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# REMARKS ON A LASALLE CONJECTURE ON GLOBAL ASYMPTOTIC STABILITY

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**Abstract.** In this paper we present some remarks on the following problem: Let $X$ be a (real or complex) Banach space, $\Omega \subset X$ be an open convex subset and $f : \Omega \to \Omega$ be an operator. We suppose that: $(i)$ $f \in C^1(X,X)$; $(ii)$ the differential of $f$ at $x$, $df(x) : X \to X$ is a Picard operator for all $x \in \Omega$; $(iii)$ the fixed point set of $f$, $F_f \neq \emptyset$. The problem is in which conditions $f$ is a Picard operator? In the case $X := \mathbb{R}^m$ or $X := \mathbb{C}^m$, this problem is in connection with a LaSalle Conjecture (J.P. LaSalle, *The Stability of Dynamical Systems*, SIAM, No. 25, 1976) and with the Belitskii-Lyubich Conjecture (G.R. Belitskii and Yu.I. Lyubich, *Matrix Norms and their Applications*, Birkhäuser, 1988).

We also formulate the following conjecture:

Let $X$ be a Banach space, $\Omega \subset X$ be an open convex subset and $f : \Omega \to \Omega$ be an operator. We suppose that: $(i)$ $f \in C^1(\Omega, X)$; $(ii)$ $df^k(x)$ is a Picard operator, $\forall\, x \in \Omega$, $\forall\, k \in \mathbb{N}^*$; $(iii)$ $F_f \neq \emptyset$. Then $f$ is a Picard operator.

Some research directions are also presented.
**Key Words and Phrases**: Banach space, differentiable nonlinear operator, fixed point, iterate, spectral radius, global asymptotic stability, Picard operator, LaSalle Conjecture, Belitskii-Lyubich Conjecture, discrete Markus-Yamabe Conjecture, Ostrowski property, stability under operator perturbation.
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## 1. Introduction

In [33], J.P. LaSalle formulated four conjectures. One of them is the following:

**LaSalle Conjecture.** *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be such that:*

$(i)$ *there exists $x^* \in \mathbb{R}^m$ with $f(x^*) = x^*$;*
$(ii)$ *$f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$;*
$(iii)$ *the spectral radius of the differential of $f$ at $x$, $\rho(df(x)) < 1$, for all $x \in \mathbb{R}^m$.*

*Then:*

$(a)$ *$F_f = \{x^*\}$, where $F_f := \{x \in \mathbb{R}^m \mid f(x) = x\}$;*
$(b)$ *$f^n(x) \to x^*$ as $n \to \infty$, $\forall\, x \in \mathbb{R}^m$.*

It is well known that (see [33], [47], ...), by definition a function $f$ as in $(a)$ and $(b)$ is a Picard function, and also, by definition a fixed point $x^*$ as in $(a)$ and $(b)$ is globally asymptotically stable.

There are many papers on the above conjecture. The results are as follow:

• counterexamples to LaSalle Conjecture: [12], [13], [14], [35], ...

• classes of functions for which LaSalle Conjecture is a theorem: [13], [2], [15], [16], [18], [35], ...

• to study the dynamic generated by a function $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, with $\rho(df(x)) < 1$, $\forall\, x \in \mathbb{R}^m$: [3], [10], [32], [33], [35], ...

The aim of this paper is to present some remarks on the LaSalle Conjecture and of the following question:

**Problem 1.1.** *Let $X$ be a (real or complex) Banach space, $\Omega \subset X$ be an open convex subset and $f : \Omega \to \Omega$ be an operator. We suppose that:*

$(i)$ $f \in C^1(\Omega, X)$;

$(ii)$ $df(x) : X \to X$ *is Picard for all* $x \in \Omega$;

$(ii)$ $F_f \neq \emptyset$.

*The problem is in which conditions $f$ is Picard operator.*

**Remark 1.1.** *It is clear that, $\rho(df(x)) < 1$, $\forall\, x \in \Omega$ implies the condition $(ii)$.*

The plan of the paper is the following:

2. Heuristic point of view
3. Metrical point of view
4. Classes of functions for which the LaSalle Conjecture is a theorem
5. Other research directions

5.1. Belitskii-Lyubich Conjecture

5.2. The case of a real Banach space

5.3. The case of a complex Banach space

5.4. Picard operators with Ostrowski property

5.5. Stability of Picard operators under operator perturbation

## 2. Heuristic point of view

Let $(X, \to)$ be an $L$-space ($(X, \tau)$ - topological space, $\stackrel{\tau}{\to}$; $(X, d)$ - metric space, $\stackrel{d}{\to}$; $(X, \|\cdot\|)$ - normed space, $\stackrel{\|\cdot\|}{\to}$, $\rightharpoonup$, ...) and $f : X \to X$ be an operator. By definition (see [47]) $f$ is Picard operator ($PO$) if:

$(i)$ $F_f = \{x^*\}$;

$(ii)$ $f^n(x) \to x^*$ as $n \to \infty$, $\forall\, x \in X$.

