Fixed Point Theory, 17(2016), No. 1, 189-200
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FRACTIONAL ITERATIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. In this paper, a class of impulsive fractional iterative functional differential equations is studied. By applying the Schauder fixed point theorem, the first existence result of solutions is presented. By applying the Picard operators methods, the second existence, uniqueness and data dependence results are also established.

Key Words and Phrases: Impulses, iterative functional differential equations, existence, uniqueness, data dependence, fixed point.
2000 Mathematics Subject Classification: 26A33, 34A37, 34G20, 47H10.

1. INTRODUCTION

In this paper, we mainly study the existence, uniqueness and data dependence of the solutions of impulsive fractional iterative functional differential equations of the form:

$$\begin{cases} {}^cD^q_{a,t}x(t) = f(t, x(t), x(x^v(t))) + \lambda,\ t \in [a,b]\backslash D,\ v \in R\setminus\{0\},\ q \in (0,1), \\ x(t_k^+) = x(t_k^-) + I_k,\ k = 1,2,\cdots,m, \\ x(t) = \varphi(t),\ t \in [a_1, a], \\ x(t) = \psi(t),\ t \in [b, b_1], \end{cases} \tag{1.1}$$

where the symbol ${}^cD^q_{a,t}$ is the Caputo fractional derivative of order q with the lower limit a, $D := \{t_1, t_2, \cdots, t_m\}$, $a = t_0 < t_1 < t_2 < \cdots < t_{m+1} = b$, and

(C_1) a, b, a_1, b_1 are real numbers and satisfy $a_1 \le a < b \le b_1$, a function $\Upsilon(z) = z^v$ satisfies $\Upsilon \in C(J, J)$ with $J = [a_1, b_1]$, where $C(J, J)$ denote the Banach space of all continuous functions from J into J with the supremum norm.

(C_2) $f \in C([a,b] \times J^2, R), a_1 \le a_1^v, b_1^v \le b_1$;

(C_3) $\varphi \in C([a_1, a], J)$ and $\psi \in C([b, b_1], J)$;

(C_4) there exist $L_f > 0$, $\nu > 0$ such that

$$|f(t, u_1, w_1) - f(t, u_2, w_2)| \le L_f(|u_1 - u_2| + |w_1 - w_2|^\nu)$$

for all $t \in [a,b],\ u_i, w_i \in J,\ i = 1,2$.

Set $PC([a,b], J) := \{x : [a,b] \to J : x_k \in C((t_k, t_{k+1}], J),\ k = 0, \cdots, m$ and there exist $x(t_k^-)$ and $x(t_k^+), k = 1, \cdots, m,$ with $x(t_k^-) = x(t_k)\}$ which is a Banach space with the norm $\|x\|_{PC} := \max\{|x_k(t)| : t \in [t_k, t_{k+1}],\ k = 0, \cdots, m\}$ where x_k is the restriction of x to $[t_k, t_{k+1}]$, $k = 0, \cdots, m$. Moreover, we define

$$PC(J, J) := \{x : J \to J : x \in PC([a, b], J) \cup C([a_1, a], J) \cup C([b, b_1], J)\}$$

and

$$PC(J, R) := \{x : J \to R : x \in PC([a, b], R) \cup C([a_1, a], R) \cup C([b, b_1], R)\}.$$

A number of papers have been recently written on fractional impulsive initial and boundary value problems [1, 2, 3, 4, 5, 6, 7, 8, 23, 24, 25, 26, 31, 32]. Meanwhile, Fečkan et al. [9], Kosmatov [11] and Wang et al. [28, 29, 30] all pointed the error in former solutions for some impulsive fractional differential equations by construct a counterexample and establish a general framework to seek a nature solution for such problems.

Motivated by [9, 11], we define what it means for the problem (1.1) to have a solution.

