n\}$  be a sequence in  $X$ , and  $\mathbf{m}$  be a nonnegative function on  $X \times X$  satisfying (2.3). If  $\{x_n\}$  is  $\mathbf{m}$ -contractive and  $d(x_n, x_{n+1}) \rightarrow 0$ , then  $\{x_n\}$  is Cauchy.*

**Corollary 3.4.** *Let  $T$  be a self-map of a metric space  $X$ , and  $\mathbf{m}$  be a nonnegative function on  $X \times X$ . Suppose there exists a point  $x \in X$  such that*

(i) *for any  $\epsilon > 0$ , there exist  $\delta > 0$  and  $N \in \mathbb{N}_0$  such that*

$$\forall p, q \geq N, \quad \mathbf{m}(T^p x, T^q x) < \delta + \epsilon \implies d(T^{p+1} x, T^{q+1} x) \leq \epsilon, \quad (3.2)$$

(ii) *condition (2.3) holds for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$  of  $\{T^n x\}$ .*

*If  $d(T^n x, T^{n+1} x) \rightarrow 0$ , then  $\{T^n x\}$  is a Cauchy sequence.*

The requirement that  $d(x_n, x_{n+1}) \rightarrow 0$  is essential in Lemma 2.1 and its subsequent results. It can, however, be replaced by other conditions.

**Proposition 3.5.** *Let  $\{x_n\}$  be a sequence in a metric space  $X$  such that*

$$\begin{cases} d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}), & (n \in \mathbb{N}_0), \\ d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), & (\text{if } x_n \neq x_{n+1}). \end{cases} \quad (3.3)$$

*If  $\mathbf{m}$  satisfies (2.3) and  $\{x_n\}$  is  $\mathbf{m}$ -contractive, then  $d(x_n, x_{n+1}) \rightarrow 0$  and, hence,  $\{x_n\}$  is Cauchy.*

For instance, if  $T : X \rightarrow X$  is contractive then (3.3) holds for  $x_n = T^n x$ .

*Proof.* If  $x_m = x_{m+1}$ , for some  $m$ , then  $x_n = x_{n+1}$  for all  $n \geq m$ , and there is nothing to prove. Assume that  $x_n \neq x_{n+1}$  for all  $n$ . Then  $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ , for every  $n$ , and thus  $d(x_n, x_{n+1}) \downarrow \epsilon$ , for some  $\epsilon \geq 0$ . If  $\epsilon > 0$ , take  $p_n = n$  and  $q_n = n + 1$  and we have, by (2.3),

$$\limsup_{n \rightarrow \infty} \mathbf{m}(x_n, x_{n+1}) \leq \limsup_{n \rightarrow \infty} d(x_n, x_{n+1}) = \epsilon.$$

Therefore,  $d(x_{n+1}, x_{n+2}) \leq \epsilon$  for  $n$  large. This is a contradiction since  $\epsilon < d(x_n, x_{n+1})$  for all  $n$ . So  $\epsilon = 0$  and  $d(x_n, x_{n+1}) \rightarrow 0$ .  $\square$

We are now in a position to state and prove our fixed point theorem.

**Theorem 3.6.** *Let  $T$  be an asymptotically regular self-map of a metric space  $X$ , and  $\mathbf{m}$  be a nonnegative function on  $X \times X$ . Suppose*

(i) *for any  $\epsilon > 0$ , there exist  $\delta > 0$  and  $N \in \mathbb{N}_0$  such that*

$$\forall x, y \in X, \quad \mathbf{m}(T^N x, T^N y) < \delta + \epsilon \implies d(T^{N+1} x, T^{N+1} y) \leq \epsilon, \quad (3.4)$$

(ii) *for every  $x \in X$ , condition (2.3) holds for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$  of the sequence  $\{T^n x\}$ .*

*Then the Picard iterates of  $T$  are Cauchy sequences. Moreover, if  $X$  is complete,  $T$  is continuous, and  $d(T^n x, T^n y) \rightarrow 0$ , for all  $x, y \in X$ , then the Picard iterates converge to a unique fixed point of  $T$ .*