Also, by definition the unique fixed point of a Picard operator is a global attractor (see [36]).

From the above definition it follows:

**Lemma 2.1.** *If $f$ is Picard operator, then:*

$(a)$ $F_f = F_{f^n} = \{x^*\}$, $\forall\, n \in \mathbb{N}^* := \{1, 2, \ldots, n, \ldots\}$;

$(b)$ *all iterates, $f^k$, $k \in \mathbb{N}^*$, of $f$ are Picard operators.*

**Lemma 2.2.** *If $f$ is continuous and there exists $k \in \mathbb{N}^*$ such that $f^k$ is Picard operator, then $f$ is Picard operator.*

From the above considerations, our first remark is the following:

**Remark 2.1.** *It is reasonably (naturally) to look at $f^k$ when we choose conditions which imply that $f$ is Picard operator. For a better understanding of this remark, here are some examples.*

**Example 2.1** (Ostrowski Theorem (see [38]))**.** *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be such that:*

(*i*) *there exists $x^* \in \mathbb{R}^m$ with $f(x^*) = x^*$;*
(*ii*) *there exists a neighborhood $V(x^*)$ of $x^*$ such that $f \in C^1(V(x^*), \mathbb{R}^m)$;*
(*iii*) *$\rho(df(x^*)) < 1$.*

*Then there exists a neighborhood $V_1(x^*)$ of $x^*$ such that $V_1(x^*) \subset V(x^*)$, $f(V_1(x^*)) \subset V_1(x^*)$ and $f\big|_{V_1(x^*)} : V_1(x^*) \to V_1(x^*)$ is a Picard operator.*

*In this case $\rho(df(x^*)) < 1$ implies that, $\rho(df^k(x^*)) < 1$, for all $k \in \mathbb{N}^*$. Indeed, we have*

$$df^k(x^*) = df(f^{k-1}(x^*))df^{k-1}(x^*) = \ldots = (df(x^*))^k.$$

*So, $\rho(df^k(x^*)) = \rho((df(x^*))^k) = (\rho(df(x^*)))^k < 1$.*

**Example 2.2** (see [29])**.** *We have a similar situation in the case of Kitchen Theorem, which is a generalization of Ostrowski Theorem for an operator $f : X \to X$ where $X$ is a (real or complex) Banach space and $f$ satisfies similar conditions.*

*We remember that if $(X, \|\cdot\|)$ is a complex Banach space and $f : X \to X$ is a bounded linear operator with the spectrum $\sigma(f)$, then (see* [4], [23], [28], [5]*)*

$$\rho(f) := \sup_{\lambda \in \sigma(f)} |\lambda| = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}^*} \|f^n\|^{\frac{1}{n}} = \inf_{|\cdot| \sim \|\cdot\|} |f|.$$

*If $X$ is a real Banach space and $f : X \to X$ is a bounded linear operator, $X_{\mathbb{C}}$ the complexification of $X$, $f_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ the complexification of $f$, then by definition, $\rho(f) := \rho(f_{\mathbb{C}})$.*

**Example 2.3** (see [23], [27], [28])**.** *Let $X$ be a Banach space and $f : X \to X$ be a bounded linear operator. If $\rho(f) < 1$, then $f$ is a Picard operator. In this example, $df^k(x) = f^k$, $\forall\ x \in X$ and $k \in \mathbb{N}^*$, and*

$$\rho(df^k(x)) = \rho(f^k) = (\rho(f))^k < 1.$$

**Example 2.4** (see [12])**.** *The function, $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by,*

$$f(x_1, x_2, x_3) := \Big(\frac{x_1}{2} + x_3(x_1 + x_2x_3)^2, \frac{x_2}{2} - (x_1 + x_2x_3)^2, \frac{x_3}{2}\Big),$$

*is a counterexample to LaSalle Conjecture, i.e.,*

$$\rho(df(x_1, x_2, x_3)) < 1, \text{ for all } (x_1, x_2, x_3) \in \mathbb{R}^3,$$

*$f(0) = 0$ and $f$ is not a Picard function. In this case, for example, $\rho(df^2(2, 0, 2)) > 1$. Indeed, we have that*

$$f'(x_1, x_2, x_3) =$$

$$= \begin{pmatrix} \frac{1}{2} + 2x_3(x_1 + x_2x_3) & 2x_3^2(x_1 + x_2x_3) & (x_1 + x_2x_3)^2 + 2x_2x_3(x_1 + x_2x_3) \\ -2(x_1 + x_2x_3) & \frac{1}{2} - 2x_3(x_1 + x_2x_3) & -2x_2(x_1 + x_2x_3) \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

*and*

$$(f^2)'(2,0,2) = f'(f(2,0,2))f'(2,0,2) = f'(9,-3,1)f'(2,0,2).$$

*By a simple calculation we have that,* $\rho((f^2)'(2,0,2)) > 1$.

