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MULTIPLE POSITIVE SOLUTIONS FOR A HIGHER ORDER BOUNDARY VALUE PROBLEM ON TIME SCALES

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Abstract. In this paper, we consider a nonlinear higher order three-point boundary value problem on time scales. We establish the criteria for the existence of one or two positive solutions for a higher order boundary value problem on time scales by using a result from the theory of fixed point index. Later, Leggett-Williams fixed-point theorem is used to investigate the existence of at least three positive solutions for a higher order boundary value problem on time scales. As an application, to demonstrate our results we also give an example.
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1. Introduction

The theory of time scales, which has received a lot of attention recently, was introduced by Hilger [12] in his Phd thesis in 1988. A result for a dynamic equation contains simultaneously a corresponding result for a differential equation, one for a difference equation, as well as results for other dynamic equations in arbitrary time scales. We refer the reader to the excellent introductory book by Bohner and Peterson [7] and the volume edited [8] edited by them.

The study of three-point boundary value problems was initiated by Neuberger [18] in 1966. The first result concerning existence of positive solutions for higher order three-point boundary value problems was given by Eloe and McKelwey [10] in 1997. They obtained sufficient conditions for the existence of at least one and two positive solutions by using the fixed point theorem in a cone. Since then, by applying the cone theory techniques, more general nonlinear three point boundary value problems have been studied by several authors. We refer the reader to [2, 14, 17, 23].

In 2001, Agarwal and O'Regan [1] discussed the following boundary value problem on a measure chain

$$u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, \;\; t \in [a, b],$$
$$u(a) = 0 = u^{\Delta}(\sigma(b))$$

and then, in Remark 2.5, they left to the reader the details of the problem

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0, \ \ t \in [a, b],$$
$$\alpha y(a) - \beta y^{\Delta}(a) = 0, \ \ y^{\Delta}(\sigma(b)) = 0, \ \ \alpha > 0, \ \ \beta \geq 0.$$

In this paper we are concerned with the existence of single and multiple positive solutions to the following nonlinear higher order three-point boundary value problem on time scales:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y(\sigma(t))), \ \ t \in [t_1, t_3] \subset \mathbb{T}, \ \ n \in \mathbb{N} \\ y^{\Delta^{2i+1}}(\sigma(t_3)) = 0, \quad \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = y^{\Delta^{2i+1}}(t_2), \end{cases} \tag{1.1}$$

for $0 \leq i \leq n-1$, where $\alpha > 0$ and $\beta > 0$ are given constants. We assume that $f : [t_1, \sigma(t_3)] \times [0, \infty) \to [0, \infty)$ is continuous. Throughout this paper we suppose $\mathbb{T}$ is any time scale and $[t_1, t_3]$ is a subset of $\mathbb{T}$ such that $[t_1, t_3] = \{t \in \mathbb{T} : t_1 \leq t \leq t_3\}$.

In recent years, there has been much research activity concerning the second order three-point boundary value problems for dynamic equations on time scales. We refer the reader to the recent papers [3, 5, 9, 13, 19, 21, 22] and references cited therein. However, there are few works on higher order three-point boundary value problems on time scales (see [4, 6, 20, 24]).

We have organized the paper as follows. In Section 2, we give some lemmas which are needed later. In Section 3, we use a result from the theory of fixed point index to show the existence of one or two positive solutions for the three-point boundary value problem (1.1). In Section 4, we establish the existence criteria of at least three positive solutions of (1.1) by using Leggett-Williams fixed-point theorem.

2. Preliminaries

The linear boundary value problem

$$-y^{\Delta^2}(t) = h(t), \ \ t \in [t_1, t_3],$$
$$y^{\Delta}(\sigma(t_3)) = 0, \quad \alpha y(t_1) - \beta y^{\Delta}(t_1) = y^{\Delta}(t_2),$$

has the unique solution

$$y(t) = \int_{t_1}^{\sigma(t_3)} (\sigma(s) + \frac{\beta}{\alpha} - t_1) h(s) \Delta s + \frac{1}{\alpha} \int_{t_2}^{\sigma(t_3)} h(s) \Delta s + \int_{t}^{\sigma(t_3)} (t - \sigma(s)) h(s) \Delta s.$$

