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COMPARISON OF SOME TYPES OF LOCALLY COVERING MAPPINGS

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Abstract. Definitions of covering property that frequently appear in publications are discussed and compared. Several examples related to these definitions and coincidence points theorems of covering and Lipschitz mappings are constructed.
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1. Introduction

For two set-valued mappings $\Psi, \Phi : X \rightrightarrows Y$ acting in some sets X and Y a coincidence point is a point $\xi \in X$ such that $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$. The covering mappings theory was developed for the investigation of coincidence points (see, for instance, [1, 2, 3, 16] in the case when X and Y are metric spaces. For example, in [1] it was proved that under additional assumptions a covering mapping Ψ and a Lipscitz mapping Φ has a coincidence point $\xi \in X$, i.e., $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$. A local analogue of this result was obtained in [3]. The properties of covering mappings are studied in detail in the above mentioned works as far as in [5, 14, 15] and some other papers. The results of the covering mappings theory are used for the investigation of ordinary differential equations, control systems, integral Volterra equations (see, for instance, [6, 7, 8, 9, 12]).

In applications it is more natural to use the property of local covering. This concept was developed in [3, 16] and some other investigations. In the papers devoted to covering and locally covering mappings there were introduced various definitions of local covering. Here we will discuss the definitions that frequently appear in publications and compare the corresponding concepts.

2. Discussion of some definitions and results

Let (X, ρ_X), (Y, ρ_Y) be metric spaces with metrics ρ_X and ρ_Y, respectively, Ψ, Φ : $X \rightrightarrows Y$ be set-valued mappings that assign closed sets $\Psi(x) \subseteq Y$ and $\Phi(x) \subseteq Y$ to every $x \in X$. Moreover, let numbers $\alpha > 0$, $\beta \geq 0$ and sets $U \subseteq X$, $V \subseteq Y$ be given.

Denote by $B_X(x,R)$ and $O_X(x,R)$ the closed and, respectively, open ball in X centered at $x \in X$ with a radius $R \geq 0$, i.e.,

$$B_X(x,R) = \{u \in X : \rho_X(x,u) \leq R\}, \quad O_X(x,R) = \{u \in X : \rho_X(x,u) < R\}.$$

Further, set

$$B_X(M,R) = \bigcup_{x \in M} B_X(x,R), \quad O_X(M,R) = \bigcup_{x \in M} O_X(x,R),$$

$$\operatorname{dist}(M,N) = \inf\{\rho_X(x,u) : x \in M, \ u \in U\}$$

for arbitrary $M, N \subset Y$. Denote by $\operatorname{gph}(\Psi)$ the graph of Ψ, i.e., $\operatorname{gph}(\Psi) = \{(x,y) : x \in X, \ y \in \Psi(x)\}$. Everywhere in the sequel we assume that $X \times Y$ is a metric space with a metric defined by formula $\rho\big((x,y),(u,v)\big) = \rho_X(x,u) + \rho_Y(u,v)$ for each (x,y), $(u,v) \in X \times Y$.

Let us recall definitions of covering from [1, 2, 3].

Definition 2.1. A set-valued mapping $\Psi : X \rightrightarrows Y$ is called α-covering with respect to the sets $U \subset X$ and $V \subset Y$, if

$$B_X(x,r) \subset U \ \Rightarrow \ B_Y\big(\Psi(x), \alpha r\big) \cap V \subset \Psi\big(B_X(x,r)\big).$$

A set-valued mapping Ψ is called locally α-covering in a neighborhood of the point (x_0, y_0), if there exist neighborhoods U and V of the points x_0 and y_0, respectively, such that Ψ is α-covering with respect to U and V. Set-valued mapping Ψ is called (globally) α-covering if it is α-covering with respect to X and Y.

As it was mentioned above, the concept of covering is used to investigate coincidence points of two mappings. For instance, Theorem 2 from [1] states that if

- **(A1):** a set-valued mapping Ψ is α-covering and has a closed graph,
- **(A2):** a set-valued mapping Φ satisfies the Lipschitz condition with Lipschitz constant $\beta < \alpha$,
- **(A3):** at least one of the graphs $\operatorname{gph}(\Psi)$ or $\operatorname{gph}(\Phi)$ is complete,

then for every $x_0 \in X$, for every $\varepsilon > 0$ there exists $\xi \in X$ such that $\Psi(\xi) \cap \Phi(\xi) \neq \emptyset$ and $\rho_X(x_0, \xi) \leq (\alpha - \beta)^{-1} \operatorname{dist}(\Psi(x_0), \Phi(x_0)) + \varepsilon$.