So, we have the second remark.

**Remark 2.2.** *In the case of the LaSalle Conjecture, in general,* $\rho(df(x)) < 1$, $\forall\ x \in \mathbb{R}^m$, *does not imply that* $\rho(df^k(x)) < 1$, $\forall\ x \in \mathbb{R}^m$, $\forall\ k \in \mathbb{N}^*$. *So, the reasonable conjecture is the following:*

**Conjecture 2.1.** *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *be such that:*

($i$) $F_f \neq \emptyset$;
($ii$) $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$;
($iii$) $\rho(df^k(x)) < 1$, $\forall\ x \in \mathbb{R}^m$, $\forall\ k \in \mathbb{N}^*$.

*Then* $f$ *is a Picard operator.*

## 3. Metrical point of view

A metrical condition which is invariant by iteration is nonexpansivity. If we put this condition on $f : \mathbb{R}^m \to \mathbb{R}^m$ with respect to a suitable norm on $\mathbb{R}^m$ we have the following result.

**Theorem 3.1.** *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *be such that:*

($i$) *there exists* $x^* \in \mathbb{R}^m$ *with,* $f(x^*) = x^*$;
($ii$) *there exists a neighborhood* $V(x^*)$ *of* $x^*$ *such that* $f \in C^1(V(x^*), \mathbb{R}^m)$;
($iii$) $\rho(df(x^*)) < 1$;
($iv$) $f$ *is nonexpansive with respect to a strict convex norm on* $\mathbb{R}^m$.

*Then:*

($a$) $F_f = F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$;
($b$) $f_\lambda^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in \mathbb{R}^m$, $\forall\ \lambda \in ]0,1[$, *where* $f_\lambda$ *is the Krasnoselskii operator,* $f_\lambda(x) := (1-\lambda)x + \lambda f(x)$.

*Proof.* ($a$). Let $\|\cdot\|$ be a strict convex norm on $\mathbb{R}^m$ such that

$$\|f(x) - f(y)\| \leq \|x - y\|,\ \forall\ x, y \in \mathbb{R}^m.$$

This condition implies that $F_f$ is a convex subset of $\mathbb{R}^m$. On the other hand, by Ostrowski Theorem, condition ($iii$) implies that $x^*$ is an isolated fixed point, i.e., $F_f = \{x^*\}$. Since, $x^* \in F_{f^n}$, $f^n$ is nonexpansive and $\rho(df^n(x^*)) < 1$, we have that $F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$.

($b$). Let for $x \in \mathbb{R}^m$, $r > 0$ be such that $x \in \overline{B}(x^*, r)$. It is clear that, $f_\lambda(\overline{B}(x^*, r)) \subset \overline{B}(x^*, r)$. By a Ishikawa theorem (see [9]), $f_\lambda : \overline{B}(0, r) \to \overline{B}(0, r)$ is asymptotically regular. But $\{f_\lambda^n\}_{n \in \mathbb{N}}$ has a convergent subsequence, $f_\lambda^{n_i}(x) \to y^* \in F_f$. Since $f_\lambda$ is nonexpansive it follows that (see [6], [9])

$$f_\lambda^n(x) \to y^* = x^*,\ \text{as } n \to \infty.$$

So, $f_\lambda$ is a Picard operator for each $\lambda \in ]0,1[$. □

**Remark 3.1.** *For more considerations on Krasnoselskii operator see:* [6], [9], [48], [54], ...

A way to have nonexpansivity for the function $f$ is to use the singular values of $df(x)$. So, we have

**Theorem 3.2.** *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *be such that:*

$(i)$ $F_f \neq \emptyset$;
$(ii)$ $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$;
$(iii)$ $\rho((df(x))^T df(x)) < 1$, $\forall\, x \in \mathbb{R}^m$.

*Then:*

$(a)$ $f$ *is contractive with respect to the* $\|\cdot\|_2$ *norm on* $\mathbb{R}^m$;
$(b)$ $F_f = F_{f^n} = \{x^*\}$, $\forall\, n \in \mathbb{N}$;
$(c)$ $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\, x \in \mathbb{R}^m$.

*Proof.* We consider on $\mathbb{R}^m$ the $\|\cdot\|_2$ norm. The condition $(iii)$ imply that (see [5], [28], [38], [42]), $\|df(x)\|_2 < 1$, $\forall\, x \in \mathbb{R}^m$. So, we have $(a)$. Since $(i)$ and $(a)$ imply that $F_f = \{x^*\}$ and $f(\overline{B}(x^*, r)) \subset \overline{B}(x^*, r)$, $\forall\, r > 0$, from the Niemytzki-Edelstein theorem (see [49], p.38) we have $(b)$ and $(c)$ □

**Remark 3.2.** *For more considerations on contractive operators see:* [7], [38], [42], [43], [49], ...

If we take on $\mathbb{R}^m$ the $\|\cdot\|_\infty$ norm, then from the Mean-Value Theorem (for a function from $\mathbb{R}^m$ to $\mathbb{R}$) we have the following result.

