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STRONG CONVERGENCE ALGORITHM FOR HIERARCHICAL FIXED POINT PROBLEMS OF A FINITE FAMILY OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we propose an iterative algorithm for hierarchical fixed point problems of a finite family of nonexpansive mappings in the setting of real Hilbert spaces. We prove that the sequence generated by the proposed method algorithm converges strongly to a fixed point of a finite family of nonexpansive mappings which is also the solution of a variational inequality. Numerical examples are presented to illustrate the proposed method and convergence result. The iterative algorithm and results presented in this paper generalize, unify and improve the previously known results of this area.

Key Words and Phrases: Hierarchical fixed point problem, fixed point problem, nonexpansive mapping, averaged mapping.

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1. Introduction

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We also assume that $T : H \to H$ is a nonexpansive operator, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in H : Tx = x\}$. It is well-known that $F(T)$ is closed and convex.

Let C be a nonempty closed convex subset of H and $S : C \to H$ be a nonexpansive mapping. The hierarchical fixed point problem (in short, HFPP) is to find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \tag{1.1}$$

It is linked with some monotone variational inequalities and convex programming problems. Various methods have been proposed to solve (1.1); see, for example, [1, 2, 3, 9, 11, 13, 15, 17, 19, 21] and the references therein.

By using W_n-mapping [16], Yao [20] introduced and analyzed an iterative method to find the approximate solutions of the following variational inequality problem defined on the set of common fixed points of nonexpansive mappings $T_i : H \to H$, $i = 1, 2, \dots, N$: Find $z \in \bigcap_{i=1}^{N} F(T_i)$ such that

$$\langle (A - \gamma f)z, y - z \rangle \geq 0, \quad \forall y \in \bigcap_{i=1}^{N} F(T_i), \tag{1.2}$$

where A is a strongly positive linear bounded operator, $f : C \to H$ is a contraction mapping and $\gamma > 0$.

By combining Korpelevich's extragradient method and the viscosity approximation method, Ceng et al. [8] introduced and analyzed implicit and explicit iterative schemes for computing a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an α-inverse-strongly monotone mapping defined on a real Hilbert space. Under suitable assumptions, they established the strong convergence of the sequences generated by the proposed schemes.

By combining Krasnoselskii-Mann type algorithm and the steepest-descent method, Buong and Duong [5] introduced an explicit iterative algorithm for finding the approximate solutions of a variational inequality problem defined over the set of common fixed points of a finite number of nonexpansive mappings in the setting of Hilbert spaces.

Recently, Zhang and Yang [22] and Bnouhachem et al. [4] considered a more general form of variational inequalities and proposed iterative methods for computing the solutions of such problems.

In this paper, motivated by the above works and related literature, we introduce an iterative algorithm for hierarchical fixed point problems of a finite family of nonexpansive mappings in the setting of real Hilbert spaces. More precisely, we use the same y_n as in [4] and we propose another way to compute x_{n+1}. We establish a strong convergence theorem for the sequence generated by the proposed method. In order to verify the theoretical assertions, some numerical examples are given. The algorithm and results presented in this paper improve and extend some recent corresponding algorithms and results; see, for example, [5, 7, 15, 17, 19, 21, 22] and the references therein.

2. Preliminaries

We present some definitions which will be used in the sequel.

Definition 2.1. A mapping $T : C \to H$ is said to be k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

If $k = 1$, then T is called nonexpansive. If $k \in (0, 1)$, then T is called contraction.

Definition 2.2. A mapping $T : C \to H$ is said to be

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

Definition 2.3. [20] A mapping $T : H \to H$ is said to be an averaged mapping if there exists $\alpha \in (0, 1)$ such that

$$T = (1 - \alpha)I + \alpha S, \tag{2.1}$$

where $I : H \to H$ is the identity mapping and $S : H \to H$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged.

Notice that the averaged mapping T is also nonexpansive and $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.

We list some fundamental lemmas that are useful in the consequent analysis.

Lemma 2.1. [6, 10] *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^N F(T_i) = F(T_1T_2 \dots T_N).$$

In particular, if $N = 2$*, we have* $F(T_1) \cap F(T_2) = F(T_1T_2) = F(T_2T_1)$.

Lemma 2.2. [12] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. If* $T : C \to C$ *is a nonexpansive mapping with* $F(T) \neq \emptyset$*, then the mapping* $I - T$ *is demiclosed at* 0*, i.e., if* $\{x_n\}$ *is a sequence in* C *weakly converges to* x *and if* $\{(I - T)x_n\}$ *converges strongly to* 0*, then* $(I - T)x = 0$.

Lemma 2.3. [18] *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \upsilon_n)a_n + \delta_n,$$

where $\{\upsilon_n\}$ *is a sequence in* $(0, 1)$ *and* δ_n *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \upsilon_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\upsilon_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [14] *Let* H *be a real Hilbert space. The following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

3. Main Results

In this section, we suggest and analyze an iterative method for finding the solution of hierarchical fixed point problem (1.1).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be nonexpansive mappings on C such that $\Omega = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, and $f : C \to C$ be a contraction mapping with a coefficient τ.

Algorithm 3.1. For arbitrarily $x_0 \in C$, the iterative sequences $\{x_n\}$ and $\{y_n\}$ are generated by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_N^n T_{N-1}^n \dots T_1^n x_n; \\ x_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n T_N^n T_{N-1}^n \dots T_1^n y_n, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $T_i^n = (1-\delta_n^i)I + \delta_n^i T_i$ and $\delta_n^i \in (0,1)$ for $i = 1, 2, \dots, N$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$,

(b) $\{\beta_n\} \subset [\sigma, 1)$ and $\lim_{n\to\infty} \beta_n = \beta < 1$,

(c) $\sum_{n=1}^{\infty} |\gamma_{n-1} - \gamma_n| < \infty$ and $\sum_{n=1}^{\infty} |\delta_{n-1}^i - \delta_n^i| < \infty$ for $i = 1, 2, \dots, N$.

We list some useful and helpful lemmas.

Lemma 3.1. *Let*

$$x_t = \alpha_t f(x_t) + \beta_t x_t + \gamma_t T_N^n T_{N-1}^n \dots T_1^n x_t, \tag{3.2}$$

where $T_i^n = (1-\delta_n^i)I + \delta_n^i T_i$ *and* $\delta_n^i \in (0,1)$ *for* $i = 1, 2, \dots, N$, $\alpha_t, \beta_t, \gamma_t \in (0,1)$, $\alpha_t + \beta_t + \gamma_t = 1$, $\lim_{t\to 0} \alpha_t = 0$ *and* $\lim_{t\to 0} \beta_t = \beta \in (0,1)$. *Then* x_t *converges strongly to* $\tilde{x} \in \Omega$ *as* $t \to 0$.