If $G(t, s)$ is Green's function for the boundary value problem

$$-y^{\Delta^2}(t) = 0, \ \ t \in [t_1, t_3],$$
$$y^{\Delta}(\sigma(t_3)) = 0, \quad \alpha y(t_1) - \beta y^{\Delta}(t_1) = y^{\Delta}(t_2),$$

then we have

$$G(t, s) = \begin{cases} H_1(t, s), \ \ t_1 \leq s \leq t_2, \\ H_2(t, s), \ \ t_2 < s \leq t_3, \end{cases} \tag{2.1}$$

where

$$H_1(t,s) = \begin{cases} \sigma(s) + \frac{\beta}{\alpha} - t_1, & \sigma(s) \le t, \\ t + \frac{\beta}{\alpha} - t_1, & t \le s, \end{cases}$$

and

$$H_2(t,s) = \begin{cases} \sigma(s) + \frac{\beta+1}{\alpha} - t_1, & \sigma(s) \le t, \\ t + \frac{\beta+1}{\alpha} - t_1, & t \le s. \end{cases}$$

To state the main results of this paper, we will need the following lemmas.

Lemma 2.1. *If $\alpha > 0$ and $\beta > 0$, then the Green's function $G(t,s)$ in (2.1) satisfies the following inequality*

$$G(t,s) \ge \frac{t - t_1}{\sigma(t_3) - t_1} G(\sigma(t_3), s)$$

for $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Proof. We proceed sequentially on the branches of the Green's function $G(t,s)$ in (2.1).

(i) Fix $s \in [t_1, t_2]$ and $\sigma(s) \le t$. Then

$$G(t,s) = \sigma(s) + \frac{\beta}{\alpha} - t_1$$

and

$$\frac{G(t,s)}{G(\sigma(t_3), s)} = 1 \ge \frac{t - t_1}{\sigma(t_3) - t_1}.$$

(ii) Take $s \in [t_1, t_2]$ and $t \le s$. Then

$$G(t,s) = t + \frac{\beta}{\alpha} - t_1$$

and

$$\frac{G(t,s)}{G(\sigma(t_3), s)} = \frac{t + \frac{\beta}{\alpha} - t_1}{\sigma(s) + \frac{\beta}{\alpha} - t_1} > \frac{t - t_1}{\sigma(t_3) - t_1}.$$

(iii) For $s \in (t_2, t_3]$ and $\sigma(s) \le t$, we have

$$G(t,s) = \sigma(s) + \frac{\beta + 1}{\alpha} - t_1$$

and

$$\frac{G(t,s)}{G(\sigma(t_3), s)} = 1 \ge \frac{t - t_1}{\sigma(t_3) - t_1}.$$

(iv) Let $s \in (t_2, t_3]$ and $t \le s$. Then

$$G(t,s) = t + \frac{\beta + 1}{\alpha} - t_1$$

and

$$\frac{G(t,s)}{G(\sigma(t_3), s)} = \frac{t + \frac{\beta+1}{\alpha} - t_1}{\sigma(s) + \frac{\beta+1}{\alpha} - t_1} > \frac{t - t_1}{\sigma(t_3) - t_1}.$$

□

Lemma 2.2. *Let* $\alpha > 0$ *and* $\beta > 0$. *Then the Green's function* $G(t,s)$ *in* (2.1) *satisfies*

$$0 < G(t,s) \leq G(\sigma(s),s)$$

for $(t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Proof. Since for $s \in [t_1, t_2]$

$$G(\sigma(t_3), s) = \sigma(s) + \frac{\beta}{\alpha} - t_1 > 0$$

and for $s \in (t_2, \sigma(t_3)]$

$$G(\sigma(t_3), s) = \sigma(s) + \frac{\beta + 1}{\alpha} - t_1 > 0,$$

we obtain $G(t,s) > 0$ from Lemma 2.1.

To show that $G(t,s) \leq G(\sigma(s),s)$, we again deal with the branches of the Green's function $G(t,s)$ in (2.1).

(*i*) Fix $s \in [t_1, t_2]$ and $\sigma(s) \leq t$. Then it is obvious that $G(t,s) = G(\sigma(s),s)$.

(*ii*) Take $s \in [t_1, t_2]$ and $t \leq s \leq \sigma(s)$.
Since $G(t,s)$ is increasing in t, $G(t,s) \leq G(\sigma(s),s)$.