Definition 1.1. A function $x \in PC(J, J)$ is said to be a solution of the problem (1.1) if $x(t) = x_k(t)$ for $t \in (t_k, t_{k+1})$ and $x_k \in C([a, t_{k+1}], J)$ satisfies the equation ${}^cD^q_{a,t}x(t) = f(t, x(t), x(x^v(t))) + \lambda$ a.e. on (a, t_{k+1}) with the restriction of $x_{k+1}(t)$ on $[a, t_{k+1})$ is just $x_k(t)$, $x(t_k^+) = x(t_k^-) + I_k$, $k = 1, 2, \cdots, m$ and $x(t) = \varphi(t), t \in [a_1, a]$, $x(t) = \psi(t), t \in [b, b_1]$.

Let (x, λ) be a solution of the problem (1.1). Then this problem is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t),\ \text{for } t \in [a_1, a], \\ \varphi(a) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, x(s), x(x^v(s)))ds \\ + \frac{\lambda}{\Gamma(q+1)} (t-a)^q + \sum_{i=1}^k I_i, \\ \text{for } t \in (t_k, t_{k+1}),\ k = 0, 1, 2, \cdots, m, \\ \psi(t),\ \text{for } t \in [b, b_1]. \end{cases} \tag{1.2}$$

Using the condition of continuity of x in $t = b$, we derive that

$$\lambda = \frac{\Gamma(q+1)(\psi(b) - \varphi(a) - \sum_{i=1}^m I_i)}{(b-a)^q} - \frac{q}{(b-a)^q} \int_a^b (b-s)^{q-1} f(s, x(s), x(x^v(s)))ds. \tag{1.3}$$

Consequently, the problem (1.1) is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t),\ \text{for } t \in [a_1, a], \\ \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, x(s), x(x^v(s)))ds \\ + \frac{(t-a)^q}{(b-a)^q} (\psi(b) - \varphi(a) - \sum_{i=1}^m I_i) + \varphi(a) + \sum_{i=1}^k I_i \\ - \frac{(t-a)^q}{(b-a)^q \Gamma(q)} \int_a^b (b-s)^{q-1} f(s, x(s), x(x^v(s)))ds, \\ \text{for } t \in (t_k, t_{k+1}),\ k = 0, 1, 2, \cdots, m, \\ \psi(t),\ \text{for } t \in [b, b_1]. \end{cases} \tag{1.4}$$

Define

$$A : PC(J, J) \to PC(J, R), \tag{1.5}$$

where

$$A(x)(t) := \begin{cases} \varphi(t), \text{ for } t \in [a_1, a], \\ \varphi(a) + \sum_{i=1}^{k} I_i + \frac{(t-a)^q}{(b-a)^q}\left(\psi(b) - \varphi(a) - \sum_{i=1}^{m} I_i\right) \\ -\frac{(t-a)^q}{(b-a)^q \Gamma(q)} \int_a^b (b-s)^{q-1} f(s, x(s), x(x^v(s)))ds \\ +\frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, x(s), x(x^v(s)))ds \\ \text{for } t \in (t_k, t_{k+1}),\ k = 0, 1, 2, \cdots, m, \\ \psi(t), \text{ for } t \in [b, b_1]. \end{cases} \tag{1.6}$$

It is clear that (x, λ) is a solution of the problem (1.1) if and only if x is a fixed point of the operator A and λ is given by (1.3). Then, all kinds of fixed point theorems can be applied to derive the existence of solutions.

2. PRELIMINARIES

We recall some basic definitions of the fractional calculus theory which are used further in this paper. For more details, see Kilbas et al. [10].

Definition 2.1. The fractional order integral of the function $h \in L^1([a,b], R)$ of order $q \in R^+$ is defined by

$$I_{a,t}^q h(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds$$

where Γ is the Gamma function.

Definition 2.2. For a function h given on the interval $[a, b]$, the qth Riemann-Liouville fractional order derivative of h, is defined by

$${}^L(D_{a,t}^q h)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1} h(s)ds,$$

here $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3. The Caputo derivative of order q for a function $f : [a, b] \to R$ can be written as

$${}^c D_{a,t}^q h(t) = {}^L D_{a,t}^q \left(h(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} h^{(k)}(a) \right),\ t > 0,\ n - 1 < q < n.$$

It is remarkable that (weakly) Picard operators methods is a powerful tool to study the nonlinear differential equations. It can be widely used to discuss existence and uniqueness and the data dependence on data of the solutions for nonlinear differential equations. For more details, one can see Mureşan [12, 13], Olaru [14], Rus et al. [15, 16, 17, 18, 19, 20, 21], Şerban et al. [22] and Wang et al. [27].