Proof. First, we show that $\{x_t\}$ is bounded. Let $w \in \Omega$. Then, we have

$$\begin{aligned} \|x_t - w\| &\leq \|\alpha_t f(x_t) + \beta_t x_t + \gamma_t T_N^n T_{N-1}^n \dots T_1^n x_t - w\| \\ &\leq \alpha_t \|f(x_t) - w\| + \beta_t \|x_t - w\| + \gamma_t \|T_N^n T_{N-1}^n \dots T_1^n x_t - w\| \\ &\leq \alpha_t \tau \|x_t - w\| + \alpha_t \|f(w) - w\| + (1-\alpha_t)\|x_t - w\| \\ &= (1 - \alpha_t(1-\tau))\|x_t - w\| + \alpha_t \|f(w) - w\| \end{aligned}$$

which implies that

$$\|x_t - x^*\| \leq \frac{\|f(w) - w\|}{1-\tau}.$$

Hence $\{x_t\}$ is bounded and consequently, we deduce that $\{T_N^n T_{N-1}^n \dots T_1^n x_t\}$ and $\{f(x_t)\}$ are bounded. It follows from (3.2) that

$$\|x_t - T_N^n T_{N-1}^n \dots T_1^n x_t\| = \|\alpha_t (f(x_t) - T_N^n T_{N-1}^n \dots T_1^n x_t) + \beta_t (x_t - T_N^n T_{N-1}^n \dots T_1^n x_t)\|.$$

Hence

$$\|x_t - T_N^n T_{N-1}^n \dots T_1^n x_t\| = \frac{\alpha_t}{1-\beta_t} \|f(x_t) - T_N^n T_{N-1}^n \dots T_1^n x_t\|.$$

Since $\lim_{t\to 0} \alpha_t = 0$ and $\lim_{t\to 0} \beta_t \in (0,1)$, we have

$$\lim_{t\to 0} \|x_t - T_N^n T_{N-1}^n \cdots T_1^n x_t\| = 0. \tag{3.3}$$

Assume $\{t_n\} \subset (0,1)$ is a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. Since $\{x_n\}$ is bounded, we may assume that $\{x_n\}$ converges weakly to a point $\tilde{x} \in C$.

By Lemma 2.2, we have $\tilde{x} \in \Omega$. Then for $\tilde{x} \in \Omega$, we get

$$\begin{aligned} \|x_t - \tilde{x}\|^2 &= \langle \alpha_t(f(x_t) - \tilde{x}) + \beta_t(x_t - \tilde{x}) + \gamma_t(T_N^n T_{N-1}^n \dots T_1^n x_t - \tilde{x}), x_t - \tilde{x}\rangle \\ &\leq \alpha_t \tau \|x_t - \tilde{x}\|^2 + \alpha_t\langle f(\tilde{x}) - \tilde{x}, x_t - \tilde{x}\rangle + \beta_t\|x_t - \tilde{x}\|^2 + \gamma_t\|x_t - \tilde{x}\|^2. \end{aligned}$$

Hence,

$$\|x_t - \tilde{x}\|^2 \leq \frac{1}{1-\tau}\langle f(\tilde{x}) - \tilde{x}, x_t - \tilde{x}\rangle.$$

In particular, we have

$$\|x_n - \tilde{x}\|^2 \leq \frac{1}{1-\tau}\langle f(\tilde{x}) - \tilde{x}, x_n - \tilde{x}\rangle. \tag{3.4}$$

Since $x_n \rightharpoonup \tilde{x}$, from (3.4), we obtain $x_n \to \tilde{x}$. □

Lemma 3.2. *Let $x^* \in \Omega$ and $\{x_n\}$ the sequence generated by Algorithm 3.1. Then,*

(a) *$\{x_n\}$ and $\{y_n\}$ are bounded,*
(b) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$
(c) $\lim_{n\to\infty} \|x_n - T_N^n T_{N-1}^n \dots T_1^n x_n\| = 0.$

Proof. Let $x^* \in \Omega$. Then, we have

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\alpha_n)(T_N^n T_{N-1}^n \dots T_1^n x_n - x^*) + \alpha_n(x_n - x^*)\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{3.5}$$

Next, we prove that the sequence $\{x_n\}$ is bounded. From (3.1) and (3.5), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|\alpha_n f(y_n) + \beta_n y_n + \gamma_n T_N^n T_{N-1}^n \dots T_1^n y_n - x^*\| \\ &\leq \alpha_n\|f(y_n) - x^*\| + \beta_n\|y_n - x^*\| + \gamma_n\|T_N^n T_{N-1}^n \dots T_1^n y_n - x^*\| \\ &\leq \alpha_n\tau\|y_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|y_n - x^*\| \\ &= (1-\alpha_n(1-\tau))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\ &\leq \max\left\{\|x_n - x^*\|, \frac{1}{1-\tau}(\|f(x^*) - x^*|)\right\}. \end{aligned}$$

By induction on n, we obtain $\|x_n - x^*\| \leq \|x_0 - x^*\|$, for $n \geq 0$ and $x_0 \in C$. Hence, $\{x_n\}$ is bounded and consequently, we deduce that $\{y_n\}$, $\{Ty_n\}$, $\{T_1 x_{n+1}\}$, $\|T_1^n x_{n+1}\|$, $\|T_2 T_1^n x_{n+1}\|$, …, $\|T_{N-1}^n \dots T_1^n x_{n+1}\|$, $\|T_N T_{N-1}^n \dots T_1^n x_{n+1}\|$, $\|T_{N-1}^{n-1} \dots T_1^{n-1} y_n\| + \|T_N T_{N-1}^{n-1} \dots T_1^{n-1} y_n\|\| + \|T_{N-1}^{n-1} \dots T_1^{n-1} x_n\| + \|T_N T_{N-1}^{n-1} \dots T_1^{n-1} x_n\|$ and $\{f(y_n)\}$ are bounded.