(*iii*) For $s \in (t_2, t_3]$ and $\sigma(s) \leq t$, it is clear that $G(t,s) = G(\sigma(s),s)$.

(*iv*) Let $s \in (t_2, t_3]$ and $t \leq s$.
Since $G(t,s)$ is increasing in t, $G(t,s) \leq G(\sigma(s),s)$. □

Lemma 2.3. *Assume* $\alpha > 0$, $\beta > 0$ *and* $s \in [t_1, t_3]$. *Then the Green's function* $G(t,s)$ *in* (2.1) *satisfies*

$$\min_{t \in [t_2, \sigma(t_3)]} G(t,s) \geq K \|G(\cdot, s)\|,$$

where

$$K = \frac{\beta + \alpha(t_2 - t_1)}{\beta + 1 + \alpha(\sigma(t_3) - t_1)} \tag{2.2}$$

and $\|\cdot\|$ *is defined by* $\|x\| = \max_{t \in [t_1, \sigma(t_3)]} |x(t)|$.

Proof. Since the Green's function $G(t,s)$ in (2.1) is nondecreasing in t, we get

$$\min_{t \in [t_2, \sigma(t_3)]} G(t,s) = G(t_2, s).$$

Moreover, it is obvious that $\|G(\cdot,s)\| = G(\sigma(s),s)$ for $s \in [t_1, t_3]$ by Lemma 2.2.
To show that $G(t_2, s) \geq KG(\sigma(s),s)$, we again deal with the branches of the Green's function $G(t,s)$ in (2.1).

(*i*) If $s \in [t_1, t_2)$, then we have

$$G(t_2, s) = \sigma(s) + \frac{\beta}{\alpha} - t_1 \geq K(\sigma(s) + \frac{\beta}{\alpha} - t_1) = KG(\sigma(s),s).$$

(*ii*) If $s = t_2$, then we obtain

$$G(t_2, s) = t_2 + \frac{\beta}{\alpha} - t_1 \geq K(\sigma(s) + \frac{\beta}{\alpha} - t_1) = KG(\sigma(s),s).$$

(iii) If $s \in (t_2, t_3]$, then we have

$$G(t_2, s) = t_2 + \frac{\beta+1}{\alpha} - t_1 \geq K(\sigma(s) + \frac{\beta+1}{\alpha} - t_1) = KG(\sigma(s), s). \qquad \square$$

If we let $G_1(t,s) := G(t,s)$ for G as in (2.1), then we can recursively define

$$G_j(t,s) = \int_{t_1}^{\sigma(t_3)} G_{j-1}(t,r)G(r,s)\Delta r$$

for $2 \leq j \leq n$ and $G_n(t,s)$ is Green's function for the homogeneous problem

$$(-1)^n y^{\Delta^{2n}}(t) = 0, \;\; t \in [t_1, t_3],$$
$$y^{\Delta^{2i+1}}(\sigma(t_3)) = 0, \quad \alpha y^{\Delta^{2i}}(t_1) - \beta y^{\Delta^{2i+1}}(t_1) = y^{\Delta^{2i+1}}(t_2), \;\; 0 \leq i \leq n-1.$$

Lemma 2.4. *Let $\alpha > 0$, $\beta > 0$. The Green's function $G_n(t,s)$ satisfies the following inequalities*

$$0 \leq G_n(t,s) \leq L^{n-1}\|G(\cdot,s)\|, \quad (t,s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$$

and

$$G_n(t,s) \geq K^n M^{n-1}\|G(\cdot,s)\|, \quad (t,s) \in [t_2, \sigma(t_3)] \times [t_1, t_3]$$

where K is given in (2.2),

$$L = \int_{t_1}^{\sigma(t_3)} \|G(\cdot,s)\|\Delta s > 0 \tag{2.3}$$

and

$$M = \int_{t_2}^{\sigma(t_3)} \|G(\cdot,s)\|\Delta s > 0. \tag{2.4}$$

Proof. Use induction on n and Lemma 2.3. $\square$

Let $\mathcal{B}$ denote the Banach space $C[t_1, \sigma(t_3)]$ with the norm

$$\|y\| = \max_{t \in [t_1, \sigma(t_3)]} |y(t)|.$$

Define the cone $P \subset \mathcal{B}$ by

$$P = \{y \in \mathcal{B} : y(t) \geq 0, \min_{t \in [t_2, \sigma(t_3)]} y(t) \geq \frac{K^n M^{n-1}}{L^{n-1}}\|y\|\}. \tag{2.5}$$

where K, L, M are given in (2.2),(2.3),(2.4), respectively.
(1.1) is equivalent to the nonlinear integral equation

$$y(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(s, y(\sigma(s)))\Delta s. \tag{2.6}$$