**Theorem 3.3.** *Let* $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, $f = (f_1, \ldots, f_m)$, *be such that*

$$\sum_{j=1}^{m} \left| \frac{\partial f_k(x)}{\partial x_j} \right| < 1, \ \forall\, x \in \mathbb{R}^m, \ k = \overline{1, m}.$$

*Then* $f$ *is contractive with respect to* $\|\cdot\|_\infty$ *norm on* $\mathbb{R}^m$. *Moreover if in addition,* $F_f \neq \emptyset$, *then* $f$ *is a Picard operator.*

## 4. Classes of functions for which LaSalle Conjecture is a theorem

### 4.1. Triangular functions

Let $f : \mathbb{R}^m \to \mathbb{R}^m$, $f(x_1, \ldots, x_m) = (f_1(x_1), f_2(x_1, x_2), \ldots, f_m(x_1, \ldots, x_m))$ be a triangular function. In [13] the authors prove that for this class of functions the LaSalle Conjecture is a theorem. Other results for triangular functions are given in [2], [16] and [18]. From the following abstract result we have a new result in which the condition, $F_f \neq \emptyset$, does not appear.

**Fiber Contraction Theorem** (see [45], [47], [49], [52]). *Let* $(X_k, d_k)$ *be a complete metric space,* $k = \overline{1, m}$. *Let* $f_k : X_1 \times \ldots \times X_k \to X_k$, $k = \overline{1, m}$ *and* $f = (f_1, \ldots, f_m) : \prod_{k=1}^{m} X_k \to \prod_{k=1}^{m} X_k$. *We suppose that:*

$(i)$ $f_1$ *is a Picard operator;*

(*ii*) $f_k(x_1,\dots,x_{k-1},\cdot): X_k \to X_k$ *is* $l_k$*-contraction,* $k=\overline{2,m}$, $\forall\ (x_1,\dots,x_{k-1}) \in \mathbb{R}^{k-1}$.

*Then:*

(*a*) $F_f = \{x^*\}$;

(*b*) *if* $f$ *is continuous in* $x^*$, *then* $f$ *is a Picard operator.*

**Theorem 4.1.** *Let* $f:\mathbb{R}^m \to \mathbb{R}^m$ *be a triangular function. We suppose that:*

(*i*) $f_1' \in C(\mathbb{R},\mathbb{R})$ *and there exists* $l_1 \in [0,1[$ *such that,* $|f_1'(x_1)| \le l_1$, $\forall\ x_1 \in \mathbb{R}$;

(*ii*) $\frac{\partial}{\partial x_k} f_k(x_1,\dots,x_{k-1},\cdot) \in C(\mathbb{R},\mathbb{R})$, $\forall\ (x_1,\dots,x_{k-1}) \in \mathbb{R}^{k-1}$, $k=\overline{2,m}$ *and there exists* $l_k \in [0,1[$ *such that*

$$\Big|\frac{\partial}{\partial x_k} f_k(x_1,\dots,x_k)\Big| \le l_k,\ \forall\ (x_1,\dots,x_k) \in \mathbb{R}^k,\ k=\overline{2,m};$$

(*iii*) $f$ *is continuous.*

*Then:*

(*a*) $F_f = F_{f^n} = \{x^*\}$, $\forall\ n \in \mathbb{N}^*$;

(*b*) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall\ x \in \mathbb{R}^m$.

*Proof.* From the Mean-Value Theorem for the functions $f_k(x_1,\dots,x_{k-1},\cdot)$ we are in the conditions of the Fiber Contraction Theorem. □

## 4.2. The class of functions $f:\mathbb{R}^m \to \mathbb{R}^m,\ f(x_1,\dots,x_m) = (x_2,\dots,x_m,h(x_1,\dots,x_m))$

In [15] the authors consider the class of functions $f:\mathbb{R}^m \to \mathbb{R}^m$ defined by, $f(x_1,\dots,x_m) := (x_2,\dots,x_m,h(x_1,\dots,x_m))$, where $h$ is a function from $\mathbb{R}^m$ to $\mathbb{R}$. A. Cima, A. Gasull and F. Mañosas ([15]) present counterexamples to the LaSalle Conjecture (i.e., the third LaSalle conjecture in [33]). On the other hand they prove that if instead of $\rho\big(\frac{\partial f_k(x)}{\partial x_j}\big) < 1$ one put $\rho\big(\big|\frac{\partial f_k(x)}{\partial x_j}\big|\big) < 1$, then the conjecture is a theorem, i.e., the fourth LaSalle conjecture is a theorem for this class of functions.