*Proof.* All conditions in Corollary 3.4 are satisfied by every point  $x$  in  $X$ . Hence  $\{T^n x\}$  is Cauchy, for every  $x$ . If  $X$  is complete, there is  $z \in X$  such that  $T^n x \rightarrow z$ . If  $T$  is continuous, then  $Tz = z$ . If  $d(T^n x, T^n y) \rightarrow 0$ , for every  $x, y \in X$ , then  $T$  has at most one fixed point.  $\square$

We remark that, by Corollary 3.4, it is enough to impose condition (3.4) on some orbit  $\{T^n x\}$  as in (3.2) to conclude that  $\{T^n x\}$  is Cauchy.

**Example 3.7.** Let  $T$  be a self-map of  $X$ , and consider the following functions:

$$\mathbf{m}_1(x, y) = \max\{d(x, Tx), d(y, Ty)\}, \tag{Bianchini [3]}$$

$$\mathbf{m}_2(x, y) = [d(x, Ty) + d(y, Tx)]/2, \tag{Chatterjea [8]}$$

$$\mathbf{m}_3(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}, \tag{Maiti and Pal [21]}$$

$$\mathbf{m}_4(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{Ciric [10]}$$

$$\mathbf{m}_5(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}, \tag{Jachymski [16]}$$

$$\mathbf{m}_6(x, y) = d(x, y) + \gamma[d(x, Tx) + d(y, Ty)], \text{ where } \gamma \geq 0 \text{ is fixed.} \tag{Proinov [26]}$$

Choose a point  $x \in X$  and set  $x_n = T^n x, n \in \mathbb{N}_0$ . If  $d(x_n, x_{n+1}) \rightarrow 0$ , then, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , we have

$$\begin{cases} \limsup_{n \rightarrow \infty} \mathbf{m}_1(x_{p_n}, x_{q_n}) = 0, \\ \limsup_{n \rightarrow \infty} \mathbf{m}_2(x_{p_n}, x_{q_n}) \leq \limsup_{n \rightarrow \infty} d(x_{p_n}, x_{q_n}), \\ \limsup_{n \rightarrow \infty} \mathbf{m}_i(x_{p_n}, x_{q_n}) = \limsup_{n \rightarrow \infty} d(x_{p_n}, x_{q_n}), \text{ for } 3 \leq i \leq 6. \end{cases}$$

We now state and prove a generalization of Proinov’s Theorem 1.7. First, a couple of notations: For a subset  $E$  of a metric space  $X$ , denote by  $\text{diam } E$  the diameter of  $E$ . If  $T$  is a self-map of  $X$  and  $x \in X$ , for every positive integer  $s \in \mathbb{N}$ , let  $\mathcal{O}_s(x) = \{T^n x : 0 \leq n \leq s\}$ . For positive integers  $s, t \in \mathbb{N}$  and real numbers  $\alpha, \beta \in [0, \infty)$ , define a function  $\mathbf{m}$  on  $X \times X$  as follows:

$$\mathbf{m}(x, y) = d(x, y) + \alpha \text{diam } \mathcal{O}_s(x) + \beta \text{diam } \mathcal{O}_t(y). \tag{3.5}$$

**Theorem 3.8.** *Let  $X$  be a complete metric space, and  $T$  be a continuous and asymptotically regular self-map of  $X$ . Define  $\mathbf{m}$  by (3.5), and suppose*

- (i)  $d(Tx, Ty) < \mathbf{m}(x, y)$ , for every  $x, y \in X$  with  $x \neq y$ ,
- (ii) for any  $\epsilon > 0$ , there exist  $\delta > 0$  and  $N \in \mathbb{N}_0$  such that

$$\forall x, y \in X, \quad \mathbf{m}(T^N x, T^N y) < \delta + \epsilon \implies d(T^{N+1} x, T^{N+1} y) \leq \epsilon. \tag{3.6}$$

*Then  $T$  has a unique fixed point  $z$ , and the Picard iterates of  $T$  converge to  $z$ .*

*Proof.* First, we prove that  $T$  has at most one fixed point. If  $Tx = x$  and  $Ty = y$  then  $\mathcal{O}_s(x) = \{x\}$  and  $\mathcal{O}_t(y) = \{y\}$  and thus

$$\mathbf{m}(x, y) = d(x, y) = d(Tx, Ty).$$

Condition (i) then implies that  $x = y$ .