It follows from (3.1) that

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|(1-\alpha_n)T_N^n T_{N-1}^n \dots T_1^n x_n + \alpha_n x_n \\ &\quad -[(1-\alpha_{n-1})T_N^{n-1} T_{N-1}^{n-1} \dots T_1^{n-1} x_{n-1} + \alpha_{n-1} x_{n-1}]\| \\ &= \|(1-\alpha_n)(T_N^n T_{N-1}^n \dots T_1^n x_n - T_N^{n-1} T_{N-1}^{n-1} \dots T_1^{n-1} x_{n-1}) \\ &\quad -(\alpha_n - \alpha_{n-1})T_N^{n-1} T_{N-1}^{n-1} \dots T_1^{n-1} x_{n-1} + \alpha_n(x_n - x_{n-1}) - (\alpha_{n-1} - \alpha_n)x_{n-1}\| \\ &\leq \|x_{n-1} - x_n\| + (1-\alpha_n)\|T_N^n T_{N-1}^n \dots T_1^n x_n - T_N^{n-1} T_{N-1}^{n-1} \dots T_1^{n-1} x_n\| \\ &\quad +|\alpha_n - \alpha_{n-1}|\|T_N^{n-1} T_{N-1}^{n-1} \dots T_1^{n-1} x_{n-1} - x_{n-1}\|. \end{aligned} \tag{3.6}$$

It follows from the definition of T_i^n that

$$\begin{aligned}
&\|T_2^nT_1^ny_n - T_2^{n-1}T_1^{n-1}y_n\| \\
&\le \|T_2^nT_1^ny_n - T_2^nT_1^{n-1}y_n\| + \|T_2^nT_1^{n-1}y_n - T_2^{n-1}T_1^{n-1}y_n\| \\
&\le \|T_1^ny_n - T_1^{n-1}y_n\| + \|T_2^nT_1^{n-1}y_n - T_2^{n-1}T_1^{n-1}y_n\| \\
&\le \|(1-\delta_n^1)y_n + \delta_n^1T_1y_n - (1-\delta_{n-1}^1)y_n - \delta_{n-1}^1T_1y_n\| \\
&+\|(1-\delta_n^2)T_1^{n-1}y_n + \delta_n^2T_2T_1^{n-1}y_n - (1-\delta_{n-1}^2)T_1^{n-1}y_n - \delta_{n-1}^2T_2T_1^{n-1}y_n\| \\
&\le |\delta_n^1 - \delta_{n-1}^1|(\|y_n\| + \|T_1y_n\|) + |\delta_n^2 - \delta_{n-1}^2|(\|T_1^{n-1}y_n\| + \|T_2T_1^{n-1}y_n\|). \qquad (3.7)
\end{aligned}$$

and from (3.7), we have

$$\begin{aligned}
&\|T_3^nT_2^nT_1^ny_n - T_3^{n-1}T_2^{n-1}T_1^{n-1}y_n\| \\
&\le \|T_3^nT_2^nT_1^ny_n - T_3^nT_2^{n-1}T_1^{n-1}y_n\| + \|T_3^nT_2^{n-1}T_1^{n-1}y_n - T_3^{n-1}T_2^{n-1}T_1^{n-1}y_n\| \\
&\le \|T_2^nT_1^ny_n - T_2^{n-1}T_1^{n-1}y_n\| + \|(1-\delta_n^3)T_2^{n-1}T_1^{n-1}y_n + \delta_n^3T_3T_2^{n-1}T_1^{n-1}y_n \\
&-(1-\delta_{n-1}^3)T_2^{n-1}T_1^{n-1}y_n - \delta_{n-1}^3T_3T_2^{n-1}T_1^{n-1}y_n\| \\
&\le |\delta_n^1 - \delta_{n-1}^1|(\|y_n\| + \|T_1y_n\|) + |\delta_n^2 - \delta_{n-1}^2|(\|T_1^{n-1}y_n\| \\
&+\|T_2T_1^{n-1}y_n\|) + |\delta_n^3 - \delta_{n-1}^3|(\|T_2^{n-1}T_1^{n-1}y_n\| + \|T_3T_2^{n-1}T_1^{n-1}y_n\|). \qquad (3.8)
\end{aligned}$$

It follows by induction on N that

$$\begin{aligned}
&\|T_N^nT_{N-1}^n\dots T_1^ny_n - T_N^{n-1}T_{N-1}^{n-1}\dots T_1^{n-1}y_n\| \\
&\le |\delta_n^1 - \delta_{n-1}^1|(\|y_n\| + \|T_1y_n\|) + |\delta_n^2 - \delta_{n-1}^2|(\|T_1^{n-1}y_n\| + \|T_2T_1^{n-1}y_n\|) \\
&+ \dots + |\delta_n^N - \delta_{n-1}^N|(\|T_{N-1}^{n-1}\dots T_1^{n-1}y_n\| + \|T_NT_{N-1}^{n-1}\dots T_1^{n-1}y_n\|). \qquad (3.9)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|T_N^nT_{N-1}^n\dots T_1^nx_n - T_N^{n-1}T_{N-1}^{n-1}\dots T_1^{n-1}x_n\| \\
&\le |\delta_n^1 - \delta_{n-1}^1|(\|x_n\| + \|T_1x_n\|) + |\delta_n^2 - \delta_{n-1}^2|(\|T_1^{n-1}x_n\| + \|T_2T_1^{n-1}x_n\|) \\
&+ \dots + |\delta_n^N - \delta_{n-1}^N|(\|T_{N-1}^{n-1}\dots T_1^{n-1}x_n\| + \|T_NT_{N-1}^{n-1}\dots T_1^{n-1}x_n\|). \qquad (3.10)
\end{aligned}$$

