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# $\Delta$-CONVERGENCE AND W-CONVERGENCE OF THE MODIFIED MANN ITERATION FOR A FAMILY OF ASYMPTOTICALLY NONEXPANSIVE TYPE MAPPINGS IN COMPLETE CAT(0) SPACES

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**Abstract.** In this paper, we show $\Delta$-convergence and $w$-convergence (in the sense of Ahmadi Kakavandi and Amini [2]) of modified Mann iteration

$$x_{n+1} = \alpha_n Py_n \oplus (1 - \alpha_n)T_n^n Py_n,\ d(y_n, x_n) \leq e_n,\ x_0 \in C,$$

to a common fixed point of the sequence $(T_n)$ of asymptotically nonexpansive type selfmappings on a closed and convex subset $C$ of a complete CAT(0) space $X$, where $(\alpha_n) \subset [0, 1]$, $(e_n) \subset \mathbb{R}^+$ and $P$ is the nearest point projection on $C$. Our results extend the results in [16, 21] in the setting of complete CAT(0) spaces.

**Key Words and Phrases**: $w$-convergence, $\Delta$-convergence, Asymptotically nonexpansive type selfmapping, Fixed point, CAT(0) space.
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## 1. Introduction

Let $C$ be a nonempty subset of a metric space $(X, d)$ and $Y$ be a nonempty subset of $C$. A mapping $T : C \to C$ is called nonexpansive respect to $Y$ if for each $x \in C$ and $y \in Y$, $d(Tx, Ty) \leq d(x, y)$. If $Y = C$, $T$ is called nonexpansive and if $Y = F(T) := \{x \in C : T(x) = x\}$, $T$ is called quasi-nonexpansive. $T$ is said to be asymptotically nonexpansive respect to $Y$ if there exists a sequence $(k_n)$ of positive real numbers such that $k_n \longrightarrow 1$ and for all $x \in C$ and $y \in Y$, $d(T^n x, T^n y) \leq k_n d(x, y)$. If $Y = C$, the mapping $T$ is called asymptotically nonexpansive and if $Y = F(T)$, $T$ is called asymptotically quasi-nonexpansive. The mapping $T$ is said to be asymptotically nonexpansive type respect to $Y$ if $\limsup_{n\to\infty} \sup_{y\in Y} (d(T^n x, T^n y) - d(x, y)) \leq 0$, for all $x \in C$. If $Y = C$, $T$ is called asymptotically nonexpansive type and if $Y = F(T)$, $T$ is called asymptotically quasi-nonexpansive type. It is clear that nonexpansive mappings (quasi-nonexpansive mappings) and asymptotically nonexpansive mappings (asymptotically quasi-nonexpansive mappings) are asymptotically nonexpansive type mappings (resp.

asymptotically quasi-nonexpansive type mappings). The sequence $(T_n)$ of selfmappings on $C$ is called a family of asymptotically nonexpansive mappings respect to $Y$ if for each $T_i$, there exists a sequence $(k_{n,i})$ of positive real numbers such that $k_{n,i} \longrightarrow 1$, as $n \to \infty$, and for all $x \in C$ and $y \in Y$, $d(T_i^n x, T_i^n y) \leq k_{n,i} d(x,y)$. If $Y = C$, the sequence $(T_n)$ is called a family of asymptotically nonexpansive mappings and if $Y = \bigcap_{n=1}^{\infty} F(T_n)$, the sequence $(T_n)$ is called a family of asymptotically quasi-nonexpansive mappings. The sequence $(T_n)$ of selfmappings on $C$ is called a family of asymptotically nonexpansive type mappings respect to $Y$ if each $T_i$ satisfies $\limsup_{n\to\infty} \sup_{y\in Y}(d(T_i^n x, T_i^n y) - d(x,y)) \leq 0$, for all $x \in C$. If $Y = C$, the sequence $(T_n)$ is called a family of asymptotically nonexpansive type mappings and if $Y = \bigcap_{n=1}^{\infty} F(T_n)$, the sequence $(T_n)$ is called a family of asymptotically quasi-nonexpansive type mappings.