Let us discuss this result. Note that the set of all coincidence points $\xi \in X$ of the set-valued mappings Ψ and Φ is the fixed points set of the mapping $\Psi^{-1}(\Phi(\cdot)) : X \rightrightarrows X$ (if the inverse $\Psi^{-1} : Y \rightrightarrows X$, $\Psi^{-1}(y) \equiv \{x \in X : y \in \Psi(x)\}$ exists). Under the assumptions **(A1)-(A3)** the inverse mapping Ψ^{-1} exists and satisfies Lipschitz inequality with Lipschitz constant α^{-1}. Thus, the set-valued mapping $\Psi^{-1}(\Phi(\cdot))$ is the contraction with Lipschitz constant $\alpha^{-1}\beta < 1$. However, in order to prove the existence of the desired coincidence point ξ one cannot apply the Nadler theorem, since the values of the set-valued mapping $\Psi^{-1}(\Phi(\cdot))$ may not be closed. Consider the corresponding example.

Example 2.1. Let $X = [0,1]$, $Y = \{0\} \cup [1, +\infty)$ $\Psi, \Phi : X \rightrightarrows Y$, $\Psi(x) = \{1/x\}$ if $x \neq 0$, $\Psi(0) = 0$, $\Phi(x) = [1, +\infty)$ for each x. The assumptions **(A1)-(A3)** are satisfied, in particular, Ψ is 1-covering. However,

$$\Psi^{-1}(\Phi(x)) = (0,1] \quad \forall \, x,$$

so, the values of $\Psi^{-1}(\Phi(\cdot))$ are not closed.

Note also that the set of all coincidence points $\xi \in X$ of the set-valued mappings Ψ and Φ is the set of all solutions of the equation $f(x) = 0$, where $f : X \to \mathbb{R}$, $f(x) = \operatorname{dist}(\Psi(x), \Phi(x))$ for all $x \in X$. However, in order to prove the existence of the desired coincidence point ξ one cannot apply theorems about solvability of the equation $f(x) = 0$ based on the iterative procedure of solution finding (see, for instance, [4], Theorem 1) because the function f may not be lower semi-continuous. Moreover, there can exist a point $x \in X$ and a sequence $\{x_n\} \subset X$ such that $x_n \to x$, $f(x_n) \to 0$ as $n \to \infty$ and $f(x) > 0$. Consider the corresponding example.

Example 2.2. Let $X = \mathbb{R}$, $Y = [0, \infty)$, $\Psi(x) = \{(x - n)^{-1} - 1\}$ for every $x \in (n, n + 1]$ and for every integer n, $\Phi(x) \equiv [1, \infty)$. The assumptions **(A1)-(A3)** are satisfied, in particular, Ψ is 1-covering. However, if a sequence of positive numbers x_n converges to 0 as $n \to \infty$, then $f(x_n)$ also converges to 0 as $n \to \infty$, since $f(x_n) = \operatorname{dist}\big(\Psi(x_n), [1, \infty)\big) = 0$ for sufficiently large n. Nevertheless, $f(0) = \operatorname{dist}\big(\{0\}, [1, \infty)\big) = 1 > 0$.

Let us turn back to Definition 2.1. Let (Z, ρ_Z) be a metric space. It is easy to observe that if a set-valued mapping $\Psi : X \rightrightarrows Y$ is α-covering and a set-valued mapping $\Theta : X \rightrightarrows Y$ is γ-covering, then their composition $\Theta(\Psi(\cdot)) : X \to Z$ is $\alpha\gamma$-covering. However, the composition of locally covering mappings is not necessarily locally covering. Consider the corresponding example.

