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ITERATIVE FIXED POINT THEOREMS AND THEIR APPLICATIONS TO ORDERED VARIATIONAL INEQUALITIES ON VECTOR LATTICES

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Abstract. In this paper, we introduce the concept of order-continuity and ordered Lipschitz conditions of maps on vector lattices, and we provide some properties of order-continuous maps. Then, by applying these properties, we prove some theorems for the existence of fixed points for maps. As applications of these results, we solve some ordered variational inequalities on vector lattices.
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1. Introduction

Fixed point theorems in topological spaces have been widely applied in solving many types of equations, variational inequalities, complementarity problems, optimization problems, equilibrium problems, and others (see [4, 5], [14] and [17]). There are two important aspects in fixed point theory in topological spaces in general, in particular in metric spaces. The first one is the existence of fixed points of maps on the considered spaces; and the second one is the approximation of an existing fixed point. Regarding to the second issue, many algorithms have been developed. The most common techniques for the estimations of fixed points are selecting some iterating processes, such as the well-known Mann scheme and Ishikawa scheme, where the continuity or a certain type of semi-continuity of the considered maps must be applied (see [2] and [14]).

For the case that the underlying spaces are equipped with an ordering relation without topological structures, such as, preordered sets, posets, lattices and vector lattices, many fixed point theorems have been proved without applying topological continuity of the considered maps (see [3], [10], [12, 13] and [16]). These theorems have been used to solve vector or ordered variational inequalities (see [6, 7, 8, 9, 10]) and to solve equilibrium problems in game theory in ordered sets (see [3] and [11]). Due to the importance of approximation of fixed points by topological convergent sequences in fixed point theory, that the underlying spaces with topological structures, it is a naturally extension to study this topic on the underlying spaces with ordering structures.

In this paper, we provide some iterative fixed point theorems for maps and their applications to ordered variational inequalities in vector lattices. This paper is organized as below. In section 2, we recall the concept of order-continuity of maps in vector lattices and prove some properties, which are similar to that in ordinary analysis. These properties lay a foundation of their applications to the proof of the existence of fixed points on vector lattices. In section 3, we prove some iterative fixed point theorems and provide a Mann type scheme for approximation of fixed points of some maps in vector lattices. In section 4, we introduce the concept of ordered Lipschitz condition of maps on vector lattices, which is applied in the proof of a theorem for the existence of fixed points. In section 5, we apply the fixed point theorems proved in section 4 to solve some ordered variational inequalities on vector lattices.

2. The order-continuity of maps in vector lattices

In this section, we recall some concepts of ordered sets and some properties of order-limits. For more details, the readers are referred to [1], [3], [12] and [14]. Then we recall the concept of order-continuity of maps on vector lattices and provide some properties that are similar to the properties of ordinary limits in analysis. These properties will be frequently used throughout this paper.

A binary relation $\succcurlyeq$ on a nonempty set P is called a lattice order, if it is reflexive, antisymmetric and transitive, such that, for every pair of elements $x, y \in P, x \vee y$ and $x \wedge y$ both exist. A nonempty set P with a lattice order $\succcurlyeq$ on it is called a lattice; and it is denoted by $(P, \succcurlyeq)$.

In this paper, all considered vector spaces are real vector spaces. A vector space X equipped with a lattice order $\succcurlyeq^X$ is called a Riesz space or a vector lattice, which is written as $(X, \succcurlyeq^X)$, if the following (order-linearity) properties hold:

1. $x \succcurlyeq^X y$ implies $x + z \succcurlyeq^X y + z$, for all $x, y, z \in X$.
2. $x \succcurlyeq^X y$ implies $\alpha x \succcurlyeq^X \alpha y$, for all $x, y \in X$ and $\alpha \geqslant 0$.

In this case, as usual, for any $x \in X, x^+ = x \vee 0, x^- = (-x) \vee 0$ and $|x| = x^+ \vee x^-$ all are well-defined.

A sequence $\{x_n\}$ in a vector lattice $(X, \succcurlyeq^X)$ is said to be order-decreasing, which is denoted by $x_n \downarrow$, whenever $m > n$ implies $x_m \preccurlyeq^X x_n$. We denote $x_n \downarrow x$, whenever $x_n \downarrow$ holds and $\wedge\{x_n\}$ exists, such that $\wedge\{x_n\} = x$. Order-increasing and the notation $x_n \uparrow$ are analogously defined for sequence $\{x_n\}$; and $x_n \uparrow x$, if and only if, $x_n \uparrow$ holds and $\vee\{x_n\}$ exists, such that $\vee\{x_n\} = x$. We refer order-decreasing or order-increasing sequences as order-monotonic sequences.