On the other hand in the theory of difference equations appears this class of functions (see [46] for example). The difference equation

$$x_{n+m} = h(x_n,\dots,x_{n+m-1}),\ n \in \mathbb{N},\ (x_0,\dots,x_{m-1}) \in \mathbb{R}^m,$$

was studied by many authors. See for example: [7], [38], [44], [46], [51], [52], [56], … The following question arises.

**Problem 4.1.** *To apply these metric results to find classes of functions for which LaSalle Conjecture is a theorem.*

*For example, the following result is given in* [15].

**Theorem 4.2.** *We suppose that:*

(*i*) $h \in C^1(\mathbb{R}^m,\mathbb{R})$;

(*ii*) *there exists* $x^* \in \mathbb{R}^m$ *with* $f(x^*) = x^*$;

(*iii*) $\sum_{j=1}^{m} \Big|\frac{\partial h(x)}{\partial x_j}\Big| < 1$, $\forall\ x \in \mathbb{R}^m$.

*Then:*

(a) $F_f = \{x^*\}$;
(b) $f^n(x) \to x^*$ as $n \to \infty$, $\forall\, x \in \mathbb{R}^m$.

Using Lemma 2.2 and Theorem 3.3 we shall give a new proof for Theorem 4.2. To do this we remark that $f^m$ satisfies the conditions of the Theorem 3.3. Indeed, first we remark that, $\big|(\tilde{h})'(u)\big| < 1, \forall\, u \in \mathbb{R}$, where $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is defined by $\tilde{h}(u) := h(u, \dots, u)$. So, $\tilde{h}$ is a contractive function. On the other hand we have that

$$(x_1, \dots, x_m) \in F_f \;\Leftrightarrow\; x_1 = \ldots = x_m = u \in F_{\tilde{h}}.$$

The contractivity of $\tilde{h}$ implies, $F_f = \{x^*\}$.

Now we shall prove that the condition $(iii)$ implies that

$$\sum_{j=1}^{m} \Big|\frac{\partial (f^m)_k(x)}{\partial x_j}\Big| < 1,\ \forall\, x \in \mathbb{R}^m,\ k = \overline{1, m}.$$

For a better understanding of the proof and for simplicity we shall present the proof in the case $m = 2$ and $m = 3$.

In the case $m = 2$ we have that

$$f^2(x_1, x_2) = ((f^2)_1, (f^2)_2) = (h(x_1, x_2), h(x_2, h(x_1, x_2))$$

and

$$\sum_{j=1}^{2} \frac{\partial (f^2)_2}{\partial x_j}(x_1, x_2) = \frac{\partial h}{\partial x_2}(x_2, h(x_1, x_2)) \cdot \frac{\partial h}{\partial x_1}(x_1, x_2) + \frac{\partial h}{\partial x_1}(x_2, h(x_1, x_2)) + \\ + \frac{\partial h}{\partial x_2}(x_2, h(x_1, x_2)) = \frac{\partial h}{\partial x_1}(x_2, h(x_1, x_2)) + \\ + \frac{\partial h}{\partial x_2}(x_2, h(x_1, x_2))\big[\frac{\partial h}{\partial x_1}(x_1, x_2) + \frac{\partial h}{\partial x_2}(x_1, x_2)\big].$$

From the condition $(iii)$ we have that

$$\sum_{j=1}^{2} \Big|\frac{\partial (f^2)_2}{\partial x_j}(x_1, x_2)\Big| \le \Big|\frac{\partial h}{\partial x_1}(x_2, h(x_1, x_2))\Big| + \\ + \Big|\frac{\partial h}{\partial x_2}(x_2, h(x_1, x_2))\Big|\Big[\Big|\frac{\partial h}{\partial x_1}(x_1, x_2)\Big| + \Big|\frac{\partial h}{\partial x_2}(x_1, x_2)\Big|\Big] < \\ < \Big|\frac{\partial h}{\partial x_1}(x_2, h(x_1, x_2))\Big| + \Big|\frac{\partial h}{\partial x_2}(x_2, h(x_1, x_2))\Big| < 1,\ \forall\ (x_1, x_2) \in \mathbb{R}^2.$$

For the case $m = 3$ we have

$$f^3(x_1, x_2, x_3) = \big(h(x_1, x_2, x_3), h(x_2, x_3, h(x_1, x_2, x_3)), \\ h(x_3, h(x_1, x_2, x_3), h(x_2, x_3, h(x_1, x_2, x_3)))\big)$$

and

$$\sum_{j=1}^{3}\frac{\partial (f^3)_2}{\partial x_j}(x_1,x_2,x_3)=\frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\cdot\frac{\partial h}{\partial x_1}(x_1,x_2,x_3)+$$
$$+\frac{\partial h}{\partial x_1}(x_2,x_3,h(x_1,x_2,x_3))+\frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\cdot\frac{\partial h}{\partial x_2}(x_1,x_2,x_3)+$$
$$+\frac{\partial h}{\partial x_2}(x_2,x_3,h(x_1,x_2,x_3))+\frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\cdot\frac{\partial h}{\partial x_3}(x_1,x_2,x_3))=$$
$$=\frac{\partial h}{\partial x_1}(x_2,x_3,h(x_1,x_2,x_3))+\frac{\partial h}{\partial x_2}(x_2,x_3,h(x_1,x_2,x_3))+$$
$$+\frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\Big[\frac{\partial h}{\partial x_1}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_2}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_3}(x_1,x_2,x_3)\Big].$$