Now, choose  $x \in X$  and set  $x_n = T^n x$ ,  $n \in \mathbb{N}_0$ . Then  $d(x_n, x_{n+1}) \rightarrow 0$  since  $T$  is asymptotically regular. It is easy to see that, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , we have

$$\mathbf{m}(x_{p_n}, x_{q_n}) \leq d(x_{p_n}, x_{q_n}) + \alpha \sum_{i=0}^{s-1} d(x_{p_n+i}, x_{p_n+i+1}) + \beta \sum_{j=0}^{t-1} d(x_{q_n+j}, x_{q_n+j+1}).$$

Since  $d(x_n, x_{n+1}) \rightarrow 0$ , we see that (2.3) holds. Hence, by Theorem 3.6, the sequence  $\{T^n x\}$  is Cauchy and, since  $X$  is complete, it converges to some point  $z \in X$ . Since  $T$  is continuous, we have  $Tz = z$ .  $\square$

We next give an example to show that Theorem 3.8 strictly extends Proinov’s Theorem 1.7.

**Example 3.9.** Take  $a_i = i$ , for  $0 \leq i < 4$ , let  $r_0 = 0$ , and  $r_n = 1/n$  for  $n \geq 1$ , and set  $x_{4n+i} = a_i + r_n$ . Let

$$X = \{x_{4n+i} : 0 \leq i < 4, n \geq 0\}.$$

Then, equipped with the Euclidean metric,  $X$  is a complete metric space. Define a mapping  $T : X \rightarrow X$  by  $T(x_\ell) = x_{2\ell}$ . Define  $\mathbf{m}(x, y)$  by setting  $s = t = 1$  and  $\alpha = \beta = 1$  in (3.5), that is,

$$\mathbf{m}(x, y) = d(x, y) + d(x, Tx) + d(y, Ty).$$

(Note that  $\mathbf{m}$  is also obtained from (1.3) by setting  $\gamma = 1$ .)

First, we show that  $T$  satisfies all conditions in Theorem 3.8. Clearly,  $T$  is continuous and, for every  $x_\ell, x_\nu \in X$ , we have

$$d(T^{n+2}x_\ell, T^{n+2}x_\nu) = d(T^n x_{4\ell}, T^n x_{4\nu}) = \left| \frac{1}{2^{n+1}\ell} - \frac{1}{2^{n+1}\nu} \right| \rightarrow 0.$$

It is a matter of calculation to see that  $d(Tx, Ty) < \mathbf{m}(x, y)$ , for all  $x, y \in X$ . The following shows that  $T$  satisfies (3.6) with  $N = 2$ :

$$\begin{aligned} d(T^3x_\ell, T^3x_\nu) &= |x_{8\ell} - x_{8\nu}| = \left| \frac{1}{8\ell} - \frac{1}{8\nu} \right| = \frac{1}{2} \left| \frac{1}{4\ell} - \frac{1}{4\nu} \right| \\ &= \frac{1}{2} |x_{4\ell} - x_{4\nu}| = \frac{1}{2} d(T^2x_\ell, T^2x_\nu). \end{aligned}$$

Next, we show that the following condition (in Proinov’s theorem) is violated:

- for every  $\epsilon > 0$ , there exist  $\delta > 0$  such that

$$\forall x, y \in X, \quad \mathbf{m}(x, y) < \delta + \epsilon \implies d(Tx, Ty) \leq \epsilon.$$