It follows from (3.6), (3.9) and (3.10) that

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&=\|\alpha_n(f(y_n)-f(y_{n-1}))+\beta_n(y_n-y_{n-1}) \\
&\quad+\gamma_n(T_N^nT_{N-1}^n\dots T_1^ny_n-T_N^{n-1}T_{N-1}^{n-1}\dots T_1^{n-1}y_{n-1}) \\
&\quad+(\alpha_n-\alpha_{n-1})f(y_{n-1})+(\beta_n-\beta_{n-1})y_{n-1} \\
&\quad+(\gamma_n-\gamma_{n-1})T_N^{n-1}T_{N-1}^{n-1}\dots T_1^{n-1}y_{n-1}\| \\
&\le(1-\alpha_n(1-\tau))\|y_n-y_{n-1}\|+\gamma_n\|T_N^nT_{N-1}^n\dots T_1^ny_n-T_N^{n-1}T_{N-1}^{n-1}\cdots T_1^{n-1}y_n\| \\
&\quad+|\alpha_n-\alpha_{n-1}|\|f(y_{n-1})\|+|\beta_n-\beta_{n-1}|\|y_{n-1}\| \\
&\quad+|\gamma_n-\gamma_{n-1}|\|T_N^{n-1}T_{N-1}^{n-1}\dots T_1^{n-1}y_{n-1}\| \\
&\le(1-\alpha_n(1-\tau))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\|f(y_{n-1})\| \\
&\quad+\|T_N^nT_{N-1}^n\dots T_1^{n-1}x_{n-1}-x_{n-1}\|) \\
&\quad+|\beta_n-\beta_{n-1}|\|y_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|T_N^nT_{N-1}^n\dots T_1^{n-1}y_{n-1}\| \\
&\quad+\gamma_n\|T_N^nT_{N-1}^n\dots T_1^ny_n-T_N^{n-1}T_{N-1}^{n-1}\dots T_1^{n-1}y_n\| \\
&\quad+(1-\alpha_n)\|T_N^nT_{N-1}^n\dots T_1^nx_n-T_N^{n-1}T_{N-1}^{n-1}\dots T_1^{n-1}x_n\| \\
&\le(1-\alpha_n(1-\tau))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|(\|f(y_{n-1})\| \\
&\quad+\|T_N^nT_{N-1}^n\dots T_1^{n-1}x_{n-1}-x_{n-1}\|) \\
&\quad+|\beta_n-\beta_{n-1}|\|y_{n-1}\|+|\gamma_n-\gamma_{n-1}|\|T_N^nT_{N-1}^n\dots T_1^{n-1}y_{n-1}\| \\
&\quad+|\delta_n^1-\delta_{n-1}^1|(\|y_n\|+\|T_1y_n\|+\|x_n\|+\|T_1x_n\|) \\
&\quad+|\delta_n^2-\delta_{n-1}^2|(\|T_1^{n-1}y_n\|+\|T_2T_1^{n-1}y_n\|+\|T_1^{n-1}x_n\|+\|T_2T_1^{n-1}x_n\|) \\
&\quad+\cdots+|\delta_n^N-\delta_{n-1}^N|(\|T_{N-1}^{n-1}\dots T_1^{n-1}y_n\|+\|T_NT_{N-1}^{n-1}\dots T_1^{n-1}y_n\|\| \\
&\quad+\|T_{N-1}^{n-1}\dots T_1^{n-1}x_n\|+\|T_NT_{N-1}^{n-1}\dots T_1^{n-1}x_n\|) \\
&\le(1-\alpha_n(1-\tau))\|x_n-x_{n-1}\| \\
&\quad+|\alpha_n-\alpha_{n-1}|(\|f(y_{n-1})\|+\|T_N^nT_{N-1}^n\dots T_1^{n-1}x_{n-1}-x_{n-1}\|+\|y_{n-1}\|) \\
&\quad+|\gamma_n-\gamma_{n-1}|(\|T_N^nT_{N-1}^n\dots T_1^{n-1}y_{n-1}\|+\|y_{n-1}\|) \\
&\quad+|\delta_n^1-\delta_{n-1}^1|(\|y_n\|+\|T_1y_n\|+\|x_n\|+\|T_1x_n\|) \\
&\quad+|\delta_n^2-\delta_{n-1}^2|(\|T_1^{n-1}y_n\|+\|T_2T_1^{n-1}y_n\|+\|T_1^{n-1}x_n\|+\|T_2T_1^{n-1}x_n\|) \\
&\quad+\cdots+|\delta_n^N-\delta_{n-1}^N|(\|T_{N-1}^{n-1}\dots T_1^{n-1}y_n\|+\|T_NT_{N-1}^{n-1}\dots T_1^{n-1}y_n\|\| \\
&\quad+\|T_{N-1}^{n-1}\dots T_1^{n-1}x_n\|+\|T_NT_{N-1}^{n-1}\dots T_1^{n-1}x_n\|) \\
&\le(1-\alpha_n(1-\tau))\|x_n-x_{n-1}\|+M(|\alpha_n-\alpha_{n-1}|+|\gamma_n-\gamma_{n-1}|+|\delta_n^1-\delta_{n-1}^1| \\
&\quad+|\delta_n^2-\delta_{n-1}^2|+\ldots+|\delta_n^N-\delta_{n-1}^N|)
\end{aligned}
$$

where

$$
\begin{aligned}
M\quad&=\max\Big\{\sup\nolimits_{n\ge1}\left(\|f(y_{n-1})\|+\|T_N^nT_{N-1}^n\dots T_1^{n-1}x_{n-1}-x_{n-1}\|+\|y_{n-1}\|\right), \\
&\sup\nolimits_{n\ge1}\left(\||T_N^nT_{N-1}^n\dots T_1^{n-1}(y_{n-1})\|+\|y_{n-1}\|\right), \\
&\sup\nolimits_{n\ge1}\left(\|y_n\|+\|T_1y_n\|+\|x_n\|+\|T_1x_n\|\right), \\
&\sup\nolimits_{n\ge1}\left(\|T_1^{n-1}y_n\|+\|T_2T_1^{n-1}y_n\|+\|T_1^{n-1}x_n\|+\|T_2T_1^{n-1}x_n\|\right)\Big\} \\
&+\cdots+\sup\nolimits_{n\ge1}\big(\|T_{N-1}^{n-1}\dots T_1^{n-1}y_n\|+\|T_NT_{N-1}^{n-1}\dots T_1^{n-1}y_n\|\| \\
&+\|T_{N-1}^{n-1}\dots T_1^{n-1}x_n\|+\|T_NT_{N-1}^{n-1}\dots T_1^{n-1}x_n\|\big).
\end{aligned}
$$

It follows by conditions (a) and (c) of Algorithm 3.1 and Lemma 2.3 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

It follows from (3.1) that

$$\begin{aligned}
&\|x_n - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\le \|x_{n+1} - x_n\| + \|x_{n+1} - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n \|f(y_n) - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\quad + \beta_n \|y_n - T_N^n T_{N-1}^n \dots T_1^n x_n\| + \gamma_n \|T_N^n T_{N-1}^n \dots T_1^n y_n - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n \|f(y_n) - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\quad + \beta_n \|y_n - T_N^n T_{N-1}^n \dots T_1^n x_n\| + \gamma_n \|y_n - x_n\| \\
&= \|x_{n+1} - x_n\| + \alpha_n \|f(y_n) - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\quad + (\beta_n \alpha_n + \gamma_n (1 - \alpha_n)) \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n \|f(y_n) - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\quad + (\beta_n \alpha_n + \gamma_n) \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\| \\
&= \|x_{n+1} - x_n\| + \alpha_n \|f(y_n) - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\quad + (1 - \beta_n)(1 - \alpha_n) \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n \|f(y_n) - T_N^n T_{N-1}^n \dots T_1^n x_n\| \\
&\quad + (1 - \beta_n) \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\|,
\end{aligned}$$

which implies that

$$\beta_n \|x_n - T_N^n T_{N-1}^n \dots T_1^n x_n\| \le \|x_{n+1} - x_n\| + \alpha_n \|f(y_n) - T_N^n T_{N-1}^n \dots T_1^n x_n\|.$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $0 < \lim_{n\to\infty} \beta_n < 1$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have

$$\lim_{n\to\infty} \|x_n - T_N^n T_{N-1}^n \dots T_1^n x_n\| = 0. \qquad \square$$

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to z, which is a unique solution of the following variational inequality:*

$$\langle z - f(z), z - x\rangle \le 0, \quad \forall x \in \Omega. \tag{3.11}$$

Proof. Given a real number $t \in (0.1)$. We define a mapping $W : C \to C$ by

$$W(x_t) = \alpha_t f(x_t) + \beta_t x_t + \gamma_t T_N^n T_{N-1}^n \dots T_1^n x_t,$$

where α_t, β_t, $\gamma_t \in (0,1)$, $\alpha_t + \beta_t + \gamma_t = 1$, $\lim_{t\to 0} \alpha_t = 0$ and $\lim_{t\to 0} \beta_t \in (0,1)$. It is easy to see that W is a contraction on C. Let $x_t \in C$ be a unique fixed point of W. Thus x_t is a unique solution of fixed point equation (3.2).