We collect some notions and results from the Picard operator theory (for more details see Rus [19, 20]).

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A = \{x \in X \mid A(x) = x\}$–the fixed point set of A;

$I(A) = \{Y \in P(X) \mid A(Y) \subseteq Y, Y \neq \emptyset\}$;

$A^{n+1} = A^n \circ A$, $A^1 = A$, $A^0 = I$, $n \in N$

$P(X) = \{Y \subseteq X \mid Y \neq \emptyset\}$;
$O_A(x) = \{x, A(x), A^2(x), \cdots, A^n(x), \cdots\}$−the A−orbit of $x \in X$;
$H : P(X) \times P(X) \to R_+ \cup \{+\infty\}$;
$H(Y, Z) = \max\left\{\sup_{y\in Y} \inf_{z\in Z} d(y, z), \sup_{z\in Z} \inf_{y\in Y} d(y, z)\right\}$ - the Pompeiu-Hausdorff functional on $P(X) \times P(X)$.

Definition 2.4. Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in N}$ converges to x^* for all $x_0 \in X$.

Definition 2.5. Let A be a Picard operator and $c > 0$. The operator A is c−Picard operator if $d(x, A^n(x)) \leq cd(x, A(x))$ for all $x \in X$, $n \in N$.

Theorem 2.6. *(Contraction principle) Let (X, d) be a complete metric space and $A : X \to X$ a γ−contraction. Then*
(i) $F_A = \{x^*\}$;
(ii) $(A^n(x_0))_{n\in N}$ converges to x^* for all $x_0 \in X$;
(iii) $d(x^*, A^n(x_0)) \leq \frac{\gamma^n}{1-\gamma} d(x_0, A(x_0))$, for all $n \in N$.

Remark 2.7. Accordingly to the Definition 2.4, the contraction principle insures that, if $A : X \to X$ is a γ−contraction on the complete metric space X, then it is a Picard operator.

Theorem 2.8. *Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose the following:*
(i) A is a contraction with contraction constant γ and $F_A = \{x_A^*\}$.
(ii) B has fixed points and $x_B^* \in F_B$.
(iii) There exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$, for all $x \in X$.
Then

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\gamma}.$$

3. EXISTENCE RESULT VIA SCHAUDER FIXED POINT THEOREM

We state the following assumptions:
(H1) The conditions (C_1)–(C_3) are satisfied.
(H2) There are m_f, $M_f \in R$ such that

$$m_f \leq f(t, u, w) \leq M_f, \ \forall\ t \in [a, b]\backslash D, u, w \in J,$$

and moreover,

$$\begin{aligned} a_1 \leq \min &\left((\varphi(a) + \sum_{i=1}^{k} I_i), (\psi(b) - \sum_{i=k+1}^{m} I_i) \right) \\ &- \max\left(0, \frac{M_f(b-a)^q}{\Gamma(q+1)}\right) + \min\left(0, \frac{m_f(b-a)^q}{\Gamma(q+1)}\right), \end{aligned}$$

and

$$\max\left((\varphi(a)+\sum_{i=1}^{k} I_i),(\psi(b)-\sum_{i=k+1}^{m} I_i)\right)$$
$$-\min\left(0,\frac{m_f(b-a)^q}{\Gamma(q+1)}\right)+\max\left(0,\frac{M_f(b-a)^q}{\Gamma(q+1)}\right)\le b_1,$$

where $k=1,2,\cdots,m$.

We are ready to state our first result in this paper.