Example 2.3. Assume that $X = Y = Z = \mathbb{R}$, $\Psi : X \rightrightarrows Y$, $\Psi(x) = \{x, 2\}$ for each $x \in \mathbb{R}$, $\Theta : Y \rightrightarrows Z$, $\Phi(y) = \{1 - |y - 1|\}$ for every $y \in \mathbb{R}$, and $x_0 = y_0 = z_0 = 0$. Then Ψ is locally 1-covering in a neighborhood of (x_0, y_0), Θ is locally 1-covering in a neighborhood of (y_0, z_0). Further, $\Theta(\Psi(x)) = \{1 - |x - 1|, 0\}$ for each $x \in \mathbb{R}$. Take an arbitrary neighborhoods U of x_0 and W of z_0. It is obvious that there exists a closed ball $B_X(x, r) \subset (0, 1)$ such that $B_X(x, r) \subset U$ and $\Theta(\Psi(B_X(x, r))) \subset V$. Then $\Theta(\Psi(B_X(x, r))) = \{z_0\} \cup [x - r, x + r]$ and $z_0 \notin [x - r, x + r]$. Thus, any neighborhood of z_0 is not included in $\Theta(\Psi(B_X(x, r)))$. So, $\Theta(\Psi(\cdot))$ is not locally α-covering in a neighborhood of (x_0, z_0) for every $\alpha > 0$.

3. Main results

Let us recall the definition of local covering from [16].

Definition 3.1. A set-valued mapping $\Psi : X \rightrightarrows Y$ is called α-covering on the set $U \subset X$ relative to the set $V \subset Y$, if

$$B_X(x, r) \subset U \Rightarrow B_Y\big(\Psi(x) \cap V, \alpha r\big) \subset \Psi\big(B_X(x, r)\big).$$

The set-valued mapping Ψ is called locally α-covering in a neighborhood of the point (x_0, y_0), if there exist neighborhoods U and V of the points x_0 and y_0, respectively, such that Ψ is α-covering on U relative to V.

Let us compare Definitions 2.1 and 3.1.

Theorem 3.1. 1) If the set-valued mapping Ψ is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 3.1, then this mapping is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 2.1.

2) Let x_0 be not an isolated point of X. If the set-valued mapping Ψ is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 2.1, then this mapping is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 3.1.

If x_0 is an isolated point of X, then a locally α-covering mapping in the sense of Definition 2.1 may not be locally α-covering in the sense of Definition 3.1. Consider the corresponding example.

Example 3.1. Let $X = \{x_0\}$, $Y = \mathbb{R}$, $\Psi : X \rightrightarrows Y$, $\Psi(x_0) = [-1, 1]$, $y_0 = 0$ and $U \subset X$, $V \subset \overline{R}$ be arbitrary neighborhoods of x_0 and y_0, respectively. For each $\alpha > 0$ set $r = 2\alpha^{-1}$. It is obvious that $B_X(x_0, r) \subset U$. Moreover,

$$B_Y(\Psi(x_0) \cap V, \alpha r) = B_Y(\Psi(x_0) \cap V, 2) \supset B_Y(y_0, 2) = [-2, 2].$$

Since $\Psi(B_X(x_0, r)) = [-1, 1]$, we have $B_Y(\Psi(x_0) \cap V, \alpha r) \not\subseteq \Psi(B_X(x_0, r))$. Therefore, Ψ is not locally α-covering in a neighborhood of a point (x_0, y_0) in the sense of Definition 3.1 for every $\alpha > 0$. However, for $V = (-1, 1)$ we have

$$B_Y(\Psi(x), \alpha r) \cap V = B_Y(\Psi(x_0), \alpha r) \cap V = (-1, 1) \subset \Psi(B_X(x_0, r)) = \Psi(B_X(x_0, r))$$

for each $B_X(x, r) \subset U$. So, Ψ is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 2.1 for every $\alpha > 0$.

Let us recall the definition of local covering from [15].

Definition 3.2. The set valued mapping $\Psi : X \rightrightarrows Y$ is called locally α-covering in a neighborhood of the point (x_0, y_0), if there exist neighborhoods $U \subset X$ and $V \subset Y$ of the points x_0 and y_0, respectively, and a number $\overline{R} > 0$ such that

$$O_Y(y, \alpha r) \subset \Psi\big(O_X(x, r)\big) \quad \forall\, x \in U, \quad \forall\, y \in \Psi(x) \cap V, \quad \forall\, r \in (0, \overline{R}). \tag{3.1}$$

Theorem 3.2. 1) If the set-valued mapping Ψ is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 3.2, then this mapping is locally $(\alpha - \varepsilon)$-covering in a neighborhood of (x_0, y_0) in the sense of Definition 2.1 for each $\varepsilon \in (0, \alpha)$.