Next, we claim that $\limsup_{n\to\infty} \langle z - f(z), z - x_n\rangle \le 0$, where $z = \lim_{t\to 0} x_t$. It follows from Lemma 3.1 that $z \in \Omega$. Putting

$$a_n(t) = \gamma_t \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\| \{2(1-\alpha_t)\|x_t - x_n\| + \gamma_t \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\|\},$$

and letting $a_n(t) \to 0$ as $n \to \infty$. By using Lemma 2.4, we obtain

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&= \|\alpha_t (f(x_t) - x_n) + \beta_t (x_t - x_n) + \gamma_t (T_N^n T_{N-1}^n \dots T_1^n x_t - x_n)\|^2 \\
&\le \|\beta_t (x_t - x_n) + \gamma_t (T_N^n T_{N-1}^n \dots T_1^n x_t - x_n)\|^2 + 2\alpha_t \langle f(x_t) - x_n, x_t - x_n\rangle \\
&\le \big((\beta_t + \gamma_t)\|x_t - x_n\| + \gamma_t \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\|\big)^2 + 2\alpha_t \langle f(x_t) - x_t, x_t - x_n\rangle \\
&\quad + 2\alpha_t \|x_n - x_t\|^2 \\
&= (1-\alpha_t)^2 \|x_t - x_n\|^2 + 2\gamma_t (1-\alpha_t)\|x_t - x_n\| \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\| \\
&\quad + \gamma_t^2 \|T_N^n T_{N-1}^n \dots T_1^n x_n - x_n\|^2 + 2\alpha_t \langle f(x_t) - x_t, x_t - x_n\rangle + 2\alpha_t \|x_n - x_t\|^2 \\
&= (1+\alpha_t^2)\|x_t - x_n\|^2 + a_n(t) + 2\alpha_t \langle f(x_t) - x_t, x_t - x_n\rangle,
\end{aligned}$$

which implies that

$$\langle x_t - f(x_t), x_t - x_n\rangle \le \frac{\alpha_t}{2}\|x_t - x_n\|^2 + \frac{a_n(t)}{2\alpha_t}.$$

It follows that

$$\limsup_{n\to\infty}\langle x_t - f(x_t), x_t - x_n\rangle \le \frac{\alpha_t M'}{2}, \tag{3.12}$$

where $M' > 0$ is a constant such that $M' \ge \|x_t - x_n\|^2$ and $t \in (0,1)$. Taking the limsup as $t \to 0$ in (3.12), we get

$$\limsup_{n\to\infty}\langle z - f(z), z - x_n\rangle \le 0.$$

Next, we show that $x_n \to z$.

$$\begin{aligned}
&\|x_{n+1} - z\|^2\\
&= \|\alpha_n(f(y_n) - z) + \beta_n(y_n - z) + \gamma_n(T_N^n T_{N-1}^n \dots T_1^n y_n - z)\|^2\\
&\le \|\beta_n(y_n - z) + \gamma_n(T_N^n T_{N-1}^n \dots T_1^n y_n - z)\|^2 + 2\alpha_n\langle f(y_n) - z, x_{n+1} - z\rangle\\
&\le (\beta_n + \gamma_n)^2\|y_n - z\|^2 + 2\alpha_n\langle f(y_n) - f(z), x_{n+1} - z\rangle + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle\\
&\le (1-\alpha_n)^2\|y_n - z\|^2 + 2\alpha_n\tau\|y_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle\\
&\le (1-\alpha_n)^2\|x_n - z\|^2 + 2\alpha_n\tau\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle\\
&\le (1-\alpha_n)^2\|x_n - z\|^2 + \alpha_n\tau(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle\\
&= (1 - (2-\tau)\alpha_n + \alpha_n^2)\|x_n - z\|^2 + \alpha_n\tau\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{1 - (2-\tau)\alpha_n + \alpha_n^2}{1 - \alpha_n\tau}\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\tau}\langle f(z) - z, x_{n+1} - z\rangle\\
&= \left(1 - \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\right)\|x_n - z\|^2\\
&\quad + \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\left(\frac{\langle f(z) - z, x_{n+1} - z\rangle}{1-\tau} + \frac{\alpha_n}{2(1-\tau)}\|x_n - z\|^2\right).
\end{aligned}$$

Let

$$\upsilon_n = \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}$$

and

$$\delta_n = \frac{2\alpha_n(1-\tau)}{1-\alpha_n\tau}\left(\frac{\langle f(z) - z, x_{n+1} - z\rangle}{1-\tau} + \frac{\alpha_n}{2(1-\tau)}\|x_n - z\|^2\right).$$

Since $\displaystyle\sum_{n=1}^{\infty}\alpha_n = \infty$ and

$$\limsup_{n\to\infty}\left\{\frac{1}{1-\tau}\langle f(z) - z, x_{n+1} - z\rangle + \frac{\alpha_n}{2(1-\tau)}\|x_n - z\|^2\right\} \le 0,$$

it follows that

$$\sum_{n=1}^{\infty}\upsilon_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\frac{\delta_n}{\upsilon_n} \le 0.$$

Thus all the conditions of Lemma 2.3 are satisfied. Hence we deduce that $x_n \to z$.

Next we show that z solves the variational inequality (3.11). It follows from (3.2) that

$$x_t - f(x_t) = -\frac{\gamma_t}{\alpha_t}(x_t - T_N^n T_{N-1}^n \dots T_1^n x_t).$$

Since $I - T_N^n T_{N-1}^n \dots T_1^n$ is monotone. Then for any $w \in \Omega$, we have

$$\begin{aligned}
&\langle x_t - f(x_t), x_t - w\rangle \\
&= -\frac{\gamma_t}{\alpha_t}\langle x_t - T_N^n T_{N-1}^n \dots T_1^n x_t, x_t - w\rangle \\
&= -\frac{\gamma_t}{\alpha_t}\langle (I - T_N^n T_{N-1}^n \dots T_1^n)(x_t) - (I - T_N^n T_{N-1}^n \dots T_1^n)(w), x_t - w\rangle \\
&\le 0. \qquad (3.13)
\end{aligned}$$

Taking the limit through $t \to 0$ ensures that z is a solution to (3.11). Since the operator $I - f$ is $1 - \tau$ strongly monotone, and we get the uniqueness of the solution of the variational inequality (3.11). This completes the proof. □

4. Examples

To illustrate Algorithm 3.1 and the convergence result, we consider the following examples.