Theorem 3.1. *Assumptions (H1) and (H2) hold. Then the problem (1.1) has a solution in $PC(J,J)$.*

Proof. In what follow we consider on $PC(J,R)$ with the norm $\|\cdot\|_{PC}$. Firstly, (H2) assures that the set $PC(J,J)$ is an invariant subset for the operator A, that is, we have

$$A(PC(J,J))\subset PC(J,J).$$

Indeed, for $t\in[a_1,a]\cup[b,b_1]$, we have $A(x)(t)\in J$. Secondly, we obtain

$$a_1\le A(x)(t)\le b_1,\ \forall\ t\in[a,b]\backslash D,$$

if and only if

$$a_1\le\min_{t\in[a,b]\backslash D}A(x)(t) \tag{3.1}$$

and

$$\max_{t\in[a,b]\backslash D}A(x)(t)\le b_1 \tag{3.2}$$

hold.

Since

$$\begin{aligned}\min_{t\in[a,b]\backslash D}A(x)(t)\quad\ge\quad&\min\left((\varphi(a)+\sum_{i=1}^{k} I_i),(\psi(b)-\sum_{i=k+1}^{m} I_i)\right)\\&-\max\left(0,\frac{M_f(b-a)^q}{\Gamma(q+1)}\right)+\min\left(0,\frac{m_f(b-a)^q}{\Gamma(q+1)}\right),\end{aligned}$$

respectively

$$\max_{t\in[a,b]\backslash D}A(x)(t)\le\max\left((\varphi(a)+\sum_{i=1}^{k} I_i),(\psi(b)-\sum_{i=k+1}^{m} I_i)\right)$$
$$-\min\left(0,\frac{m_f(b-a)^q}{\Gamma(q+1)}\right)+\max\left(0,\frac{M_f(b-a)^q}{\Gamma(q+1)}\right),$$

where $k=1,2,\cdots,m$.

Clearly, (3.1) and (3.2) are equivalent with the conditions appearing in (H2).

Thus, the operator

$$A:PC(J,J)\to PC(J,J).$$

Thirdly, we check A is a completely continuous operator. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J, J)$. Then for each $t \in J$, we have that

$$\begin{aligned} &|(Ax_n)(t) - (Ax)(t)| \\ \leq &\begin{cases} 0, \text{ for } t \in [a_1, a], \\ \frac{2(b-a)^q}{\Gamma(q+1)} \|f(\cdot, x_n(\cdot), x_n(x_n^v(\cdot)) - f(\cdot, x_n(\cdot), x(x^v(\cdot)))\|_{PC}, \text{ for } t \in [a, b]\backslash D, \\ 0, \text{ for } t \in [b, b_1]. \end{cases} \end{aligned}$$

Since $f \in C([a, b] \times J^2, R)$, we have that

$$\|Ax_n - Ax\|_{PC} \to 0 \ as \ n \to \infty.$$

Now, consider $a_1 \leq s_1 < s_2 \leq a$. Then,

$$|(Ax)(s_2) - (Ax)(s_1)| = |\varphi(s_2) - \varphi(s_1)|.$$

Similarly, for $b \leq s_1 < s_2 \leq b_1$,

$$|(Ax)(s_2) - (Ax)(s_1)| = |\psi(s_2) - \psi(s_1)|.$$

On the other hand, for $t_k \leq s_1 < s_2 \leq t_{k+1}$, $k = 0, 1, \cdots, m$,

$$\begin{aligned} |(Ax)(s_2) - (Ax)(s_1)| \leq & \frac{(s_2 - s_1)^q}{(b-a)^q} |\psi(b) - \varphi(a) - \sum_{i=1}^{m} I_i| \\ & + \frac{2(s_2 - s_1)^q \max\{|m_f|, |M_f|\}}{\Gamma(q+1)}. \end{aligned}$$

Together with the Arzela-Ascoli theorem and A is a continuous operator, we can conclude that A is a completely continuous operator.