2) If the set-valued mapping Ψ is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 2.1, then this mapping is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 3.2.

In [8] there was introduced a definition of covering on a family. Let us present a set-valued analogue of this definition. Assume that a set $\mathfrak{A} \subset X \times \mathbb{R}_+$ is given.

Definition 3.3. We will say that the set-valued mapping $\Psi : X \rightrightarrows Y$ α-covers the set V on the family $\mathfrak{A}$, if

$$(x, r) \in \mathfrak{B} \;\Rightarrow\; B_Y(\Psi(x), \alpha r) \cap V \subset \Psi\big(B_X(x, r)\big).$$

An analogue of the presented definition for single-valued mappings was introduced in [8]. Definition 3.3 is sufficiently broad, since it generalizes various definitions of covering which frequently appear in publications. For example,

1) if $\mathfrak{A} = \{(x, r) \,:\, B_X(x, r) \subset U\}$, then the set-valued mapping Ψ α-covers the set V on the family $\mathfrak{A}$ if and only if this mapping is α-covering with respect to the sets U and V (see Definition 2.1);

2) if $\mathfrak{B} = X \times \mathbb{R}_+$, $W = Y$, then the set-valued mapping Ψ α-covers the set V on the family $\mathfrak{A}$ if and only if this mapping is α-covering (see Definition 2.1);

3) in [14] if $\mathfrak{A} \neq \emptyset$ is such that

$$(x, r) \in \mathfrak{A} \;\text{ and }\; r' + \rho_X(x, x') \le r \;\;\Rightarrow\;\; (x', r') \in \mathfrak{A},$$

then the set-valued mapping Ψ that α-covers the V on the family $\mathfrak{A}$ is called α-covering the set V on the complete system $\mathfrak{B}$.

These statements follow directly from the introduced definitions, so we omit their proofs.

To complete this section we will compare Definition 3.3 and the concept of ordered covering from [10, 11]. Let $(X, \preceq)$, $(Y, \preceq)$ be partially ordered sets, $W \subset Y$. For arbitrary $x \in X$ denote

$$\Omega_X(x) = \{u \in X : \, u \preceq x\}.$$

Definition 3.4. The set-valued mapping $\Psi : X \rightrightarrows Y$ is called orderly covering the set W, if

$$O_Y\big(\Psi(x)\big) \cap W \subset \Psi\big(O_X(x)\big) \quad \forall\, x \in X.$$

This definition was introduced in [11]. In this paper it was proved that orderly covering and monotone mappings have a coincidence point under certain assumptions.

Let now (X, ρ_X), (Y, ρ_Y) be metric spaces, $\mathfrak{A} \subset X \times \mathbb{R}_+$. Define a partial order in $\mathfrak{A}$ as follows:

$$(x, r) \preceq (\overline{x}, \overline{r}) \;\Leftrightarrow\; \rho_X(x, \overline{x}) + r \le \overline{r}.$$

We define a partial order $\preceq$ in $Y \times \mathbb{R}_+$ analogically. For the set-valued mapping $\Psi : X \rightrightarrows Y$ define the set-valued mapping $\Psi_\alpha : \mathfrak{A} \rightrightarrows Y \times \mathbb{R}_+$ by formula

$$\Psi_\alpha(x, r) = \{(y, \alpha r) : \, y \in \Psi(x)\} \quad \forall\, (x, r) \in \mathfrak{A}.$$

Proposition 3.1. If the set-valued mapping $\Psi : X \rightrightarrows Y$ α-covers the set V on the family $\mathfrak{A}$, then the set-valued mapping $\Psi_\alpha : \mathfrak{A} \rightrightarrows Y \times \mathbb{R}_+$ orderly covers the set $W = V \times \mathbb{R}_+$.

This proposition follows directly from Definitions 3.3 and 3.4, so we omit its proof.

4. Proofs of the main results

In order to prove Theorem 3.1 let us consider the following auxiliary statement.