Example 4.1. Let $\alpha_n = \frac{1}{2(n+1)}$, $\beta_n = \frac{1}{4}$ and $\gamma_n = \frac{3n+1}{4(n+1)}$.

It is easy to show that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfies the conditions (a), (b) and (c). Let $\delta_n^i = \frac{n+i}{n+i+1}$ for $i = 1, 2$.

$$\begin{aligned}
\sum_{n=1}^{\infty} |\delta_{n-1}^i - \delta_n^i| &= \sum_{n=1}^{\infty}\left|\frac{n-1+i}{n+i} - \frac{n+i}{n+i+1}\right| \\
&= \sum_{n=1}^{\infty}\left|\frac{1}{(n+i)(n+1+i)}\right| \\
&< \infty.
\end{aligned}$$

Then, the sequence $\{\delta_{n-1}^i\}$ satisfies condition (c).

Let $\mathbb{R}$ be the set of real numbers, and let the mapping $T_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ be defined by

$$T_1(x) = \sin(x), \quad T_2(x) = \frac{x}{3}, \quad \forall x \in \mathbb{R},$$

let the mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x}{14}, \forall x \in \mathbb{R}.$$

It is easy to show that $T_i (i = 1, 2)$ are nonexpansive mappings, and f is contraction mapping with a coefficient $\frac{1}{7}$. It is clear that

$$\Omega = \bigcap_{i=1}^{2} F(T_i) = \{0\}.$$

We compare Algorithm 3.1 with the algorithm given in [4]. For the algorithm given in [4], we take the mapping $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = \frac{2x+3}{7}$, $\forall x \in \mathbb{R}$. It is easy to show that T is a 1-Lipschitzian mapping and $\frac{1}{7}$-strongly monotone. In all the tests for the algorithm given in [4], we take $\alpha_n = \frac{1}{2(n+1)}$, $\beta_n = \frac{1}{n^3}$ and $\gamma_n = \frac{1}{4}$, $\rho = \frac{1}{30}$, $\mu = \frac{1}{7}$, $\eta = \frac{1}{7}$, $k = 1$ and $\tau = \frac{1}{7}$. It is easy to show that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 \leq \rho\tau < \nu$, where $\nu = \mu\left(\eta - \frac{\mu k^2}{2}\right)$. We also use the same $T_i, \{\delta_n^i\}$ for $i = 1, 2$. All codes were written in Matlab, the values of $\{y_n\}$ and $\{x_n\}$ with different n are reported in Table 4.1.

Table 4.1: Numerical results for Example 4.1

	$x_1 = 30$				$x_1 = -30$			
	Algorithm 3.1		Algorithm of [4]		Algorithm 3.1		Algorithm of [4]	
	y_n	x_n	y_n	x_n	y_n	x_n	y_n	x_n
n=1	11.392213	30.000000	30.000000	30.000000	-11.392213	-30.000000	-30.000000	-30.000000
n=2	0.843022	3.935512	5.487062	29.696429	-0.843022	-3.935512	-5.492110	-29.727041
n=3	0.232880	0.445593	0.820945	11.494050	-0.232880	-0.445593	-0.826800	-11.525865
n=4	0.063751	0.127733	0.122208	3.477624	-0.063751	-0.127733	-0.118849	-3.505247
n=5	0.016569	0.034800	0.176840	0.954470	-0.016569	-0.034800	-0.179105	-0.971114
n=6	0.004113	0.008975	0.068600	0.365579	-0.004113	-0.008975	-0.071490	-0.381635
n=7	0.000983	0.002212	0.024409	0.138283	-0.000983	-0.002212	-0.027026	-0.153204
n=8	0.000228	0.000526	0.008126	0.048993	-0.000228	-0.000526	-0.010336	-0.062332
n=9	0.000051	0.000121	0.002343	0.014924	-0.000051	-0.000121	-0.004194	-0.026714
n=10	0.000011	0.000027	0.000363	0.002422	-0.000011	-0.000027	-0.001927	-0.012877

Remark 4.1. Table 4.1 and Figures 4.1 and 4.2 show that the sequences $\{y_n\}$ and $\{x_n\}$ converge to 0, where $\{0\} = \Omega$. Table 4.1 shows that the convergence of Algorithm 3.1 is faster than the algorithm given in [4].

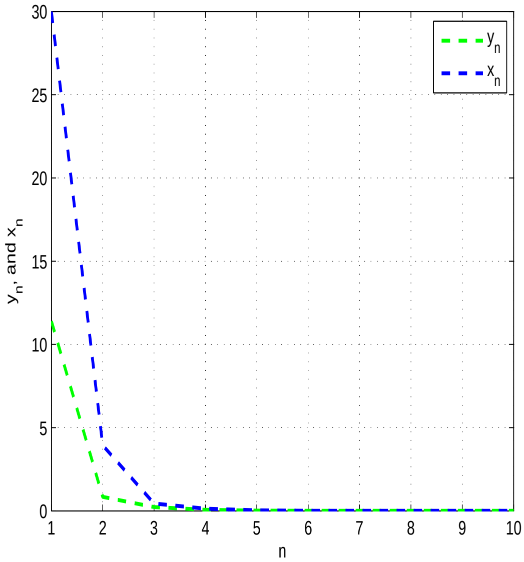
Example 4.2. In this example we take the same mappings and parameters as in Example 4.1 except T_i and f.

Let the mapping $T_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3)$ be defined by $T_1(x) = \cos(1 - x)$, $T_2(x) = \sin(x - 1) + 1$, $T_3(x) = \frac{-2x+5}{3}$, $\forall x \in \mathbb{R}$, let the mapping $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{2x+5}{7}$, $\forall x \in \mathbb{R}$. It is easy to show that $T_i (i = 1, 2, 3)$ are nonexpansive mappings, and f is a contraction mapping with a coefficient $\frac{2}{7}$. It is clear that $\Omega = \bigcap_{i=1}^{3} F(T_i) = \{1\}$. Let $\delta_n^i = \frac{n+i}{n+i+1}$ for $i = 1, 2, 3$.