It is obvious that the set $PC(J, J) \subseteq PC(J, R)$ is a bounded convex closed subset of the Banach space $PC(J, R)$. Thus, the operator A has a fixed point due to the well known Schauder's fixed point theorem. This completes the proof. □

To end this section, we consider the following problem:

$$\begin{cases} {}^cD_{0,t}^{\frac{1}{2}} x(t) = \mu x(x(t)) + \lambda, \ t \in [0, 1] \setminus \{\frac{1}{5}\}, \ \mu > 0, \ \lambda \in R, \\ x(t) = 0, \ t \in [-\frac{1}{3}, 0], \\ x(t) = 1, \ t \in [1, \frac{4}{3}], \\ x(\frac{1}{5}^+) - x(\frac{1}{5}^-) = \frac{1}{7}, \end{cases} \tag{3.3}$$

where $x \in PC([-\frac{1}{3}, \frac{4}{3}], [-\frac{1}{3}, \frac{4}{3}])$.

First of all notice that according to the Theorem 3.1 we have $v = 1$, $q = \frac{1}{2}$, $a = 0, b = 1$, $\psi(b) = 1, \varphi(a) = 0$ and $f(t, u_1, u_2) = \mu u_2, I_1 = \frac{1}{7}$, $t_1 = \frac{1}{5}$. Moreover, $a_1 = -\frac{1}{3}$ and $b_1 = \frac{4}{3}$ can be taken. Therefore, from the relation

$$m_f \leq f(t, u_1, u_2) \leq M_f, \ \forall \ t \in [0, 1], \ u_1, u_2 \in [-\frac{1}{3}, \frac{4}{3}],$$

we can choose $m_f = -\frac{\mu}{3}$ and $M_f = \frac{4\mu}{3}$. For these data it can be easily verified that the conditions (H2) from the Theorem 3.1 are equivalent with $\mu \leq \frac{2\Gamma(\frac{3}{2})}{7}$. Then the problem (10) has in $PC([-\frac{1}{3}, \frac{4}{3}], [-\frac{1}{3}, \frac{4}{3}])$ at least a solution provided $\mu \leq \frac{\sqrt{\pi}}{7}$.

4. Further results

In Section 2, we only obtain the existence result. In order to obtain the uniqueness result, we need to introduce the following notation:

$$PC_L^q(J,J)=\{x\in PC(J,J):|x(t_1)-x(t_2)|\le L|t_1-t_2|^q, L>0\},$$

for all $t_1,t_2\in J$. Remark that $PC_L^q(J,J)\subseteq PC(J,R)$ is also a complete metric space with respect to the metric, $d(x_1,x_2):=\|x_1(\cdot)-x_2(\cdot)\|_{PC}$. Consider the operator

$$A: PC_L^q(J,J)\to PC(J,R)$$

where the definition of $A(x)(\cdot)$ is the same as (1.6).

In addition to (H2), we also list the necessary additional assumptions:

(H1') The conditions (C_1) and (C_2) are satisfied but in addition $v\in(0,1], b_1\le 1, \nu\ge\frac{1}{vq}$.

(H1") $\varphi\in C_L^q([a_1,a],J)$, $\psi\in C_L^q([b,b_1],J)$.

(H3) $\frac{3\max\{|m_f|,|M_f|\}}{\Gamma(q+1)}+\frac{|\psi(b)-\varphi(a)-\sum_{i=1}^m I_i|}{(b-a)^q}\le L$.

(H4) $\left(1+2^{\nu-1}L+2^{\nu-1}\right)\frac{2L_f(b-a)^q}{\Gamma(q+1)}<1$.

Theorem 4.1. *Assumptions (H1'), (H1"), (H2), (H3) and (H4) hold. Then the problem (1.1) has in $PC_L^q(J,J)$ a unique solution. Moreover, the operator $A: PC_L^q(J,J)\to PC_L^q(J,R)$ is a $c-$Picard operator with*

$$c=\frac{1}{1-\left(1+2^{\nu-1}L+2^{\nu-1}\right)\frac{2L_f(b-a)^q}{\Gamma(q+1)}}.$$

Proof. First of all we prove that $PC_L^q(J,J)$ is an invariant subset for A. Indeed, for $t\in[a_1,a]\cup[b,b_1]$, we have $A(x)(t)\in J$. Similar to the proof of Theorem 3.1, we obtain $a_1\le A(x)(t)\le b_1,\ \forall\, t\in[a,b]\backslash D$, by virtue of (H2).