Definition 2.2. A sequence $\{x_n\}$ in a vector lattice $(X, \succcurlyeq^X)$, is said to order-converge to a vector x, which is denoted by $x_n \underset{\rightarrow}{o} x$, whenever there exists another sequence $\{\xi_n\}$ in $(X, \succcurlyeq^X)$ with $\xi_n \downarrow 0$ such that

$$|x_n - x| \preccurlyeq^X \xi_n \text{ holds, for each } n. \tag{2.1}$$

In this case, x is called an order-limit of the sequence $\{x_n\}$. Next lemma provides the connections between the notions $\uparrow, \downarrow$, and order-limits.

Lemma 2.3. *Let $\{x_n\}$ be a sequence in a vector lattice $(X, \succcurlyeq^X)$. Either $x_n \downarrow x$ or $x_n \uparrow x$ implies $x_n \underset{\to}{o} x$.*

Proof. Suppose $x_n \uparrow x$; that is, $x_n \uparrow$ and $\vee\{x_n\} = x$. Take $\xi_n = x - x_n$, for all n. It is clear to see that $\xi_n \downarrow$ holds. Under the hypothesis $\vee\{x_n\} = x$, we have

$$\wedge\{\xi_n\} = \wedge\{x - x_n\} = x - \vee\{x_n\} = x - x = 0.$$

Noticing $\xi_n = x - x_n \succcurlyeq^X 0$, for all n, we have

$$|x_n - x| = x - x_n = \xi_n, \text{ for each } n.$$

It implies $x_n \underset{\to}{o} x$. In case if $x_n \downarrow x$, taking $\xi_n = x_n - x$, then under the hypothesis $\wedge\{x_n\} = x$, we have $\xi_n \downarrow$ and

$$\wedge\{x_n\} = \wedge\{x_n - x\} = \wedge\{x_n\} - x = x - x = 0.$$

Noticing $\xi_n = x_n - x \succcurlyeq^X 0$, for all n, we have

$$|x_n - x| = x_n - x = \xi_n, \text{ for each } n.$$

It implies $x_n \underset{\to}{o} x$.

Lemma 2.3. *Let $\{x_n\}, \{y_n\}$ be two sequences in an arbitrary vector lattices $(X, \succcurlyeq^X)$. The following properties hold:*

1. *$x_n \downarrow 0$ implies $ax_n \downarrow 0$, for every real number $a \geqslant 0$.*
2. *$x_n \downarrow 0$ and $y_n \downarrow 0$ imply $(x_n + y_n) \downarrow 0$.*
3. *$x_n \underset{\to}{o} x$ and $y_n \underset{\to}{o} y$ imply $ax_n + by_n \underset{\to}{o} ax + by$, for any real numbers a and b.*

Proof. To show Part 1, we need to prove that $ax_n \downarrow$ holds and $\wedge\{ax_n\} = 0$. It is trivial for $a = 0$. So we assume that $a > 0$. From the order linearity of vector lattices, the condition $x_n \downarrow 0$ and $a > 0$ imply $ax_n \downarrow$ and $ax_n \succcurlyeq^X 0$, for every n. So 0 is a lower bound of the sequence $\{ax_n\}$. Assume, by the way of contribution, that $\{ax_n\}$ has a lower bound $z \succ^X 0$; that is, $ax_n \succcurlyeq^X z$, which is equivalent to $x_n \succcurlyeq^X a^{-1}z$, for all n. It implies that $a^{-1}z$ is a lower bound of the set $\{x_n\}$. It is clear that $a^{-1}z \succ^X 0$. This is a contradiction to the hypothesis $\wedge\{x_n\} = 0$ (Note that $x_n \downarrow 0$ means that $x_n \downarrow$ and $\wedge\{x_n\} = 0$).

To show Part 2, we first clearly see that $(x_n + y_n) \downarrow$ and $(x_n + y_n) \succcurlyeq^X 0$, for all n. Assume, on the contrary, that $\{x_n + y_n\}$ has a lower bound $z \succ^X 0$; that is, $x_n + y_n \succcurlyeq^X z$, for all n. From the conditions $x_n \downarrow$ and $y_n \downarrow$, it implies that either $x_n \succcurlyeq^X 2^{-1}z$ or $y_n \succcurlyeq^X 2^{-1}z$, for all n; that is, $2^{-1}z$ is a lower bound of either the set $\{x_n\}$ or the set $\{y_n\}$. Since $a^{-1}z \succ^X 0$, this is a contradiction to either the hypothesis $\wedge\{x_n\} = 0$ or the hypothesis $\wedge\{y_n\} = 0$.