We can define the operator $A : P \to \mathcal{B}$ by

$$Ay(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(s, y(\sigma(s)))\Delta s, \tag{2.7}$$

where $y \in P$. Then (2.6) can be written as $y = Ay$. Therefore solving (2.6) in P is equivalent to finding fixed points of the operator A. If $y \in P$, then by Lemma 2.4 we have

$$\begin{aligned}
\min_{t\in[t_2,\sigma(t_3)]} Ay(t) &= \int_{t_1}^{\sigma(t_3)} \min_{t\in[t_2,\sigma(t_3)]} G_n(t,s) f(s, y(\sigma(s)))\Delta s \\
&\geq \frac{K^n M^{n-1}}{L^{n-1}} \int_{t_1}^{\sigma(t_3)} \max_{t\in[t_1,\sigma(t_3)]} |G_n(t,s)| f(s, y(\sigma(s)))\Delta s \\
&= \frac{K^n M^{n-1}}{L^{n-1}} \|Ay\|.
\end{aligned}$$

Thus $Ay \in P$ and therefore $AP \subset P$. In addition, $A : P \to P$ is completely continuous by a standard application of the Arzela-Ascoli Theorem.

We will apply the following well-known result of the fixed point theorems to prove the existence of one or two positive solutions to the (1.1).

Lemma 2.5. [11, 15] *Let P be a cone in a Banach space $\mathcal{B}$, and let D be an open, bounded subset of $\mathcal{B}$ with $D_P := D \cap P \neq \emptyset$ and $\overline{D}_P \neq P$. Assume that $A : \overline{D}_P \to P$ is a compact map such that $y \neq Ay$ for $y \in \partial D_P$. The following result hold.*

(i) If $\|Ay\| \leq \|y\|$ for $y \in \partial D_P$, then $i_P(A, D_P) = 1$.

(ii) If there exists an $b \in P \setminus \{0\}$ such that $y \neq Ay + \lambda b$ for all $y \in \partial D_P$ and all $\lambda > 0$, then $i_P(A, D_P) = 0$.

(iii) Let U be open in P such that $\overline{U}_P \subset D_P$. If $i_P(A, D_P) = 1$ and $i_P(A, U_P) = 0$, then A has a fixed point in $D_P \setminus \overline{U}_P$. The same result holds if $i_P(A, D_P) = 0$ and $i_P(A, U_P) = 1$.

Now, to prove the existence of at least three positive solutions for the (1.1), we state the Leggett-Williams fixed point theorem [16].

Theorem 2.6. *Let P be a cone in the real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\}$$

$$P(\psi, a, b) := \{x \in P : a \leq \psi(x),\ \|x\| \leq b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \leq \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \leq r$ such that the following condition hold,

(i) $\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ and $\psi(Au) > q$ for all $u \in P(\psi, q, l)$;

(ii) $\|Au\| < p$ for $\|u\| \leq p$;

(iii) $\psi(Au) > q$ for $u \in P(\psi, q, r)$ with $\|Au\| > l$,

then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

3. One or two positive solutions

For the cone P given in (2.5) and any positive real number r, define the convex set

$$P_r := \{y \in P : \|y\| < r\}$$

and the set

$$\Omega_r := \{y \in P : \min_{t\in[t_2,\sigma(t_3)]} y(t) < er\}$$

where

$$e := \frac{K^n M^{n-1}}{L^{n-1}} \in (0,1) \tag{3.1}$$

and K, L, and M are defined in (2.2),(2.3), and (2.4), respectively. The following results are proved in [15].