Now, consider $a_1\le s_1<s_2\le a$. Then,

$$|A(x)(s_2)-A(x)(s_1)| \;=\; |\varphi(s_2)-\varphi(s_1)| \;\le\; L|s_1-s_2|^q,$$

as $\varphi\in C_L^q([a_1,a],J)$, due to (H1"). Similarly, for $b\le s_1<s_2\le b_1$,

$$|A(x)(s_2)-A(x)(s_1)| \;=\; |\psi(s_2)-\psi(s_1)| \;\le\; L|s_1-s_2|^q,$$

that follows from (H1"), too.

On the other hand, for $t_k\le s_1<s_2\le t_{k+1}$, $k=0,1,\cdots,m$,

$$\begin{aligned}
&|A(x)(s_2)-A(x)(s_1)|\\
\le\;& \frac{\max\{|m_f|,|M_f|\}}{\Gamma(q)}\left|\int_a^{s_2}(s_2-s)^{q-1}ds-\int_a^{s_1}(s_1-s)^{q-1}ds\right|\\
&+\frac{|\psi(b)-\varphi(a)-\sum_{i=1}^m I_i|}{(b-a)^q}|(s_2-a)^q-(s_1-a)^q|\\
&+\frac{\max\{|m_f|,|M_f|\}\left|\int_a^b(b-s)^{q-1}ds\right|}{(b-a)^q\Gamma(q)}|(s_2-a)^q-(s_1-a)^q|\\
\le\;& \left(\frac{3\max\{|m_f|,|M_f|\}}{\Gamma(q+1)}+\frac{|\psi(b)-\varphi(a)-\sum_{i=1}^m I_i|}{(b-a)^q}\right)|s_2-s_1|^q,
\end{aligned}$$

where we use the inequality $r^q - s^q \le |r-s|^q$ for all $0 < q < 1$. Therefore, due to (H3), the function $A(x)$ is L–Lipschitz in t. Thus, according to the above, we have $PC_L^q([a_1, a], J) \in I(A)$.

From the condition (H4) it follows that A is an L_A–contraction with

$$L_A := \left(1 + 2^{\nu-1}L + 2^{\nu-1}\right) \frac{2L_f(b-a)^q}{\Gamma(q+1)}.$$