From this it follows that

$$\sum_{j=1}^{3}\Big|\frac{\partial (f^3)_2}{\partial x_j}(x_1,x_2,x_3)\Big|<1.$$

Also we have

$$\sum_{j=1}^{3}\frac{\partial (f^3)_3}{\partial x_j}(x_1,x_2,x_3)\ =\frac{\partial h}{\partial x_2}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot$$
$$\cdot\frac{\partial h}{\partial x_1}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_3}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot$$
$$\cdot\frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\cdot\frac{\partial h}{\partial x_1}(x_1,x_2,x_3)+$$
$$+\frac{\partial h}{\partial x_2}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot\frac{\partial h}{\partial x_2}(x_1,x_2,x_3)+$$
$$+\frac{\partial h}{\partial x_3}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot\frac{\partial h}{\partial x_1}(x_2,x_3,h(x_1,x_2,x_3))+$$
$$+\frac{\partial h}{\partial x_3}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot\frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\cdot$$
$$\cdot\frac{\partial h}{\partial x_2}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_1}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))+$$
$$+\frac{\partial h}{\partial x_2}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot\frac{\partial h}{\partial x_3}(x_1,x_2,x_3)+$$
$$+\frac{\partial h}{\partial x_3}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot\frac{\partial h}{\partial x_2}(x_2,x_3,h(x_1,x_2,x_3))+$$
$$+\frac{\partial h}{\partial x_3}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot\frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\cdot$$
$$\cdot\frac{\partial h}{\partial x_3}(x_1,x_2,x_3)=\frac{\partial h}{\partial x_2}(x_3,h(x_1,x_2,x_3),h(x_2,x_3,h(x_1,x_2,x_3)))\cdot$$
$$\cdot\big[\frac{\partial h}{\partial x_1}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_2}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_3}(x_1,x_2,x_3)\big]+$$

$$+\frac{\partial h}{\partial x_3}(x_3, h(x_1,x_2,x_3), h(x_2,x_3,h(x_1,x_2,x_3)))\cdot \frac{\partial h}{\partial x_3}(x_2,x_3,h(x_1,x_2,x_3))\cdot$$
$$\cdot\big[\frac{\partial h}{\partial x_1}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_2}(x_1,x_2,x_3)+\frac{\partial h}{\partial x_3}(x_1,x_2,x_3)\big]+$$
$$+\frac{\partial h}{\partial x_3}(x_3, h(x_1,x_2,x_3), h(x_2,x_3,h(x_1,x_2,x_3)))\cdot\big[\frac{\partial h}{\partial x_1}(x_2,x_3,h(x_1,x_2,x_3))+$$
$$+\frac{\partial h}{\partial x_2}(x_2,x_3,h(x_1,x_2,x_3))\big]+\frac{\partial h}{\partial x_1}(x_3, h(x_1,x_2,x_3), h(x_2,x_3,h(x_1,x_2,x_3))).$$

From this we have that

$$\sum_{j=1}^{3}\big|\frac{\partial (f^3)_3}{\partial x_j}(x_1,x_2,x_3)\big|<1,\ \forall\ (x_1,x_2,x_3)\in\mathbb{R}^3.$$

So, $f^m$ is Picard operator. Now the proof follows from Lemma 2.2.

## 5. Other research directions

### 5.1. Belitskii-Lyubich Conjecture

In [5] (p. 41) G.R. Belitskii and Yu.I. Lyubich formulated the following conjecture:

*Let $\mathbb{K}:=\mathbb{R}$ or $\mathbb{C}$, $\Omega\subset\mathbb{K}^m$ be open subset, $\Omega_1\subset\mathbb{K}^m$ be a compact convex subset with $\Omega_1\subset\Omega$. Let $f:\Omega\to\mathbb{K}^m$ be a function. We suppose that:*

*(i)* $f\in C^1(\Omega,\mathbb{K}^m)$;
*(ii)* $f(\Omega_1)\subset\Omega_1$;
*(iii)* $\rho(df(x))<1$, $\forall\ x\in\Omega_1$.

*Then $f\big|_{\Omega_1}:\Omega_1\to\Omega_1$ is a Picard operator.*

**Commentaries:**

(1) From Brouwer fixed point theorem it follows that, $F_f\neq\emptyset$.