To show Part 3, from $x_n \underset{\to}{o} x$ and $y_n \underset{\to}{o} y$, there are sequences $\{\xi_n\}$ and $\{z_n\}$ in X with $x_n \downarrow 0$ and $x_n \downarrow 0$, such that

$$|x_n - x| \preccurlyeq^X \xi_n \text{ and } |y_n - y| \preccurlyeq^X \zeta_n \text{ both hold, for each } n. \tag{2.2}$$

For any real numbers a and b, we have

$$\begin{aligned}|ax_n - ax| &= |a(x_n - x)| \\ &= (a(x_n - x)) \wedge (-a(x_n - x)) \\ &= (|a|(x_n - x)) \wedge (-|a|(x_n - x)) \\ &= (|a|(x_n - x)) \wedge (|a|(-(x_n - x))) \\ &= |a|((x_n - x) \wedge (-(x_n - x))) \\ &= |a| \cdot |x_n - x|.\end{aligned}$$

Then from (2.2), it implies

$$\begin{aligned}|(ax_n + by_n) - (ax + by)| &= |(ax_n - ax) + (by_- nby)| \\ &\preccurlyeq^X |ax_n - ax| + |by_n - by| \\ &\preccurlyeq^X |a| \cdot |x_n - x| + |b| \cdot |y_n - y| \\ &\preccurlyeq^X |a|\xi_n + |b|\zeta_n, \text{ for each } n.\end{aligned}$$

By Parts 1 and 2, and from the conditions $\xi_n \downarrow 0$ and $\zeta_n \downarrow 0$, it implies $(|a|\xi_n + |b|\zeta_n) \downarrow 0$. From the above order-inequalities, it follows that $ax_n + by_n \underset{\rightarrow}{o} ax + by$.

Lemma 2.3. *If a sequence $\{x_n\}$ in a vector lattice $(X, \succcurlyeq^X)$ is order-convergent, then its order-limit is unique.*

Proof. Let x and y be order-limits of $\{x_n\}$. Then there are sequence $\{\xi_n\}$ and $\{\zeta_n\}$ in X with $\xi_n \downarrow 0$ and $\zeta_n \downarrow 0$ such that

$$|x_n - x| \preccurlyeq^X \xi_n \text{ and } |x_n - y| \preccurlyeq^X \zeta_n, \text{ for each } n.$$

Then it follows that

$$|x - y| \preccurlyeq^X \xi_n + \zeta_n, \text{ for each } n.$$

So, $|x - y|$ is a lower bound of $\{\xi_n + \zeta_n\}$. Applying Part 2 of Lemma 2.4, we have $0 \preccurlyeq^X |x - y| \preccurlyeq \wedge(\xi_n + \zeta_n) = 0$. It implies $y - x = 0$.

Definition 2.6. *A vector lattice $(X, \succcurlyeq^X)$ is said to be generalized Archimedean if and only if for any given element $x \succcurlyeq^X 0$ and any decreasing sequence of positive numbers $\{a_n\}$ with limit 0, we have*

$$a_n x \downarrow 0.$$

Lemma 2.7. *Let $(X, \succcurlyeq^X)$ be a generalized Archimedean vector lattice. Then for any given element $x \succcurlyeq^X 0$ and any decreasing sequence of positive numbers $\{a_n\}$ with limit 0, we have*

$$a_n x \underset{\rightarrow}{o} 0.$$

This lemma immediately follows from Definition 2.6 and Lemma 2.3.

Let $(X, \succcurlyeq^X), (U, \succcurlyeq^U)$ be vector lattices and let C be a nonempty subset of X. Let $T : C \to 2^U \setminus \{\emptyset\}$ be a set-valued map. T is said to be order-increasing upward, if $x \preccurlyeq^X y$ in C implies that, for any $z \in T(x)$, there is a $w \in T(y)$ such that $z \preccurlyeq^U w$. T is said to be order-increasing downward, if$x \preccurlyeq^X y$ in C implies that, for any $w \in T(y)$, there is a $z \in T(x)$ such that $z \preccurlyeq^U w$. If T is both order-increasing upward and downward, then T is said to be order-increasing. A single-valued map T from C to

U is said to be order-preserving (order-reversing) if and only if for any elements x, y in C, $x \succcurlyeq^X y$ implies $Tx \succcurlyeq^U Ty (Tx \preccurlyeq^U Ty)$.