Indeed, for all $t \in [a_1, a] \cup [b, b_1]$, we

$$|A(x_1)(t) - A(x_2)(t)| = 0.$$

Moreover, for $t \in [a, b]\backslash D$ we get

$$\begin{aligned}
&|A(x_1)(t) - A(x_2)(t)| \\
&\le \frac{(t-a)^q}{\Gamma(q)(b-a)^q} \int_a^b (b-s)^{q-1} \left| f(s, x_1(s), x_1(x_1^v(s))) - f(s, x_2(s), x_2(x_2^v(s))) \right| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left| f(s, x_1(s), x_1(x_1^v(s))) - f(s, x_2(s), x_2(x_2^v(s))) \right| ds \\
&\le \frac{L_f}{\Gamma(q)} \int_a^b (b-s)^{q-1} \left[|x_1(s) - x_2(s)| + |x_1(x_1^v(s)) - x_2(x_2^v(s))|^\nu \right] ds \\
&\quad + \frac{L_f}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left[|x_1(s) - x_2(s)| + |x_1(x_1^v(s)) - x_2(x_2^v(s))|^\nu \right] ds \\
&\le \frac{L_f}{\Gamma(q)} \int_a^b (b-s)^{q-1} \Big[|x_1(s) - x_2(s)| + 2^{\nu-1} |x_1(x_1^v(s)) - x_1(x_2^v(s))|^\nu \\
&\qquad + 2^{\nu-1} |x_1(x_2^v(s)) - x_2(x_2^v(s))|^\nu \Big] ds \\
&\quad + \frac{L_f}{\Gamma(q)} \int_a^t (t-s)^{q-1} \Big[|x_1(s) - x_2(s)| + 2^{\nu-1} |x_1(x_1^v(s)) - x_1(x_2^v(s))|^\nu \\
&\qquad + 2^{\nu-1} |x_1(x_2^v(s)) - x_2(x_2^v(s))|^\nu \Big] ds \\
&\le \frac{L_f}{\Gamma(q)} \int_a^b (b-s)^{q-1} \left[\|x_1 - x_2\|_{PC} + 2^{\nu-1} L\, |x_1(s) - x_2(s)|^{\nu v q} + 2^{\nu-1} \|x_1 - x_2\|_{PC}^\nu \right] ds \\
&\quad + \frac{L_f}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left[\|x_1 - x_2\|_{PC} + 2^{\nu-1} L\, |x_1(s) - x_2(s)|^{\nu v q} + 2^{\nu-1} \|x_1 - x_2\|_{PC}^\nu \right] ds \\
&\le \frac{L_f}{\Gamma(q)} \int_a^b (b-s)^{q-1} \left[\|x_1 - x_2\|_{PC} + 2^{\nu-1} L \|x_1 - x_2\|_C^{\nu v q} + 2^{\nu-1} \|x_1 - x_2\|_{PC}^\nu \right] ds \\
&\quad + \frac{L_f}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left[\|x_1 - x_2\|_{PC} + 2^{\nu-1} L \|x_1 - x_2\|_C^{\nu v q} + 2^{\nu-1} \|x_1 - x_2\|_{PC}^\nu \right] ds \\
&\le \frac{2(b-a)^q L_f}{\Gamma(q+1)} \left(1 + 2^{\nu-1}L + 2^{\nu-1}\right) \|x_1 - x_2\|_{PC} = L_A \|x_1 - x_2\|_{PC},
\end{aligned}$$

where we use the fact $\|x_1 - x_2\|_{PC} \le 1$, $\nu \ge 1$, $\nu v q \ge 1$ and the inequalities

$$(r+s)^\nu \le 2^{\nu-1}(r^\nu + s^\nu) \text{ and } |r^v - s^v| \le |r-s|^v,$$

for nonnegative r, s and $v \in (0,1]$. So we get

$$\|A(x_1) - A(x_2)\|_{PC} \leq L_A \|x_1 - x_2\|_{PC}. \tag{4.1}$$

So, A is a $c-$Picard operator, with

$$c = \frac{1}{1 - (1 + 2^{\nu-1}L + 2^{\nu-1}) \frac{2L_f (b-a)^q}{\Gamma(q+1)}}.$$

This completes the proof. □

Next, we consider the problem (1.1) and suppose the conditions of Theorem 4.1 are satisfied. Denote by $x(\cdot; \varphi, \psi, f)$ the solution of the problem (1.1).

We need the following assumptions:

(H5) There exists $\eta_1 > 0$, such that $|\varphi_1(t) - \varphi_2(t)| \leq \eta_1,\ t \in [a_1, a]$, and $|\psi_1(t) - \psi_2(t)| \leq \eta_1,\ t \in [b, b_1]$.

(H6) There exists $\eta_2 > 0$ such that $|f_1(t,u,w) - f_2(t,u,w)| \leq \eta_2$, $\forall t \in [a,b]$, $u, w \in J$.

Theorem 4.2. *Assumptions (H5) and (H6) hold. Let $\varphi_i, \psi_i, f_i, i = 1,2$, be as in Theorem 4.1. Then we have*

$$|x(t; \varphi_1, \psi_1, f_1) - x(t; \varphi_2, \psi_2, f_2)| \leq \frac{3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2}{1 - (1 + 2^{\nu-1}L + 2^{\nu-1}) \frac{2L_f (b-a)^q}{\Gamma(q+1)}},$$

and

$$|\lambda_1^* - \lambda_2^*| \leq \frac{2\Gamma(q+1)}{(b-a)^q}\eta_1 + \eta_2,$$

where $L_f = \min\{L_{f_1}, L_{f_2}\}$, and λ_i^*, are the parameters of the solutions of the corresponding solutions $x(\cdot; \varphi_i, \psi_i, f_i)(i = 1,2)$, $L_f = \min\{L_{f_1}, L_{f_2}\}$.