(2) In the paper [53], M.-H. Shih and J.-W. Wu have given a counterexample in the case $\mathbb{K}:=\mathbb{R}$ and $m:=2$. For example, let $\Omega_1:=\{(x_1,x_2)\in\mathbb{R}^2\mid |x_1|+|x_2|\le 1\}$ and $f:\Omega_1\to\Omega_1$ be defined by, $f(x_1,x_2):=(\varphi(x_2),\varphi(x_1))$, where

$$\varphi(t):=\begin{cases}4(t-\frac{1}{2})^2 & \text{for } \frac{1}{2}\le t\le 1;\\ 0 & \text{for } |t|\le\frac{1}{2};\\ 4(t+\frac{1}{2})^2 & \text{for } -1\le t\le -\frac{1}{2}.\end{cases}$$

We remark that:

*(i)* $F_f=\{(0,0)\}$;
*(ii)* $\rho(f'(x_1,x_2))=0$, $\forall\ (x_1,x_2)\in\Omega_1$.

On the other hand, $F_{f^2}=\{(0,0),(0,1),(1,0)\}$. This implies that $f$ is not a Picard function (see Lemma 2.1).

In this counterexample,

$$\rho((f^2)'(0,1))=4>1.$$

Indeed we have

$$(f^2)'(0,1) = f'(f(0,1))f'(0,1) = f'(1,0)f'(0,1) = \\ = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}.$$

(3) Shih and Wu ([53]) prove that the Belitskii-Lyubich Conjecture is a theorem in the case $\mathbb{K} := \mathbb{C}$.

(4) In the same paper Shih and Wu give the following result:

**Theorem 5.1.** *Let $X$ be a complex Banach space, $\Omega \subset X$ be a nonempty, bounded, open and convex subset and $f : \Omega \to \Omega$ be compact and holomorphic function with $f(x^*) = x^*$. Then $x^*$ is globally asymptotically stable if and only if, $\rho(df(x^*)) < 1$.*

(5) From the Kitchen Theorem ([29]) and our heuristic point of view (see §2) the following open problem arises:

**Conjecture 5.1.** *Let $X$ be a real Banach space, $\Omega \subset X$ be an open subset, $\Omega_1 \subset \Omega$ be bounded, closed and convex and $f \in C^1(\Omega, X)$. We suppose that:*

(*i*) $f(\Omega_1) \subset \Omega_1$;
(*ii*) $f\big|_{\Omega_1} : \Omega_1 \to \Omega_1$ *is a compact operator;*
(*iii*) $\rho(df^k(x)) < 1$, $\forall\, x \in \Omega_1$, $\forall\, k \in \mathbb{N}^*$.

*Then $f\big|_{\Omega_1} : \Omega_1 \to \Omega_1$ is a Picard operator.*

(6) References: [11], [24], [29], [30], [40], [53].

## 5.2. The case of a real Banach space

From the above considerations in this paper the following open problem arises:

**Conjecture 5.2.** *Let $X$ be a real Banach space and $f : X \to X$ be an operator. We suppose that:*

(*i*) $f \in C^1(X, X)$;
(*ii*) $\rho(df^k(x)) < 1$, $\forall\, x \in X$, $\forall\, k \in \mathbb{N}^*$.
(*iii*) *there exists $x^* \in X$ with $f(x^*) = x^*$.*

*Then $f$ is a Picard operator.*

**Commentaries:**

(1) We think it is useful to look to the following problems:

($A$) There exist counterexamples to LaSalle Conjecture which satisfy the conditions of Conjecture 5.2?

($B$) In which conditions the following implication holds:

$$f \in C^1(X, X),\ \rho(df(x)) < 1,\ \forall\, x \in X\ \Rightarrow \rho(df^k(x)) < 1,\ \forall\, x \in X,\ \forall\, k \in \mathbb{N}^*?$$

($C$) There exist some connections between Conjecture 5.1 and Conjecture 5.2?

(2) References: [12], [13], [14], [23], [28], [29], [35], [36], [39], [41].

## 5.3. The case of a complex Banach space

The LaSalle Conjecture take the following form in this case.

*Let $X$ be a complex Banach space and $f : X \to X$ be an operator. We suppose that:*

$(i)$ *$f$ is holomorphic operator;*
$(ii)$ *$\rho(df(x)) < 1$, $\forall\, x \in X$;*
$(iii)$ *$F_f \neq \emptyset$.*

*Then $f$ is a Picard operator.*

**Commentaries:**

(1) As in the real case, the LaSalle Conjecture is a theorem for a triangular function, $f : \mathbb{C}^m \to \mathbb{C}^m$.

(2) It is useful to study the connections between LaSalle Conjecture and Belitskii-Lyubich Conjecture in a omplex Banach space.

(3) References: [11], [53], [1], [24], [30], [32], [40], [45], [55], [31], [57], [58].