Definition 2.8. *Let $(X, \succcurlyeq^X), (U, \succcurlyeq^U)$ be two vector lattices and C a nonempty subset of X. A single-valued map $T : C \to 2^U \setminus \{\emptyset\}$ is said to be (sequentially) order-continuous if and only if, for any sequence $\{x_n\}$ with $x_n \xrightarrow{o} x$ in X, for some $x \in C$, implies $Tx_n \xrightarrow{o} Tx$ in U.*

The following definition is an extension of the order-continuity from single-valued maps to set-valued map.

Definition 2.9. *Let $(X, \succcurlyeq^X), (U, \succcurlyeq^U)$ be two vector lattices and C a nonempty subset of X. A set-valued map $T : C \to 2^U \setminus \{\emptyset\}$ is said to be order-continuous whenever, for any order-convergent sequence $\{x_n\}$ in X, with $x_n \xrightarrow{o} x$, for some $x \in C$, and $u_n \in Tx_n$, for $n = 1, 2, \cdots$, if $\{u_n\}$ is also an order-convergent sequence in U, then $u_n \xrightarrow{o} u$ in U, for some $u \in Tx$.*

A nonempty subset C of a vector lattice $(X, \succcurlyeq^X)$ is said to be chain-complete if and only if for any chain $\{x_\alpha\}$ in $C, \vee\{x_\alpha\}$ and $\wedge\{x_\alpha\}$ both exist. Next, we define a special case of chain-complete subsets, which is frequently applied in the sequel.

Definition 2.10. *Let C be a nonempty subset of a vector lattice $(X, \succcurlyeq^X)$. C is said to be (sequentially) conditionally chain-complete if and only if for any order-monotonic sequence $\{x_n\}$ in C, the following properties hold:*

1. If $x_n \downarrow$ and $\{x_n\}$ has a lower bound, then $\wedge\{x_n\}$ exists and $\wedge\{x_n\} \in C$; that is $x_n \downarrow \wedge\{x_n\}$ in C;

2. If $x_n \uparrow$ and $\{x_n\}$ has an upper bound, then $\vee\{x_n\}$ exists and $\vee\{x_n\} \in C$; that is, $x_n \uparrow \vee\{x_n\}$ in C.

It is clearly to see that the first case in Definition 2.10 implies $x_n \xrightarrow{o} \wedge\{x_n\}$ and the second one implies $x_n \xrightarrow{o} \vee\{x_n\}$. We define the order-Cauchy completeness in vector lattices, analogously to the Cauchy completeness in ordinal analysis.

Definition 2.11. *Let $(X, \succcurlyeq^X)$ be a vector lattice. A sequence $\{x_n\}$ in X is called an order-Cauchy sequence if there is a sequence $\{\xi_n\}$ in X with $\xi_n \downarrow 0$ such that, for every positive integer n, we have*

$$|x_n - x_m| \preccurlyeq^X \xi_n, \text{ for all } m \geqslant n. \tag{2.3}$$

Definition 2.12. *A nonempty subset C of a vector lattice $(X, \succcurlyeq^X)$ is said to be order-Cauchy complete if every order-Cauchy sequence $\{x_n\} \subset C$ order-converges to a vector x in C; that is, $\{x_n\}$ has an order-limit $x \in C$.*

The following results are similar to the corresponding results related to Cauchy sequences in ordinary analysis.

Lemma 2.13. *In an arbitrary vector lattice, we have*

1. Every order-convergent sequence is an order-Cauchy sequence;

2. Every order-Cauchy sequence is order-bounded.

The proof of this lemma is straightforward, and it is omitted here.

3. Several iterative fixed point theorems in vector lattices

In [10], Li provided several fixed point theorems for set-valued maps on chain-complete posets, which are extensions of the Abian-Brown Fixed Point Theorem. In

[12], Ok proved several iterative fixed point theorems for single-valued order-increasing (order-preserving) maps on finite posets, where the order-continuity is not used. In this section, we apply the order-continuity property to prove some iterative fixed point theorems for order-increasing maps on vector lattices.