Take  $\epsilon = a_2 - a_0$ . Let  $u_\ell = x_{4\ell}$  and  $v_\ell = x_{4(\ell+1)+1}$ . Then

$$d(Tu_\ell, Tv_\ell) = |x_{4(2\ell)} - x_{4(2\ell+2)+1}| = a_2 - a_0 + r_{2\ell} - r_{2\ell+2} > \epsilon,$$



and

$$\begin{aligned} \mathbf{m}(u_\ell, v_\ell) &= |u_\ell - v_\ell| + |u_\ell - Tu_\ell| + |v_\ell - Tv_\ell| \\ &= |x_{4\ell} - x_{4(\ell+1)+1}| + |x_{4\ell} - x_{4(2\ell)}| + |x_{4(\ell+1)+1} - x_{4(2\ell+2)+2}| \\ &= (a_1 - a_0 + r_\ell - r_{\ell+1}) + (r_\ell - r_{2\ell}) + (a_2 - a_1 + r_{\ell+1} - r_{2\ell+2}) \\ &= a_2 - a_0 + 2r_\ell - r_{2\ell} - r_{2\ell+2}. \end{aligned}$$

If  $\delta_\ell = 2r_\ell - r_{2\ell} - r_{2\ell+2}$ , we have

$$\epsilon < d(Tu_\ell, Tv_\ell) < \mathbf{m}(u_\ell, v_\ell) \leq \epsilon + \delta_\ell.$$

Since  $\delta_\ell \rightarrow 0$  as  $\ell \rightarrow \infty$ , we see that  $T$  does not satisfy (3.6) for  $\epsilon = a_2 - a_0$ .

We conclude this section by showing that the following theorem of Geraghty [14] is a special case of our fixe point theorem.

**Theorem 3.10** (Geraghty [14]). *Let  $T$  be a contractive self-map of a complete metric space  $X$ , let  $x \in X$ , and set  $x_n = T^n x$ ,  $n \in \mathbb{N}_0$ . Then  $\{x_n\}$  converges to a unique fixed point of  $T$  if and only if, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , with  $x_{p_n} \neq x_{q_n}$ , condition*

$$\lim_{n \rightarrow \infty} \frac{d(Tx_{p_n}, Tx_{q_n})}{d(x_{p_n}, x_{q_n})} = 1,$$

*implies  $d(x_{p_n}, x_{q_n}) \rightarrow 0$ .*

**Proposition 3.11.** *Let  $\{x_n\}$  be a sequence in a metric space  $X$  such that  $d(x_{p+1}, x_{q+1}) \leq d(x_p, x_q)$ , for all  $p, q \in \mathbb{N}_0$ . Then the following statements are equivalent:*

(i) *for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , with  $x_{p_n} \neq x_{q_n}$ , condition*

$$\lim_{n \rightarrow \infty} \frac{d(x_{p_n+1}, x_{q_n+1})}{d(x_{p_n}, x_{q_n})} = 1, \tag{3.7}$$

*implies  $d(x_{p_n}, x_{q_n}) \rightarrow 0$ .*

(ii) *for every  $\epsilon > 0$ , for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup d(x_{p_n}, x_{q_n}) \leq \epsilon$  then*

$$\limsup_{n \rightarrow \infty} d(x_{p_n+1}, x_{q_n+1}) < \epsilon. \tag{3.8}$$

(iii) *for every  $\epsilon > 0$ , there exist  $\delta > 0$ ,  $\eta \in (0, \epsilon)$ , and  $N \in \mathbb{N}_0$ , such that*

$$\forall p, q \geq N, \quad d(x_p, x_q) < \epsilon + \delta \implies d(x_{p+1}, x_{q+1}) \leq \eta. \tag{3.9}$$

*Proof.* (i)  $\implies$  (ii): Assume that, for two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , we have

$$\limsup_{n \rightarrow \infty} d(x_{p_n+1}, x_{q_n+1}) = \limsup_{n \rightarrow \infty} d(x_{p_n}, x_{q_n}) > 0.$$

Since  $d(x_{p_n+1}, x_{q_n+1}) \leq d(x_{p_n}, x_{q_n})$  for all  $n$ , by passing through subsequences, if necessary, we can assume that

$$\lim_{n \rightarrow \infty} d(x_{p_n+1}, x_{q_n+1}) = \lim_{n \rightarrow \infty} d(x_{p_n}, x_{q_n}) > 0.$$

Therefore, we get (3.7). Since  $d(x_{p_n}, x_{q_n})$  does not converge to 0, we conclude that the sequence does not satisfy (i).