Lemma 3.1. *The set Ω_r has the following properties.*
(i) Ω_r is open relative to P.
(ii) $P_{er} \subset \Omega_r \subset P_r$
(iii) $y \in \partial\Omega_r$ if and only if $\min_{t\in[t_2,\sigma(t_3)]} y(t) = er$.
(iv) If $y \in \partial\Omega_r$, then $er \le y(t) \le r$ for $t \in [t_2, \sigma(t_3)]$.

As in [21], for convenience, we introduce the following notations. Let

$$f^r_{er} := \min\left\{\min_{t\in[t_2,\sigma(t_3)]} \frac{f(t,y)}{r} : y \in [er, r]\right\}$$

$$f^r_0 := \max\left\{\max_{t\in[t_1,\sigma(t_3)]} \frac{f(t,y)}{r} : y \in [0, r]\right\}$$

$$f^a := \limsup_{y\to a} \max_{t\in[t_1,\sigma(t_3)]} \frac{f(t,y)}{y}$$

$$f_a := \liminf_{y\to a} \min_{t\in[t_2,\sigma(t_3)]} \frac{f(t,y)}{y} \quad (a := 0^+, \infty).$$

In the next two lemmas, we give conditions on f guaranteeing that $i_P(A, P_r) = 1$ or $i_P(A, \Omega_r) = 0$.

Lemma 3.2. *For L in (2.3), if the conditions*

$$f^r_0 \le \frac{1}{L^n} \text{ and } y \ne Ay \text{ for } y \in \partial P_r,$$

hold, then $i_P(A, P_r) = 1$.

Proof. If $y \in \partial P_r$, then using Lemma 2.4, we have

$$\begin{aligned} Ay(t) &= \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(s, y(\sigma(s)))\Delta s \\ &\le \|f(.,y)\| L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(\cdot,s)\|\Delta s \\ &\le \frac{r}{L^n} L^n = r = \|y\|. \end{aligned}$$

It follows that $\|Ay\| \le \|y\|$ for $y \in \partial P_r$. By Lemma 2.5(i), we get $i_P(A, P_r) = 1$. □

Lemma 3.3. *Let*

$$N := \left(\int_{t_2}^{\sigma(t_3)} \min_{t \in [t_2, \sigma(t_3)]} G_n(t,s)\Delta s \right)^{-1}. \tag{3.2}$$

If the conditions

$$f_{er}^r \ge Ne \text{ and } y \ne Ay \text{ for } y \in \partial\Omega_r,$$

hold, then $i_P(A, \Omega_r) = 0$.

Proof. Let $b(t) \equiv 1$ for $t \in [t_1, \sigma(t_3)]$, then $b \in \partial P_1$. Assume there exist $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$ such that $y_0 = Ay_0 + \lambda_0 b$. Then for $t \in [t_2, \sigma(t_3)]$ we have

$$\begin{aligned} y_0(t) &= Ay_0(t) + \lambda_0 b(t) \\ &\ge \int_{t_2}^{\sigma(t_3)} G_n(t,s) f(s, y_0(\sigma(s)))\Delta s + \lambda_0 \\ &\ge Ner \int_{t_2}^{\sigma(t_3)} \min_{t \in [t_2, \sigma(t_3)]} G_n(t,s)\Delta s + \lambda_0 \\ &= er + \lambda_0. \end{aligned}$$

But this implies that $er \ge er + \lambda_0$, a contradiction. Hence, $y_0 \ne Ay_0 + \lambda_0 b$ for $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$, so by Lemma 2.5(ii), we get $i_P(A, \Omega_r) = 0$. □

Theorem 3.4. *Let* L, e, *and* N *be as in* (2.3), (3.1), *and* (3.2), *respectively. Suppose that one of the following conditions holds.*

($C1$) *There exist constants* $c_1, c_2, c_3 \in \mathbb{R}$ *with* $0 < c_1 < c_2 < ec_3$ *such that*

$$f_{ec_1}^{c_1}, f_{ec_3}^{c_3} \ge Ne, f_0^{c_2} \le \frac{1}{L^n}, \text{ and } y \ne Ay \text{ for } y \in \partial P_{c_2}.$$

($C2$) *There exist constants* $c_1, c_2, c_3 \in \mathbb{R}$ *with* $0 < c_1 < ec_2$ *and* $c_2 < c_3$ *such that*

$$f_0^{c_1}, f_0^{c_3} \le \frac{1}{L^n}, f_{ec_2}^{c_2} \ge Ne, \text{ and } y \ne Ay \text{ for } y \in \partial\Omega_{c_2}.$$