Lemma 4.1. If x_0 is not an isolated point of X, then there exists a sequence of numbers $R_j > 0$, $j = 1, 2, ...$, such that $R_j \to 0$ as $j \to \infty$ and

$$B_X(x, r) \subset B_X(x_0, R_j) \;\Rightarrow\; r \le 3R_j.$$

for every j.

Proof. Since x_0 is not an isolated point of X, there exists a sequence $\{x_j\} \in X$ such that $x_j \to x_0$ as $j \to \infty$ and $x_j \neq x$, $j = 1, 2, \ldots$. Set $R_j = 2^{-1}\rho_X(x_j, x)$, $j = 1, 2, \ldots$. Take an arbitrary ball $B_X(x, r) \subseteq B_X(x_0, R_j)$. Since $x_j \notin B_X(x_0, R_j)$ and $B_X(x, r) \subset B_X(x_0, R_j)$, we have $x_j \notin B_X(x, r)$, and, therefore, $\rho_X(x, x_j) > r$. Thus,

$$r < \rho_X(x, x_j) \le \rho_X(x, x_0) + \rho_X(x_0, x_j) \le R_j + 2R_j = 3R_j. \qquad \square$$

Proof of Theorem 3.1. 1) Let Ψ be locally α-covering in a neighborhood of (x_0, y_0)in the sense of Definition 3.1. Then there exists $R > 0$ such that

$$B_X(x, r) \subset B_X(x_0, R) \;\Rightarrow\; B_Y(\Psi(x) \cap B_Y(y_0, \alpha R), \alpha r) \subset \Psi(B_X(x, r)), \tag{4.1}$$

i.e., Ψ is α-covering on $B_X(x_0, R)$ relative to $B_Y(y_0, \alpha R)$.

At first let us assume that x_0 is not an isolated point of X. Then according to Lemma 4.1 there exists positive number $\overline{R} \le 4^{-1}R$ such that

$$B_X(x, r) \subset B_X(x_0, \overline{R}) \;\Rightarrow\; r \le 3\overline{R}. \tag{4.2}$$

Let us show that Ψ is α-covering with respect to $B_X(x_0,\overline{R})$ and $B_Y(y_0,\alpha\overline{R})$. Take an arbitrary ball $B_X(x,r) \subset B_X(x_0,\overline{R})$. For each $z \in B_Y(\Psi(x),\alpha r) \cap B_Y(y_0,\alpha\overline{R})$ we have $\rho_Y(y_0,z) \le \alpha\overline{R}$ and there exists $y \in \Psi(x)$ such that $\rho_Y(y,z) \le \alpha r$. So,

$$\rho_Y(y_0,y) \le \rho_Y(y_0,z) + \rho_Y(z,y) \le \alpha(\overline{R}+r) \le \alpha(\overline{R}+3\overline{R}) \le \alpha\left(\frac{1}{4}R + \frac{3}{4}R\right) \le \alpha R.$$

Hence, $y \in \Psi(x) \cap B_Y(y_0,\alpha R)$ and, therefore, $z \in B_Y(\Psi(x)\cap B_Y(y_0,\alpha R),\alpha r)$. Thus, (4.1) implies that $z \in \Psi(B_X(x,r))$ for every $z \in B_Y(\Psi(x),\alpha r) \cap B_Y(y_0,\alpha\overline{R})$. So, $B_Y(\Psi(x),\alpha r) \cap B_Y(y_0,\alpha\overline{R}) \subset \Psi\big(B_X(x,r)\big)$ and, therefore Ψ is α-covering with respect to $B_X(x_0,\overline{R})$ and $B_Y(y_0,\alpha\overline{R})$. Hence, Ψ is locally α-covering in a neighborhood of (x_0,y_0) in the sense of Definition 2.1.