Theorem 3.1. *Let $(X, \succcurlyeq^X)$ be a vector lattice and let C be a chain-complete nonempty subset of X. Let $T : C \to 2^C \setminus \{\emptyset\}$ be an order-continuous and order-increasing upward set-valued map. If there is an $x_0 \in C$ and $u_0 \in Tx_0$ satisfying $x_0 \preccurlyeq^X u_0$, then T has a fixed point.*

Proof. For the given points $x_0 \in C, u_0 \in Tx_0$ with $x_0 \preccurlyeq^X u_0$, since $T : C \to 2^C \setminus \{\emptyset\}$ is order-increasing upward, there is $u_1 \in Tu_0$ such that $u_0 \preccurlyeq^X u_1$. By iterating this process, for every positive integer n, there is $u_{n+1} \in Tu_n$ such that $u_n \preccurlyeq^X u_{n+1}$. Hence we get an order-increasing sequence $\{u_n\} \subset C$, with $u_{n+1} \in Tu_n$, for $n = 0, 1, 2, \cdots$, such that $u_n \uparrow$ holds.

Since C is a chain complete subset of X, then $\vee\{u_n\}$ exists, which is denoted by x^*. It follows that $u_n \uparrow x^*$. From Lemma 2.3, u_n order-converges to an element $x^* \in C$. That is

$$u_n \xrightarrow{o} x^*.$$

From $u_{n+1} \in Tu_n$, for $n = 0, 1, 2, \cdots$, and by applying the order-continuity of the set-valued map T, this order-convergent sequence $\{u_{n+1}\}$ must order-converges to an element in Tx^*, say y, that is

$$u_{n+1} \xrightarrow{o} y \in Tx^*.$$

From Lemma 2.5, we get $x^* = y \in Tx^*$. Hence x^* is a fixed point of T.

It is well-known that the Lipschitz condition has been widely applied in the ordinal analysis, fixed point theory, differential equations, etc, to prove the existence and the uniqueness of fixed point of maps or the solutions to some equations. In this section, we introduce the ordered Lipschitz condition on vector lattices and prove some existence theorems for fixed points of maps with this condition.

Definition 3.2. *Let C be a nonempty subset of a vector lattice $(X, \succcurlyeq^X)$. A single-valued map $T : C \to X$ is said to have ordered Lipschitz property on C whenever there exists a positive number $k < 1$, such that*

$$|Tx - Ty| \preccurlyeq^X k|x - y|, \textit{ for all } x, y \in C. \tag{3.1}$$

Similarly to ordinary analysis, it is worthy of noting that if a single valued map has the ordered Lipschitz property, then it is order-continuous.

Theorem 3.3. *Let $(X, \succcurlyeq^X)$ be a generalized Archimedean vector lattice and let C be an order-Cauchy complete subset of X. If a map $T : C \to C$ holds the ordered Lipschitz property on C with a positive number $k < 1$, then T has a unique fixed point.*

Proof. Let k be the given positive number with $k < 1$ satisfying (3.1) for the map T. Then, taking any $x_0 \in C$, we have

$$|T^{n+1}x_0 - T^n x_0| \preccurlyeq^X k|T^n x_0 - T^{n-1}x_0|, \text{ for all positive integer } n.$$

Iterating the above order-inequality yields

$$|T^{n+1}x_0 - T^n x_0| \preccurlyeq^X k^n|Tx_0 - x_0|, \text{ for all positive integer } n. \tag{3.2}$$

Applying the order-triangle inequality (see Theorem 1.9 in [1]) and (3.2), for any positive integer i, we get

$$\begin{aligned} |T^{n+i}x_0 - T^n x_0| &\preccurlyeq^X |T^{n+i}x_0 - T^{n+i-1}x_0| + |T^{n+i-1}x_0 - T^{n+i-2}x_0| \\ &\quad + \cdots + |T^{n+1}x_0 - T^n x_0| \\ &\preccurlyeq^X k^{n+i}|Tx_0 - x_0| + k^{n+i-1}|Tx_0 - x_0| + \cdots + k^n|Tx_0 - x_0| \\ &= k^n(k^i + k^i - 1 + \cdots + 1)|Tx_0 - x_0| \\ &\preccurlyeq^X \frac{k^n}{1-k}|Tx_0 - x_0|, \text{ for all positive integer } n. \end{aligned} \tag{3.3}$$

Take $\xi_n = \frac{k^n}{1-k}|Tx_0 - x_0|$, for every positive integer n. Since $(X, \succcurlyeq^X)$ is generalized Archimedean, from Lemma 2.7, it follows that $\xi_n \downarrow 0$. From (3.3), we have

$$|T^m x_0 - T^n x_0| \preccurlyeq^X \xi_n, \text{ for all positive integer } n \text{ and for all } m \geqslant n.$$