Then (1.1) *has two positive solutions. Additionally, if in* ($C2$) *the condition* $f_0^{c_1} \leq \frac{1}{L^n}$ *is replaced by* $f_0^{c_1} < \frac{1}{L^n}$*, then* (1.1) *has a third positive solution in* P_{c_1}*.*

Proof. Assume that ($C1$) holds. We show that either A has a fixed point in $\partial\Omega_{c_1}$ or in $P_{c_2} \setminus \overline{\Omega}_{c_1}$. If $y \neq Ay$ for $y \in \partial\Omega_{c_1}$, then by Lemma 3.3, we have $i_P(A, \Omega_{c_1}) = 0$. Since $f_0^{c_2} \leq \frac{1}{L^n}$ and $y \neq Ay$ for $y \in \partial P_{c_2}$, from Lemma 3.2 we get $i_P(A, P_{c_2}) = 1$. By Lemma 3.1(ii) and $c_1 < c_2$, we have $\overline{\Omega}_{c_1} \subset \overline{P}_{c_1} \subset P_{c_2}$. From Lemma 2.5($iii$), A has a fixed point in $P_{c_2} \setminus \overline{\Omega}_{c_1}$. If $y \neq Ay$ for $y \in \partial\Omega_{c_3}$, then from Lemma 3.3 $i_P(A, \Omega_{c_3}) = 0$. By Lemma 3.1(ii) and $c_2 < ec_3$, we get $\overline{P}_{c_2} \subset P_{ec_3} \subset \Omega_{c_3}$. From Lemma 2.5($iii$), A has a fixed point in $\Omega_{c_3} \setminus \overline{P}_{c_2}$. The proof is similar when ($C2$) holds and we omit it here. □

Corollary 3.5. *If there exist a constant* $c > 0$ *such that one of the following conditions holds:*

($H1$) $N < f_0, f_\infty \leq \infty$, $f_0^c \leq \frac{1}{L^n}$*, and* $y \neq Ay$ *for* $y \in \partial P_c$*.*

($H2$) $0 \leq f^0, f^\infty < \frac{1}{L^n}$, $f_{ec}^c \geq Ne$*, and* $y \neq Ay$ *for* $y \in \partial\Omega_c$*.*

Then (1.1) *has two positive solutions.*

Proof. Since ($H1$) implies ($C1$) and ($H2$) implies ($C2$), the result follows. □

As a special case of Theorem 3.4 and Corollary 3.5, we have the following two results.

Theorem 3.6. *Assume that one of the following conditions holds.*

($C3$) *There exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < c_2$ *such that*

$$f_{ec_1}^{c_1} \geq Ne \text{ and } f_0^{c_2} \leq \frac{1}{L^n}.$$

($C4$) *There exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < ec_2$ *such that*

$$f_0^{c_1} \leq \frac{1}{L^n} \text{ and } f_{ec_2}^{c_2} \geq Ne.$$

Then (1.1) *has a positive solution.*

Corollary 3.7. *Assume that one of the following conditions holds:*

($H3$) $0 \leq f^\infty < \frac{1}{L^n}$ *and* $N < f_0 \leq \infty$*.*

($H4$) $0 \leq f^0 < \frac{1}{L^n}$ *and* $N < f_\infty \leq \infty$*.*

Then (1.1) *has a positive solution.*

Example 3.8. *Let* $\mathbb{T} = \mathbb{R}$*. Consider the following boundary value problem*

$$y''(t) + \frac{y^2}{y^2+1} = 0, \quad t \in [1, 5],$$

$$y'(5) = 0, \quad y(1) - 2y'(1) = y'(3),$$

where $n = t_1 = \alpha = 1$, $t_2 = 3$, $t_3 = 5$, $\beta = 2$ *and* $f(t, y) = \frac{y^2}{y^2+1}$*. The Green's function* $G(t, s)$ *of this problem is*

$$G(t, s) = \begin{cases} H_1(t, s), & 1 \leq s \leq 3, \\ H_2(t, s), & 3 < s \leq 5, \end{cases}$$