Assume now that x_0 is an isolated point of X. Then there exists $\overline{R} > 0$ such that $B_X(x_0,\overline{R}) = \{x_0\}$. It is obvious that $B_X(x_0,\overline{R}) \subset B_X(x_0,R)$. So, (4.1) implies

$$B_Y(\Psi(x_0)\cap B_Y(y_0,\alpha R),\alpha\overline{R}) \subset \Psi(B_X(x_0,\overline{R})) = \Psi(x_0).$$

Since $y_0 \in \Psi(x_0) \cap B_Y(y_0,\alpha R)$, we have

$$B_Y(y_0,\alpha\overline{R}) \subset \Psi(x_0).$$

Take an arbitrary ball $B_X(x,r) \subset B_X(x_0,\overline{R})$. Obviously $B(x,r) = \{x_0\}$. Thus, if follows from the above inclusion that

$$\Psi(B_X(x,r)) \cap B_Y(y_0,\alpha\overline{R}) = \Psi(x_0) \cap B_Y(y_0,\alpha\overline{R}) \subset \Psi(x_0) = \Psi(B_X(x,r)).$$

So, Ψ is α-covering with respect to $B_X(x_0,\overline{R})$ and $B_Y(y_0,\alpha\overline{R})$. Therefore, Ψ is locally α-covering in a neighborhood of (x_0,y_0) in the sense of Definition 2.1.

2) Let Ψ be locally α-covering in a neighborhood of (x_0,y_0) in the sense of Definition 2.1. Then there exists $R > 0$ such that

$$B_X(x,r) \subset B_X(x_0,R) \Rightarrow B_Y(\Psi(x),\alpha r) \cap B_Y(y_0,\alpha R) \subset \Psi(B_X(x,r)), \tag{4.3}$$

i.e., Ψ is α-covering with respect to $B_X(x_0,R)$ and $B_Y(y_0,\alpha R)$. Lemma 4.1 implies that there exists $\overline{R} \le 4^{-1}R$ such that (4.2) holds.

Let us show that Ψ is α-covering on $B_X(x_0,\overline{R})$ relative to $B_Y(y_0,\alpha\overline{R})$. Take an arbitrary ball $B_X(x,r) \subset B_X(x_0,\overline{R})$. For each point $z \in B_Y(\Psi(x)\cap B_Y(y_0,\alpha\overline{R}),\alpha r)$ there exists $y \in \Psi(x)$ such that $\rho_Y(y_0,y) \le \alpha\overline{R}$ and $\rho_Y(y,z) \le \alpha r$. Therefore, $z \in B_Y(\Psi(x),\alpha r)$ and, moreover,

$$\rho_Y(y_0,z) \le \rho_Y(y_0,y) + \rho_Y(y,z) \le \alpha(\overline{R}+r) \le \alpha(\overline{R}+3\overline{R}) \le \alpha\left(\frac{1}{4}R + \frac{3}{4}R\right) \le \alpha R.$$

So, $z \in B_Y(\Psi(x),\alpha r) \cap B_Y(y_0,\alpha r)$. In virtue of (4.3) we have $z \in \Psi(B_X(x,r))$ for every $z \in B_Y\big(\Psi(x)\cap B_Y(y_0,\alpha\overline{R}),\alpha r\big)$. Hence, $B_Y\big(\Psi(x)\cap B_Y(y_0,\alpha\overline{R}),\alpha r\big) \subset \Psi\big(B_X(x,r)\big)$ and so Ψ is α-covering on $B_X(x_0,\overline{R})$ relative to $B_Y(y_0,\alpha\overline{R})$. Therefore, Ψ is locally α-covering in a neighborhood of (x_0,y_0) in the sense of Definition 3.1. □

Proof of Theorem 3.2. 1) Assume that Ψ is locally α-covering in a neighborhood of (x_0,y_0) in the sense of Definition 3.2. Then there exists $\overline{R} > 0$ such that (3.1) holds for $U = O_X(x_0,\overline{R})$, $V = O_X(y_0,\alpha\overline{R})$.

First let us assume that x_0 is not an isolated point of X. Lemma 4.1 implies that there exists positive $R < 4^{-1}\overline{R}$ such that (4.2) holds. Take an arbitrary $\varepsilon \in (0, \alpha)$ and show that Ψ is $(\alpha - \varepsilon)$-covering with respect to $B_X(x_0, R)$ and $B_Y(y_0, \alpha R)$.