It implies that $\{T^n x_0\}$ is an order-Cauchy sequence in the order-Cauchy complete subset C. Hence, $\{T^n x_0\}$ has an order-limit, say x^* C; that is,

$$T^n x_0 \underrightarrow{o} x^*.$$

Since the map $T : C \to C$ has the ordered Lipschitz property, so it is order-continuous. It implies

$$T^{n+1} x_0 \underrightarrow{o} Tx^*.$$

From Lemma 2.5, the uniqueness of order-limit, or from the ordered Lipschitz property of the map T, it implies $Tx^* = x^*$. Thus x^* is a fixed point of T. The uniqueness of the fixed point of T can be proven by using standard methods.

Theorem 3.4. *Let $(X, \succcurlyeq^X)$ be a generalized Archimedean vector lattice and let C be an order-Cauchy complete convex subset of X. Suppose that a map $T : C \to C$ is order-continuous and has the ordered Lipschitz property on C with a positive number $k < 1$. For any given $x_0 \in C$, and for an arbitrary number $a \in (0, 1)$, an iterative sequence is defined by*

$$x_n = aTx_{n-1} + (1-a)x_{n-1}, \text{ for all positive integern.}$$

Then the sequence $\{x_n\}$ is an order-Cauchy sequence and its order-limit is the unique fixed point of T.

Proof. From the selection of the sequence $\{x_n\}$ given in this theorem, we have

$$x_{n+1} - x_n = a(Tx_n - Tx_{n-1}) + (1-a)(x_n - x_{n-1}). \tag{3.4}$$

From Part 4 of Theorem 1.3 in [1], for every $x, y \in X$, and for any $\alpha \geqslant 0$, we have $\alpha(x \vee y) = (\alpha x) \vee (\alpha y)$, and $\alpha(x \wedge y) = (\alpha x) \wedge (\alpha y)$. Taking $y = 0$, it implies $\alpha x^+ = (\alpha x)^+$ and $\alpha x^{-1} = (\alpha x)^{-1}$. Then

$$\alpha|x| = \alpha(x^+ + x^{-1}) = \alpha x^+ + \alpha x^{-1} = (\alpha x)^+ + (\alpha x)^{-1} = |\alpha x|. \tag{3.5}$$

Applying the order-triangle inequality and by (3.3), (3.4), (3.5), from the ordered Lipschitz condition of T, we get

$$\begin{aligned}|x_{n+1}-x_n| &\preccurlyeq^X a|Tx_n - Tx_{n-1}| + (1-a)|x_n - x_{n-1}| \\ &\preccurlyeq^X ak|x_n - x_{n-1}| + (1-a)|x_n - x_{n-1}| \\ &= (ak+(1-a))|x_n - x_{n-1}|, \text{ for all positive integer } n.\end{aligned} \tag{3.6}$$

Iterating the process (3.6) obtains

$$|x_{n+1}-x_n| \preccurlyeq^X (ak+(1-a))^n|x_1 - x_0|, \text{ for all positive integer } n. \tag{3.7}$$

Since $0 < ak + (1-a) < 1$, from (3.7), similarly to the proof of Theorem 3.3, we can show that $\{x_n\}$ is an order-Cauchy sequence in the order-Cauchy complete subset C. Hence, $\{x_n\}$ has an order-limit, say $x^* \in C$; that is,

$$x_n \xrightarrow{o} x^*.$$

Since $T : C \to C$ is order-continuous, from Lemma 2.4, it implies

$$x_n = aTx_{n-1} + (1-a)x_{n-1} \xrightarrow{o} aTx^* + (1-a)x^*.$$

From Lemma 2.5, we obtain

$$x^* = aTx^* + (1-a)x^*.$$

It immediately follows that $Tx^* = x^*$. Thus x^* is a fixed point of T. The rest of the proof is similar to the proof of Theorem 3.3 and is omitted.

4. Applications to ordered-variational inequalities on vector lattices

Let $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$ be two vector lattices. X^+ and U^+ denote the positive cones of $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$, respectively; that is,

$$X^+ = \{x \in X : x \succcurlyeq^X 0\} \text{ and } U^+ = \{u \in U : u \succcurlyeq^U 0\},$$

where, as usual, without confusion, both of the origins of X and U are denoted by 0.