Proof. Consider the operators $A_{\varphi_i, \psi_i, f_i}, i = 1,2$. From Theorem 4.1 these operators are contractions. Additionally, for $t \in [a,b] \setminus D$, we have

$$\begin{aligned}
& |A_{\varphi_1,\psi_1,f_1}(x) - A_{\varphi_2,\psi_2,f_2}(x)| \\
\leq\ & \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} |f_1(s, x(s), x(x^v(s))) - f_2(s, x(s), x(x^v(s))| ds \\
& + \frac{(t-a)^q}{(b-a)^q} |\psi_1(b) - \varphi_1(a) - \psi_2(b) + \varphi_2(a)| + |\varphi_1(a) - \varphi_2(a)| \\
& + \frac{(t-a)^q}{(b-a)^q \Gamma(q)} \int_a^b (b-s)^{q-1} |f_1(s, x(s), x(x^v(s))) - f_2(s, x(s), x(x^v(s))| ds \\
\leq\ & \frac{\eta_2}{\Gamma(q)} \int_a^t (t-s)^{q-1} ds + \frac{2\eta_1 (t-a)^q}{(b-a)^q} + \eta_1 + \frac{\eta_2 (t-a)^q}{(b-a)^q \Gamma(q)} \int_a^b (b-s)^{q-1} ds \\
=\ & \frac{2\eta_2 (t-a)^q}{\Gamma(q+1)} + \frac{2\eta_1 (t-a)^q}{(b-a)^q} + \eta_1.
\end{aligned}$$

Now, the proof follows from Theorem 2.8, with

$$A := A_{\varphi_1,\psi_1,f_1},\ B := A_{\varphi_2,\psi_2,f_2},\ \eta := 3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2,$$

and

$$\gamma := L_A = \left(1 + 2^{\nu-1}L + 2^{\nu-1}\right)\frac{2L_f(b-a)^q}{\Gamma(q+1)}.$$

Next, (4.1) holds for both A_i with L_{f_i}.

Without loss of generality, we may suppose that $L_{f_1} = \min\{L_{f_1}, L_{f_2}\}$.

Consequently, we obtain

$$\begin{aligned}\|x_1^* - x_2^*\|_{PC} &= \|A_1(x_1^*) - A_2(x_2^*)\|_{PC}\\ &\le \|A_1(x_2^*) - A_1(x_2^*)\|_{PC} + \|A_1(x_2^*) - A_2(x_2^*)\|_{PC}\\ &\le L_{A_1}\|x_1^* - x_2^*\|_{PC} + \|A_1(x_2^*) - A_2(x_2^*)\|_{PC},\end{aligned}$$

where $x_i^* := x(\cdot;\varphi_i,\psi_i,f_i)(i=1,2)$, which implies the fist statement.

Moreover, we get

$$\begin{aligned}&|\lambda_1^* - \lambda_2^*|\\ \le\ & \frac{\Gamma(q+1)(|\psi_1(b) - \psi_2(b)| + |\varphi_1(a) - \varphi_2(a)|)}{(b-a)^q}\\ &+\frac{q}{(b-a)^q}\int_a^b (b-s)^{q-1}\,|f_1(s,x(s),x(x^v(s))) - f_2(s,x(s),x(x^v(s)))|\,ds\\ \le\ & \frac{2\Gamma(q+1)}{(b-a)^q}\eta_1 + \eta_2.\end{aligned}$$

The proof is completed. □

Acknowledgements. The authors thank the referees for their careful reading of the manuscript and insightful comments. We also acknowledge the valuable comments and suggestions from the editors. Finally, the first authors work was supported by Unite Foundation of Guizhou Province ([2015]7640) and Outstanding Scientific and Technological Innovation Talent Award of Education Department of Guizhou Province ([2014]240) and the third authors work was supported by National Natural Science Foundation of China (11261011).

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Received: September 28, 2012; Accepted: January 13, 2013.