Table 4.2: Numerical results for Example 4.2

	$x_1 = 20$				$x_1 = -40$			
	Algorithm 3.1		Algorithm of [4]		Algorithm 3.1		Algorithm of [4]	
	y_n	x_n	y_n	x_n	y_n	x_n	y_n	x_n
n=1	5.346649	20.000000	20.000000	20.000000	-8.189901	-40.000000	-40.000000	-40.000000
n=2	1.292003	2.309806	3.314436	19.911168	0.870725	-1.889556	-3.511899	-39.809921
n=3	1.004779	1.077550	0.821715	7.456106	0.998646	0.970275	0.728508	-12.572762
n=4	1.000035	1.001112	1.245361	2.480472	0.999990	0.999686	1.401251	-2.596434
n=5	1.000000	1.000008	1.027777	1.553479	1.000000	0.999998	1.106490	0.400880
n=6	1.000000	1.000000	0.996338	1.158989	1.000000	1.000000	1.005119	0.929753
n=7	1.000000	1.000000	0.998314	1.036860	1.000000	1.000000	1.000795	0.986125
n=8	1.000000	1.000000	0.999618	1.007825	1.000000	1.000000	1.000154	0.996999
n=9	1.000000	1.000000	0.999927	1.001556	1.000000	1.000000	1.000036	0.999252
n=10	1.000000	1.000000	0.999990	1.000232	1.000000	1.000000	1.000012	0.999738

Remark 4.2. Table 4.2 and Figures 4.3 and 4.4 show that the sequences $\{y_n\}$ and $\{x_n\}$ converge to 1, where $\{1\} = \Omega$. Table 4.2 shows that the convergence of Algorithm 3.1 is faster than the algorithm given in [4].



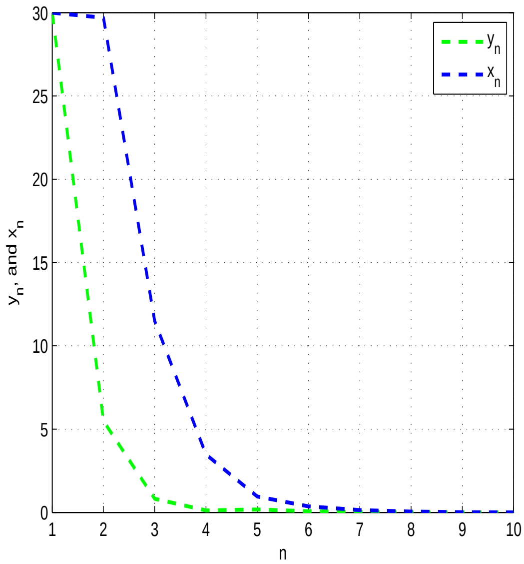


Figure 4.1: The convergence of $\{y_n\}$ and $\{x_n\}$ for Algorithm 3.1 and $\{x_n\}$ for the algorithm given in [4] with initial value $x_1 = 30$.

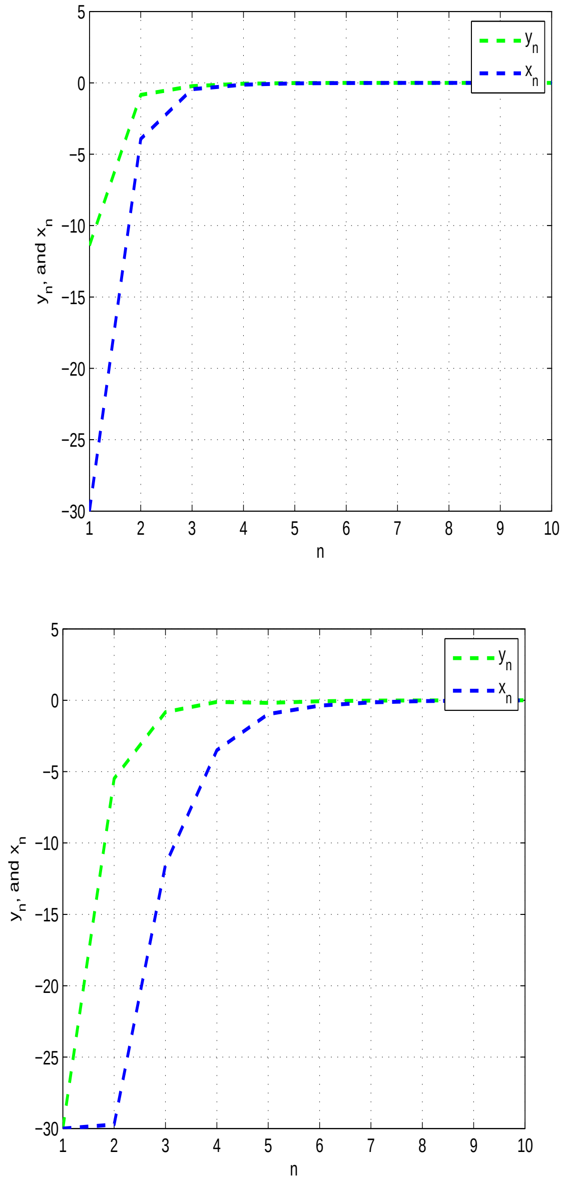


Figure 4.2: The convergence of $\{y_n\}$ and $\{x_n\}$ for Algorithm 3.1 and $\{x_n\}$ for the algorithm given in [4] with initial value $x_1 = -30$.

