

## ITERATIVE PROCEDURES FOR LEFT BREGMAN STRONGLY RELATIVELY NONEXPANSIVE MAPPINGS WITH APPLICATION TO EQUILIBRIUM PROBLEMS

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**Abstract.** Our purpose in this paper is to prove strong convergence theorems using the modified Mann type iteration for approximation of a fixed point of a left Bregman strongly relatively nonexpansive mapping in the framework of reflexive real Banach spaces. We also discuss the approximation of a fixed point of a left Bregman strongly nonexpansive mapping which is also solution to a finite system of equilibrium problems in reflexive real Banach spaces. Our results complement many known recent results in the literature.

**Key Words and Phrases:** Left Bregman strongly relatively nonexpansive mapping, left Bregman projection, equilibrium problem, Banach spaces.

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