(ii)  $\Rightarrow$  (iii): Assume there is  $\epsilon > 0$  such that, for every  $n \in \mathbb{N}_0$ , there exist  $p_n$  and  $q_n$  with  $q_n > p_n \geq n$  such that

$$d(x_{p_n}, x_{q_n}) < \epsilon + \frac{1}{n} \quad \text{and} \quad \epsilon - \frac{1}{n} \leq d(x_{p_n+1}, x_{q_n+1}).$$

Then

$$\epsilon \leq \limsup_{n \rightarrow \infty} d(x_{p_n+1}, x_{q_n+1}) \leq \limsup_{n \rightarrow \infty} d(x_{p_n}, x_{q_n}) \leq \epsilon.$$

(iii)  $\Rightarrow$  (i): Assume that, for two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , condition (3.7) holds and also

$$\epsilon = \limsup_{n \rightarrow \infty} d(x_{p_n}, x_{q_n}) > 0.$$

For this  $\epsilon$ , by (iii), there exist  $\delta > 0$ ,  $\eta \in (0, \epsilon)$ , and  $N \in \mathbb{N}_0$ , such that (3.9) holds true. There is  $N_1 > N$  such that, for  $n \geq N_1$ , we have  $d(x_{p_n}, x_{q_n}) < \epsilon + \delta$  and thus  $d(x_{p_n+1}, x_{q_n+1}) \leq \eta$ . On the other hand (3.7) implies that, for every  $r < 1$ , there is  $N_2 > N_1$  such that, for  $n \geq N_2$ ,

$$rd(x_{p_n}, x_{q_n}) \leq d(x_{p_n+1}, x_{q_n+1}) \leq \eta.$$

If  $n \rightarrow \infty$  we get  $r\epsilon \leq \eta$ . If  $r \rightarrow 1$ , we get  $\epsilon \leq \eta$  which is absurd.  $\square$

Now, to see why Theorem 3.10 follows from the results in this section, take a point  $x \in X$  and set  $x_n = T^n x$ ,  $n \in \mathbb{N}_0$ . If  $\{x_n\}$  satisfies the condition in Theorem 3.10, then, by Proposition 3.11, the sequence  $\{x_n\}$  is  $d$ -contractive (in the sense of Definition 3.2). On the other hand,  $T$  being contractive implies that  $\{x_n\}$  satisfies (3.3). Hence, by Proposition 3.5, we have  $d(x_n, x_{n+1}) \rightarrow 0$ . Theorem 3.3 now implies that  $\{x_n\}$  is a Cauchy sequence.

#### 4. ASYMPTOTIC CONTRACTIONS OF MEIR-KEELER TYPE

In 2003, Kirk [18] introduced the notion of asymptotic contraction on a metric space, and proved a fixed-point theorem for such contractions (see also [1]). In 2006, Suzuki [32] introduced the notion of asymptotic contraction of Meir-Keeler type, and proved a fixed-point theorem for such contractions, which is a generalization of both Meir and Keeler's theorem [24] and Kirk's theorem [18]. A year later, Suzuki [33] introduced the following notion of asymptotic contractions which is, in some sense, the final definition of asymptotic contractions (see [33, Theorem 6]).

**Definiton 4.1** (Suzuki [33]). A mapping  $T$  on a metric space  $X$  is said to be an *asymptotic contraction of the final type* if

- (i)  $d(T^n x, T^n y) \rightarrow 0$ , for all  $x, y \in X$ ,
- (ii) for every  $x \in X$  and  $\epsilon > 0$ , there exist  $\delta > 0$  and  $\nu \in \mathbb{N}$  such that

$$\forall p, q \in \mathbb{N}, \quad \epsilon < d(T^p x, T^q x) < \epsilon + \delta \implies d(T^{p+\nu} x, T^{q+\nu} x) \leq \epsilon.$$

Then they proved the following result.