Let $\mathcal{L}(X, U)$ denote the collection of all linear operators from X to U. A linear operator f from X to U is called a positive (linear) operator from X to U whenever $x \succcurlyeq^X 0$ in X implies $f(x) \succcurlyeq^U 0$ in U. The collection of all positive (linear) operators from X to U is denoted by $\mathcal{L}^+(X, U)$. One can see that a linear operator from X to U is order-preserving if and only if, it is positive.

It is clear that $\mathcal{L}(X, U)$ is also a real vector space. We define a binary relation $\succcurlyeq^{\mathcal{L}}$ on $\mathcal{L}(X, U)$ as follows: for every $f, g \in \mathcal{L}(X, U)$

$$f \succcurlyeq^{\mathcal{L}} g \text{ if and only if } f(x) \succcurlyeq^U g(x), \text{ for all } x \in X^+. \tag{4.1}$$

It has been shown (see Page 10 in [1]) that if both $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$ are vector lattices (Riesz spaces), then $\succcurlyeq^{\mathcal{L}}$ is a partial order on $\mathcal{L}(X, U)$; and therefor, $(\mathcal{L}(X, U), \succcurlyeq^{\mathcal{L}})$ is an ordered vector space. But, it is worthy of noting that it may not be a vector lattice. In [11], Xie, Li, and Yang applied the Choquet-Kendall Theorem and deeply studied the criteria for $(\mathcal{L}(X, U), \succcurlyeq^{\mathcal{L}})$ to be an ordered vector space or a vector lattice.

Moreover, if $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$ are just ordered vector spaces and not vector lattices; that is, the partial orders $\succcurlyeq^X$ and $\succcurlyeq^U$ are not lattice order, then the binary relation $\succcurlyeq^L$ on $\mathcal{L}(X,U)$, defined by (4.1), holds the reflexive and transitive properties; and it may not have the antisymmetric property. So it may not be a partial ordering relation on $\mathcal{L}(X,U)$. This argument can be demonstrated by the following example.

Example 4.1. In $\mathcal{R}^3$, take K to be the closed convex cone as below:

$$K = \{(x,y,0) \in \mathcal{R}^3 : x \geqslant 0 \text{ and } y \geqslant 0\}.$$

Let $\succcurlyeq_K$ be the partial order on $\mathcal{R}^3$ induced by the closed convex cone K. Since K is not a base of $\mathcal{R}^3$, then according to Choquet-Kendall Theorem, $\succcurlyeq_K$ is not a lattice order on $\mathcal{R}^3$. Hence, $(\mathcal{R}^3, \succcurlyeq_K)$ is an ordered vector space, but not a vector lattice. Then we define linear operators f and g as following:

$$f(x_1,x_2,x_3) = (2x_1,2x_2,3x_3) \text{ and } g(x_1,x_2,x_3) = (2x_1,2x_2,4x_1),$$

for any $x = (x_1,x_2,x_3) \in \mathcal{R}^3$.

So it is clearly seen that $f, g \in \mathcal{L}(\mathcal{R}^3,\mathcal{R}^3)$. Notice that in the ordered vector space $(\mathcal{R}^3, \succcurlyeq_K)$, we have $(\mathcal{R}^3)^+ = K$. From (11), we see that $f \succcurlyeq^{\mathcal{L}} g$ and $g \succcurlyeq^{\mathcal{L}} f$ both hold. But it is clear that $f \neq g$. Hence, $\succcurlyeq^{\mathcal{L}}$ does not have the antisymmetric property; and therefore, $\succcurlyeq^{\mathcal{L}}$ is not a partial order on $\mathcal{L}(\mathcal{R}^3,\mathcal{R}^3)$.

Definition 4.2. *Let $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$ be two vector lattices. Let C be a nonempty subset of X and let $F : C \to \mathcal{L}(X,U)$ be a map. Then*