where

$$H_1(t,s) = \begin{cases} s+1, & s \le t, \\ t+1, & t \le s, \end{cases}$$

and

$$H_2(t,s) = \begin{cases} s+2, & s \le t, \\ t+2, & t \le s. \end{cases}$$

Then we obtain

$$L = 20 \text{ and } N = \frac{1}{10}, \ e = \frac{4}{7}, \ f^0 = 0 = f^\infty,$$

$$f_{ec}^c = \frac{16c}{16c^2+49} \text{ and } f_0^c = \frac{c}{c^2+1}.$$

If we take $c = 1$, then the condition $(H2)$ of Corollary 3.5 is satisfied. Hence, the boundary value problem has two positive solutions such that $\min_{t\in[3,5]} y(t) \neq \frac{4}{7}$.

If we take $c_1 = 0.01$ and $c_2 = 16$, then $0 < c_1 < ec_2$ and the condition $(C4)$ of Theorem 3.6 is satisfied. Thus, the boundary value problem has a positive solution.

4. Three positive solutions

We will use the Leggett-Williams fixed point theorem to prove the next theorem.

Theorem 4.1. [16] *Let $\alpha > 0$, $\beta > 0$. Suppose that there exist numbers*

$$0 < p < q < \frac{qL^{n-1}}{K^n M^{n-1}} \le r$$

such that the function f satisfies the following conditions:

(i) $f(t,y) \le \dfrac{r}{L^n}$ *for* $t \in [t_1, \sigma(t_3)]$ *and* $y \in [0,r]$,

(ii) $f(t,y) > \dfrac{q}{K^n M^n}$ *for* $t \in [t_2, \sigma(t_3)]$ *and* $y \in \left[q, \dfrac{qL^{n-1}}{K^n M^{n-1}}\right]$,

(iii) $f(t,y) < \dfrac{p}{L^n}$ *for* $t \in [t_1, \sigma(t_3)]$ *and* $y \in [0,p]$,

where K, L, M are as defined in (2.2), (2.3), (2.4)*, respectively. Then* (1.1) *has at least three positive solutions y_1, y_2 and y_3 satisfying*

$$\max_{t\in[t_1,\sigma(t_3)]} y_1(t) < p, \quad q < \min_{t\in[t_2,\sigma(t_3)]} y_2(t),$$

$$p < \max_{t\in[t_1,\sigma(t_3)]} y_3(t) \text{ with } \min_{t\in[t_2,\sigma(t_3)]} y_3(t) < q.$$

Proof. The conditions of Theorem 2.6 will be shown to be satisfied. Define the non-negative continuous concave functional $\psi : P \to [0,\infty)$ to be

$$\psi(y) := \min_{t\in[t_2,\sigma(t_3)]} y(t)$$

and the cone P as in (2.5).
We have $\psi(y) \leq \|y\|$ for all $y \in P$. If $y \in \overline{P_r}$, then $0 \leq y \leq r$ and

$$f(t, y) \leq \frac{r}{L^n}$$

from the hypothesis (i). Then by Lemma 2.4, we get

$$\begin{aligned} \|Ay\| &= \int_{t_1}^{\sigma(t_3)} \max_{t\in[t_1,\sigma(t_3)]} G_n(t,s) f(s, y(\sigma(s)))\Delta s \\ &\leq L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(\cdot, s)\| f(s, y(\sigma(s)))\Delta s \\ &\leq r. \end{aligned}$$

This proves that $A : \overline{P_r} \to \overline{P_r}$.
Since $K < 1$ and $\frac{M}{L} < 1$,

$$y(t) \equiv \frac{qL^{n-1}}{K^n M^{n-1}} \in P\left(\psi, q, \frac{qL^{n-1}}{K^n M^{n-1}}\right)$$

and

$$\psi\left(\frac{qL^{n-1}}{K^n M^{n-1}}\right) > q.$$

Then

$$\left\{ y \in P(\psi, q, \frac{qL^{n-1}}{K^n M^{n-1}}) : \psi(y) > q \right\} \neq \emptyset.$$

For all $y \in P(\psi, q, \frac{qL^{n-1}}{K^n M^{n-1}})$, we have

$$q \leq \min_{t\in[t_2,\sigma(t_3)]} y(t) \leq \|y\| \leq \frac{qL^{n-1}}{K^n M^{n-1}} \text{ for } t \in [t_2, \sigma(t_3)].$$