**Theorem 4.2** (Suzuki [33]). *Let  $X$  be a metric space and let  $T$  be an asymptotic contraction of the final type on  $X$ . Then  $\{T^n x\}$ , for every  $x$ , is a Cauchy sequence.*

We present a short proof of the above theorem using the results in section 2. But, first, let us make the following definition.

**Definiton 4.3.** Let  $T$  be a mapping on a metric space  $X$ , and  $\mathbf{m}$  a nonnegative function on  $X \times X$ . We call  $T$  an *asymptotic  $\mathbf{m}$ -contraction* if

- (i)  $d(T^n x, T^n y) \rightarrow 0$ , for all  $x, y \in X$ ,
- (ii) for every  $x \in X$  and  $\epsilon > 0$ , there exist  $\delta > 0$ ,  $\nu \in \mathbb{N}$ , and  $N \in \mathbb{N}_0$  such that

$$\forall p, q \geq N, \quad \mathbf{m}(T^p x, T^q x) < \epsilon + \delta \implies d(T^{p+\nu} x, T^{q+\nu} x) \leq \epsilon.$$

**Theorem 4.4.** Let  $T$  be an asymptotic  $\mathbf{m}$ -contraction on  $X$ . If  $\mathbf{m}$  satisfies (2.3), for some  $x \in X$ , then  $\{T^n x\}$  is a Cauchy sequence.

*Proof.* Let  $x_n = T^n x$ ,  $n \in \mathbb{N}_0$ . Then the following are equivalent (the proof is similar to that of Lemma 3.1 and hence is omitted).

- (i) for every  $\epsilon > 0$ , there exist  $\nu \in \mathbb{N}$ ,  $\delta > 0$  and  $N \in \mathbb{N}_0$  such that

$$\forall p, q \geq N, \quad \mathbf{m}(x_p, x_q) < \epsilon + \delta \implies d(x_{p+\nu}, x_{q+\nu}) \leq \epsilon,$$

- (ii) for every  $\epsilon > 0$ , there exists  $\nu \in \mathbb{N}$  such that, for any two subsequences  $\{x_{p_n}\}$  and  $\{x_{q_n}\}$ , if  $\limsup \mathbf{m}(x_{p_n}, x_{q_n}) \leq \epsilon$ , then, for some  $N$ ,

$$d(x_{p_n+\nu}, x_{q_n+\nu}) \leq \epsilon, \quad (n \geq N).$$

Condition (i) in Definition 4.3 implies that  $d(x_n, x_{n+1}) \rightarrow 0$ . Now part (iii) of Corollary 2.3 shows that  $\{x_n\}$  is Cauchy.  $\square$

At first look, because of replacing the distance function  $d(x, y)$  with a more general function  $\mathbf{m}(x, y)$ , such as the one defined by (3.5), it may seem that Definition 4.3 and Theorem 4.4 are extensions of Suzuki's Definition 4.1 and Theorem 4.2, respectively. However, we have the following result which confirms that Definition 4.1 is the final definition of asymptotic contractions.

**Theorem 4.5.** Let  $X$  be a complete metric space. If  $T$  is an asymptotic  $\mathbf{m}$ -contraction, for some  $\mathbf{m}$  satisfying (2.3) for all  $x$ , then  $T$  is an asymptotic contraction of the final type.

*Proof.* For every  $x \in X$ , by Theorem 4.4, the sequence  $\{T^n x\}$  is Cauchy, and, since  $X$  is complete, there is  $z \in X$  such that  $T^n x \rightarrow z$ . Since  $d(T^n x, T^n y) \rightarrow 0$ , for all  $x, y \in X$ , the point  $z$  is unique. Now, Theorem 8 in [33] shows that  $T$  is an asymptotic contraction of the final type.  $\square$

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