1. the ordered variational inequality problem associated with C, F and U, denoted by $OVI(C,F,U)$, is to find an $x^* \in C$ such that

$$F(x^*)(x - x^*) \succcurlyeq^U 0, \text{ for all } x \in C. \tag{4.2}$$

2. the extended ordered variational inequality problem associated with C, F and U, denoted by $EOVI(C,F,U)$, is to find an $y* \in C$ such that

$$F(y*)(y - y*) \not\prec^U 0, \text{ for all } y \in C. \tag{4.3}$$

Let C be a nonempty subset of a vector lattice $(X, \succcurlyeq^X)$. For an arbitrary linear operator $f \in \mathcal{L}(X,U)$, if the smallest (the minimum) element of the set $\{f(t) \in U : t \in C\}$, with respect to the order $\succcurlyeq^U$ on U, exists, which is denoted by $\min\{f(t) \in U : t \in C\}$, then we define a set-valued map $\psi_C : \mathcal{L}(X,U) \to 2^C \setminus \{\emptyset\}$ by

$$\psi_C f = \{s \in C : f(s) = \min\{f(t) \in U : t \in C\}\}, \text{ for every } f \in \mathcal{L}(X,U). \tag{4.4}$$

Theorem 4.3. *Let $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$ be vector lattices. Let C be a chain-complete nonempty subsets of X. Let $F : C \to \mathcal{L}(X,U))$ be a map with the properties*

v1. $\psi_C F(x) \neq \emptyset$, for every $x \in C$; that is, the map $\psi_C F : C \to \setminus\{\emptyset\}$ is well-defined;

v2. $\psi_C F : C \to 2^C \setminus \{\emptyset\}$ is an order-continuous and order-increasing upward set-valued map;

v3. There are elements $x_0 \in C$ and $u_0 \in \psi_C F(x_0)$ satisfying $x_0 \preccurlyeq^X u_0$.

Then the problem $OVI(C,F,U)$ has a solution.

Proof. Define a set-valued map $T : C \to 2^C \setminus \{\emptyset\}$ as follows:

$$T(x) = \psi_C F(x), \text{ for all } x \in C.$$

From condition v1, it follows that $T : C \to 2^C \setminus \{\emptyset\}$ is well-defined. By condition v2, T is an order-continuous and order-increasing upward set-valued map. The elements

$x_0, u_0 \in C$, given in condition v3, satisfy $u_0 \in Tx_0$ and $x_0 \preccurlyeq^X u_0$. Then T satisfies all conditions in Theorem 3.1. It follows that T has a unique fixed point in C, say x^*, which satisfies $x^* \in T(x^*)$; that is,

$$x^* \in \psi_C F(x^*).$$

From (4.4), we obtain $F(x^*)(x^*) = \min\{F(x^*)(t) \in U : t \in U\}$. It implies

$$F(x^*)(x^*) \preccurlyeq^U F(x^*)(x), \text{ for all } x \in C.$$

Since $F(x^*) \in \mathcal{L}(X, U)$, from the order-linearity of the partial order $\preccurlyeq^U$ on vector lattice $(U, \succcurlyeq^U)$, it yields

$$F(x^*)(x - x^*) \succcurlyeq^U 0, \text{ for all } x \in C.$$

It follows that (4.2) holds for x^*. Hence x^* is a solution to the problem $OVI(C, F, U)$.

To consider the problem $EOVI(C, F, U)$, we introduce a notation similar to the one defined in (4.4). Let C be a nonempty subset of a vector lattice $(X, \succcurlyeq^X)$. For an arbitrary linear operator $f \in \mathcal{L}(X, U)$, the set of minimal elements of $\{f(t) \in U : t \in C\}$, with respect to the order $\succcurlyeq^U$ in U, is denoted by $Min\{f(t) \in U : t \in C\}$. Then we define a set-valued map $\psi_C : \mathcal{L}(X, U) \to 2^C \setminus \{\emptyset\}$ as

$$\psi_C f = \{s \in C : f(s) \in Min\{f(t) \in U : t \in C\}\}, \text{ for every } f \in \mathcal{L}(X, U). \tag{4.5}$$

Theorem 4.4. *Let $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U))$ be vector lattices. Let C be a chain-complete nonempty subsets of X. Let $F : C \to \mathcal{L}(X, U)$ be a map satisfying the conditions*

V1. $\psi_C F(x) \neq \emptyset$, for every $x \in C$; that is, the map $\psi_C F : C \to 2^C \setminus \{\emptyset\}$ is well-defined;

V2. $\psi_C F : C \to 2^C \setminus \{\emptyset\}$ is an order-continuous and order-increasing upward set-valued map;

V3. There are elements $y_0 \in C$ and $v_0 \in \psi_C F(y_0)$ with $y_0 \preccurlyeq^X v_0$.

Then the problem $EOVI(C, F, U)$ has a solution.

The proof of Theorem 4.4 is similar to the proof of Theorem 4.3 by applying Theorem 3.1, (4.3) and the operator ψ_C defined in (4.5); and it is omitted here.

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