For an arbitrary ball $B_X(x, r) \subset B_X(x_0, R)$ and a point $z \in B_Y(\Psi(x), (\alpha - \varepsilon)r) \cap B_X(y_0, \alpha R)$ inequality $\rho_Y(z, y_0) \leq \alpha R$ holds and there exists $y \in \Psi(x)$ such that $\rho_Y(y, z) \leq (\alpha - \varepsilon)r$. Therefore,

$$\rho_Y(y, y_0) \leq \rho_Y(y, z) + \rho_Y(z, y_0) < \alpha r + \alpha\overline{R} \leq 4\alpha R < \alpha\overline{R},$$

and, thus, $y \in V \cap \Psi(x)$. Moreover, $x \in U$. Further, it follows from (3.1) that $O_Y(y, \alpha r) \subset \Psi(O_X(x, r))$. Inequality $\rho_Y(y, z) \leq (\alpha - \varepsilon)r$ implies $z \in O_Y(y, \alpha r)$. Since $\Psi(O_X(x, r)) \subset \Psi(B_X(x, r))$, we obtain $z \in \Psi(B_X(x, r))$ for each $z \in B_Y(\Psi(x), (\alpha - \varepsilon)r) \cap B_X(y_0, \alpha R)$. Hence,

$$B_Y(\Psi(x), (\alpha - \varepsilon)r) \cap B_X(y_0, \alpha R) \subset \Psi(B_X(x, r)).$$

Therefore, Ψ is $(\alpha - \varepsilon)$-covering with respect to $B_X(x_0, R)$ and $B_Y(y_0, \alpha R)$. So, Ψ is locally $(\alpha - \varepsilon)$-covering in a neighborhood of (x_0, y_0) in the sense of Definition 2.1.

Assume now that x_0 is an isolated point of X. Without loss of generality we will assume that there exists $\overline{R} > 0$ such that $B_X(x_0, \overline{R}) = \{x_0\}$ and (3.1) holds. Then

$$O_Y(y_0, \alpha r) \subset \Psi(x_0) \quad \forall\, r < \overline{R}.$$

Take arbitrary numbers $\varepsilon > 0$, $R < \overline{R}$ and a ball $B_X(x, r) \subset B_X(x_0, R)$. It is obvious that $B_X(x, r) = \{x_0\}$, $x = x_0$. Thus,

$$B_Y(\Psi(x), (\alpha - \varepsilon)r) \cap B_X(y_0, \alpha R) \subset B_X(y_0, \alpha R) \subset \Psi(x_0) = \Psi(B_X(x, r)).$$

Therefore, Ψ is $(\alpha - \varepsilon)$-covering with respect to the balls $B_X(x_0, R)$ and $B_Y(y_0, \alpha R)$. So, this mapping is locally $(\alpha - \varepsilon)$-covering in a neighborhood of (x_0, y_0) in the sense of Definition 3.2.

2) Let Ψ be locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 2.1. Then there exists $R > 0$ such that (4.1) holds. Set

$$\overline{R} = \frac{R}{2}, \quad U = O_X(x_0, \overline{R}), \quad V = O_Y(y_0, \alpha\overline{R})$$

and prove that (3.1) holds.

Take arbitrary $r \in (0, \overline{R})$, $x \in U$, $y \in V \cap \Psi(x)$, $z \in O_Y(y, \alpha r)$. Set $\overline{r} = \alpha^{-1}\rho_Y(y, z)$. Then $\overline{r} < r$, $z \in B_Y(\Psi(x), \alpha\overline{r})$ and

$$\rho_Y(y_0, z) \leq \rho_Y(y_0, y) + \rho_Y(y, z) < \alpha\overline{R} + \alpha r < 2\alpha\overline{R} < \alpha R.$$

Therefore, $z \in B_Y(\Psi(x), \alpha\overline{r}) \cap B_Y(y_0, \alpha R)$. Moreover, $\rho_X(x_0, x) + \overline{r} < R + R < \overline{R}$. Thus, $B_X(x, \overline{r}) \subset B_X(x_0, \overline{R})$. Formula (4.1) implies

$$z \in B_Y\big(\Psi(x), \alpha\overline{r}\big) \cap B_Y(y_0, \alpha R) \subset \Psi\big(B_X(x, \overline{r})\big) \subset \Psi\big(O_X(x, r)\big).$$

for every $z \in O_Y(y, \alpha r)$. So, $O_Y(y, \alpha r) \subset \Psi\big(O_X(x, r)\big)$. Therefore, Ψ is locally α-covering in a neighborhood of (x_0, y_0) in the sense of Definition 3.2. □

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