### 5.4. Picard operators with Ostrowski property

Let $(X, +, \mathbb{K}, \|\cdot\|)$ be a Banach space, $f : X \to X$ be a Picard operator.
By definition, $f$ has the Ostrowski property (limit shadowing property in [17], [21], [48], [49], [50], [59]; plus-global stability in [16]) if the following implication holds ($F_f = \{x^*\}$):

$$y_n \in X,\ \|y_{n+1} - f(y_n)\| \to 0 \text{ as } n \to \infty \ \Rightarrow y_n \to x^* \text{ as } n \to \infty.$$

The problem is in which conditions a Picard operator has the Ostrowski property?

**Commentaries:**

(1) The notion operator with the Ostrowski property arise from Ostrowski Theorem on contraction (see [38], p. 394). In [37] the authors prove this property for Schröder-Perov contraction (see [49], [42]). Other examples of generalized contractions were given in [49], [50], [6], [26], [37]. The following problem is an open one: Which generalized contractions have the Ostrowski property?

(2) Let $f \in C^1(X, X)$ be such that

$(i)$ $F_f \neq \emptyset$;
$(ii)$ $\rho(df^k(x)) < 1$, $\forall\, x \in X$, $\forall\, k \in \mathbb{N}^*$.

In which conditions the operator $f$ is a $PO$ with Ostrowski property?

(3) References: [6], [16], [17], [21], [26], [37], [38], [48], [49], [59].

### 5.5. Stability of Picard operators under operator perturbations

Let $(X, +, \mathbb{K}, \|\cdot\|)$ be a Banach space, $f : X \to X$ be an operator. There exist notions of fixed points and of iteration processes stability under operator perturbations. The problem is what we understand by stability, under operator perturbations, of a global asymptotic stable fixed point? In other words, what we understand by stability, under operator perturbation of a Picard operator?

**Commentaries:**

(1) From the dynamical system point of view the problem is the following:
Let $f \in C^1(X, X)$ be such that:

(*i*) $\rho(df(x)) < 1$, $\forall\, x \in X$;

(*ii*) $f$ is a Picard operator.

In which conditions the discrete dynamical system, $(X, f)$ is structurally stable?

(2) Let $(X, +, \mathbb{K}, \|\cdot\|)$ be a Banach space, $f, g : X \to X$ be two operators. We suppose that:

(*i*) $f$ is Picard operator ($F_f = \{x^*\}$);

(*ii*) $\|f(x) - g(x)\| \leq \eta$, $\forall\, x \in X$, for some $\eta \in \mathbb{R}_+^*$.

The problem is to give an estimate of $\|g^n(x) - x^*\|$.

(3) Let $f, g : X \to X$ be such that:

(*i*) $f, g \in C^1(X, X)$;

(*ii*) $\rho(df^k(x)) < 1$, $\forall\, x \in X$, $\forall\, k \in \mathbb{N}^*$.

In which conditions we have that

$$\rho(d(f+g)^k(x)) < 1,\ \forall\, x \in X,\ \forall\, k \in \mathbb{N}^*?$$

(4) Following K. Goebel (1967), an operator $g : (X, \|\cdot\|) \to (X, \|\cdot\|)$ is called a strong contraction if for every $\varepsilon > 0$ there exists a norm, $\|\cdot\|_\varepsilon$, on $X$ equivalent with $\|\cdot\|$ such that

$$\|g(x) - g(y)\| \leq \varepsilon \|x - y\|_\varepsilon,\ \forall\, x, y \in X.$$

Let $f : X \to X$ be a Picard operator and $g : X \to X$ be a strong contraction. The problem is in which conditions on $f$, $f + g$ is Picard operator?

For example, let $X := C[0, 1]$ with *max* norm and $f, g : X \to X$. We suppose that:

(*i*) there exists $l \in [0, 1[$ such that:

$$|f(x)(t) - f(y)(t)| \leq l|x(t) - y(t)|,\ \forall\, x, y \in X,\ t \in [0, 1];$$

(*ii*) $g(x)(t) := \int_0^t K(t, s)x(s)ds$ with $K \in C([0, 1] \times [0, 1])$, $\|K\| \leq 1$.

Then:

(*a*) $f$ is a $l$-contraction with respect to

$$\|x\|_\varepsilon := \max_{0 \leq t \leq 1} \left(|x(t)| e^{-\frac{t}{\varepsilon}}\right)$$

for all $\varepsilon > 0$;

(*b*) $g$ is a strong contraction $(C[0, 1], \|\cdot\|)$;

(*c*) $f + g$ is $(l + \varepsilon)$-Lipschitz with respect to $\|\cdot\|_\varepsilon$, for all $\varepsilon > 0$, i.e., $f + g$ is a Picard operator.

(5) References: [6], [8], [19], [20], [22], [25], [31], [34], [39], [43], [48], [50].

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