Using the hypothesis (ii) and Lemma 2.4, we obtain

$$\begin{aligned} \psi(Ay) &= \int_{t_1}^{\sigma(t_3)} \min_{t\in[t_2,\sigma(t_3)]} G_n(t,s) f(s, y(\sigma(s)))\Delta s \\ &\geq K^n M^{n-1} \int_{t_2}^{\sigma(t_3)} \|G(\cdot, s)\| f(s, y(\sigma(s)))\Delta s \\ &> q. \end{aligned}$$

Hence, the condition (i) of Theorem 2.6 is satisfied.

If $\|y\| \leq p$, then $f(t,y) < \frac{p}{L^n}$ for $t \in [t_1, \sigma(t_3)]$ from the hypothesis (iii). We find

$$\begin{aligned} \|Ay\| &= \int_{t_1}^{\sigma(t_3)} \max_{t\in[t_1,\sigma(t_3)]} G_n(t,s) f(s, y(\sigma(s)))\Delta s \\ &\leq L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(\cdot,s)\| f(s,y(\sigma(s)))\Delta s \\ &< p. \end{aligned}$$

Consequently, the condition (ii) of Theorem 2.6 is satisfied.
For the condition (iii) of Theorem 2.6, we suppose that $y \in P(\psi, q, r)$ with

$$\|Ay\| > \frac{qL^{n-1}}{K^n M^{n-1}}.$$

Then, from Lemma 2.4 we obtain

$$\psi(Ay) = \min_{t\in[t_2,\sigma(t_3)]} Ay(t) \geq \frac{K^n M^{n-1}}{L^{n-1}} \|Ay\| > q.$$

Since all conditions of the Leggett-Williams fixed point theorem are satisfied, (1.1) has at least three positive solutions y_1, y_2 and y_3 such that

$$\begin{gathered} \max_{t\in[t_1,\sigma(t_3)]} y_1(t)) < p, \quad q < \min_{t\in[t_2,\sigma(t_3)]} y_2(t), \\ p < \max_{t\in[t_1,\sigma(t_3)]} y_3(t) \text{ with } \min_{t\in[t_2,\sigma(t_3)]} y_3(t) < q. \end{gathered}$$ □

Example 4.2. *Let* $\mathbb{T} = \{\frac{1}{5^n} : n \in \mathbb{N}\} \cup \{0\} \cup [3,5]$. *Taking* $n = 1$, $t_1 = \frac{1}{5}$, $t_2 = 3$, $t_3 = 5$, $\alpha = \frac{1}{2}$, $\beta = 2$ *and* $f(t,y) = \frac{y^2}{y^2+1}$, *we investigate the existence of at least three positive solutions of this problem by using Theorem 4.1. The Green's function* $G(t,s)$ *of this problem is*

$$G(t,s) = \begin{cases} H_1(t,s), & \frac{1}{5} \leq s \leq 3, \\ H_2(t,s), & 3 < s \leq 5, \end{cases}$$

where

$$H_1(t,s) = \begin{cases} \sigma(s) + \frac{19}{5}, & \sigma(s) \leq t, \\ t + \frac{19}{5}, & t \leq s, \end{cases}$$

and

$$H_2(t,s) = \begin{cases} \sigma(s) + \frac{29}{5}, & \sigma(s) \leq t, \\ t + \frac{29}{5}, & t \leq s. \end{cases}$$

Then we have $K = \frac{29}{54}$, $L = \frac{118}{5}$ *and* $M = \frac{98}{5}$.

If we take $p = 0.04$, $q = 6$ *and* $r = 24$ *then* $0 < p < q < \frac{q}{K} \leq r$ *and the conditions* $(i), (ii), (iii)$ *of Theorem 4.1 are satisfied. Thus, the three-point boundary value problem has at least three positive solutions* y_1, y_2 *and* y_3 *satisfying*

$$\max_{t\in[\frac{1}{5},5]} y_1(t) < p, \quad q < \min_{t\in[3,5]} y_2(t),$$

$$p < \max_{t\in[\frac{1}{5},5]} y_3(t) \textit{ with } \min_{t\in[3,5]} y_3(t) < q.$$

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