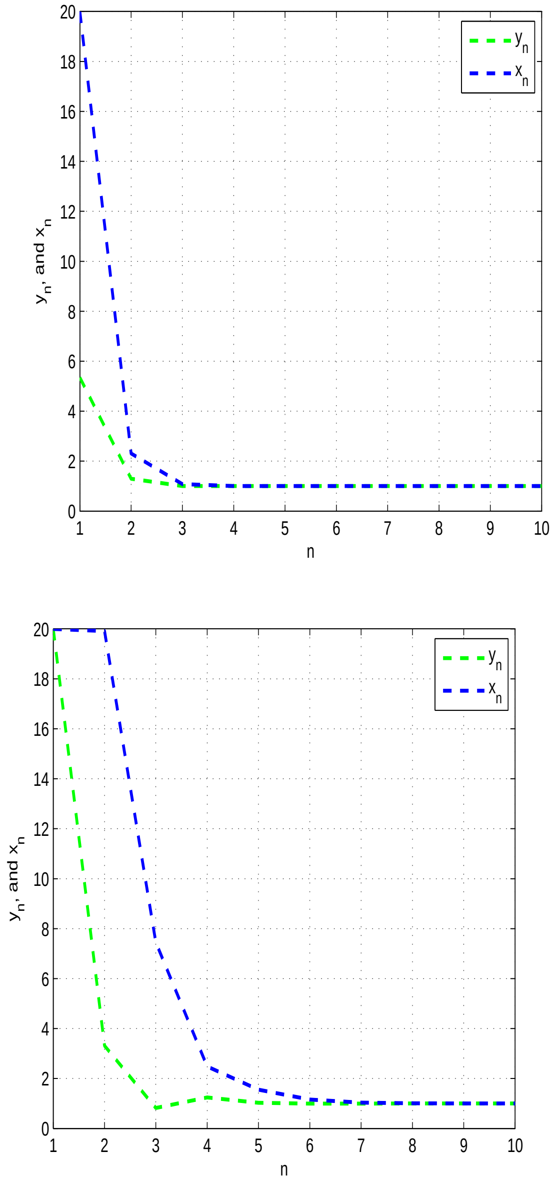
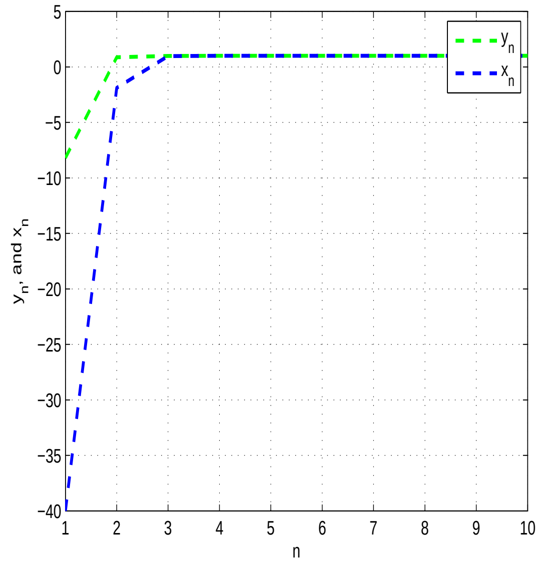


Figure 4.3: The convergence of $\{y_n\}$ and $\{x_n\}$ for Algorithm 3.1 and $\{x_n\}$ for the algorithm given in [4] with initial value $x_1 = 20$.



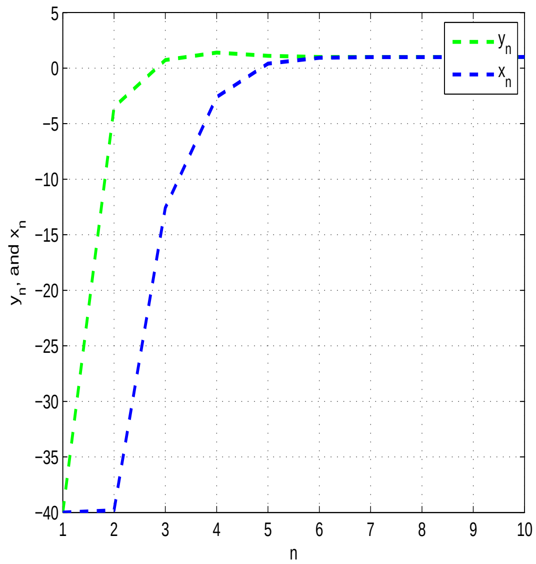


Figure 4.4: The convergence of $\{y_n\}$ and $\{x_n\}$ for Algorithm 3.1 and $\{x_n\}$ for the algorithm given in [4] with initial value $x_1 = -40$.

References

[1] S.A. Al-Mezel, Q.H. Ansari, L.-C. Ceng, *Hybrid viscosity approach for hierarchical fixed point problems*, Appl. Anal., **94(1)**(2015), 2-23.

[2] A. Bnouhachem, *An iterative method for common solutions of equilibrium problems and hierarchical fixed point problems*, Fixed Point Theory Appl., **2014**(2014), Article ID 194.

[3] A. Bnouhachem, S. Al-Homidan, Q.H. Ansari, *An iterative algorithm for system of generalized equilibrium problems and fixed point problem*, Fixed Point Theory Appl., **2014**(2014), Article ID 235.

[4] A. Bnouhachem, Q.H. Ansari, J.C. Yao, *An iterative algorithm for hierarchical fixed point problems for a finite family of nonexpansive mappings*, accepted.

[5] N. Buong, L.T. Duong, *An explicit iterative algorithm for a class of variational inequalities in Hilbert spaces*, J. Optim. Theory Appl., **151**(2011), 513-524.

[6] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl., **20**(2004), 103-120.

[7] L.-C. Ceng, Q.H. Ansari, J.C. Yao, *Some iterative methods for finding fixed points and for solving constrained convex minimization problems*, Nonlinear Anal., **74(16)**(2011), 5286-5302.

[8] L.-C. Ceng, A.R. Khan, Q.H. Ansari, J.-C. Yao, *Viscosity approximation methods for strongly positive and monotone operators*, Fixed Point Theory, **10(1)**(2009), 35-71.

[9] F. Cianciaruso, G. Marino, L. Muglia, Y. Yao, *On a two-steps algorithm for hierarchical fixed point problems and variational inequalities*, J. Inequal. Appl., **2009**(2009), 1-13.

[10] P.L. Combettes, *Solving monotone inclusions via compositions of nonexpansive averaged operators*, Optimization, **53**(2004), 475-504.

[11] G. Crombez, *A hierarchical presentation of operators with fixed points on Hilbert spaces*, Numer. Funct. Anal. Optim., **27**(2006), 259-277.

[12] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[13] P.E. Mainge, A. Moudafi, *Strong convergence of an iterative method for hierarchical fixed-point problems*, Pacific J. Optim., **3(3)**(2007), 529-538.

[14] G. Marino, H.K. Xu, *Convergence of generalized proximal point algorithms*, Comm. Pure Appl. Anal., **3**(2004), 791-808.

[15] N. Suzuki, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl., **325**(2007), 342-352.

[16] W. Takahashi, S. Atsushiba, *Strong convergence theorems for a finite family of nonexpansive mappings and applications*, Indian J. Math., **41(3)**(1999), 435-453.

[17] M. Tian, *A general iterative algorithm for nonexpansive mappings in Hilbert spaces*, Nonlinear Anal., **73**(2010), 689-694.

[18] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240-256.

[19] H.K. Xu, *Viscosity method for hierarchical fixed point approach to variational inequalities*, Taiwanese J. Math., **14(2)**(2010), 463-478.

[20] Y. Yao, *A general iterative method for a finite family of nonexpansive mappings*, Nonlinear Anal., **66**(2007), 2676-2678.

[21] Y. Yao, Y.J. Cho, Y.C. Liou, *Iterative algorithms for hierarchical fixed points problems and variational inequalities*, Math. Computer Model., **52 (9-10)** (2010), 1697-1705.

[22] C. Zhang, C. Yang, *A new explicit iterative algorithm for solving a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings*, Fixed Point Theory Appl., **2014**(2014), Article ID 60.

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