Mann [15], for approximation fixed point of nonexpansive mapping $T$, suggested the iterative sequence given by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$. He proved the weak convergence of this sequence under the appropriate conditions on $(\alpha_n) \subset [0,1]$. Since then many authors worked on Mann iteration and extended the results in Hilbert and Banach spaces. Schu in [18, 19] proved weak and strong convergence of the modified Mann iteration $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n$ for the asymptotically nonexpansive mappings in Hilbert and Banach spaces. Nanjaras and Panyanak [16] extended the results of Schu [18, 19] to CAT(0) spaces. Let us to introduce the CAT(0) spaces.

Let $(X,d)$ be a metric space and $x, y \in X$. A geodesic path joining $x$ to $y$ is an isometry $c : [0, d(x,y)] \longrightarrow X$ such that $c(0) = x, c(d(x,y)) = y$. The image of a geodesic path joining $x$ to $y$ is called a geodesic segment between $x$ and $y$. The metric space $(X,d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$.

A geodesic space $(X,d)$ is a CAT(0) space if satisfies the following inequality:
*CN−inequality*: If $x, y_0, y_1, y_2 \in X$ such that $d(y_0,y_1) = d(y_0,y_2) = \frac{1}{2}d(y_1,y_2)$, then

$$d^2(x,y_0) \leq \frac{1}{2}d^2(x,y_1) + \frac{1}{2}d^2(x,y_2) - \frac{1}{4}d^2(y_1,y_2).$$

It is known that a CAT(0) space is a uniquely geodesic space. For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [4, 6, 10, 11]. Some examples of CAT(0) spaces are pre-Hilbert spaces (see [4]), $\mathbb{R}$-trees (see [12]), Euclidean buildings (see [5]), the complex Hilbert ball with a hyperbolic metric (see [9]), Hadamard manifolds and many others.
Let $X$ be a CAT(0) space and $x, y \in X$. We write $(1-t)x \oplus ty$ for the unique point $z$ in the geodesic segment joining from $x$ to $y$ such that $d(z,x) = td(x,y)$ and $d(z,y) = (1-t)d(x,y)$. Set $[x,y] = \{(1-t)x \oplus ty : t \in [0,1]\}$, a subset $C$ of $X$ is called convex if $[x,y] \subseteq C$, for all $x, y \in C$.

A notion of convergence in complete CAT(0) spaces was introduced by Lim [14] that is called $\Delta$-convergence as follows:
Let $(X,d)$ be a complete CAT(0) space, $(x_n)$ be a bounded sequence in $X$ and $x \in X$. Set $r(x,(x_n)) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius of $(x_n)$ is given by $r((x_n)) = \inf\{r(x,(x_n)) : x \in X\}$ and the asymptotic center of $(x_n)$ is the set

$A((x_n)) = \{x \in X : r(x,(x_n)) = r((x_n))\}$. It is known that in the complete CAT(0) spaces, $A((x_n))$ consists exactly one point (see [13]). A sequence $(x_n)$ in the complete CAT(0) space $(X,d)$ is said Δ-convergent to $x \in X$ if $A((x_{n_k})) = \{x\}$ for every subsequence $(x_{n_k})$ of $(x_n)$. It is well-known that in all CAT(0) spaces every bounded sequence has a Δ-convergent subsequence. The concept of Δ-convergence that has been studied by many authors (e.g. [8, 7]), extends the notion of weak convergence of Hilbert spaces to CAT(0) spaces.

Another approach for extension of weak convergence to CAT(0) spaces proposed by Ahmadi Kakavandi and Amini [2], based on the concept of quasilinearization of Berg and Nikolaev [3]. They denoted a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle \cdot,\cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

Ahmadi Kakavandi and Amini [2] have introduced the concept of dual space in complete CAT(0) spaces, based on the work of Berg and Nikolaev [3]. Introducing of a dual space for a CAT(0) space implies a concept of weak convergence respect to the dual space which is named $w-$ *convergence* in [2]. In [2], authors also showed that $w$-convergence is stronger than Δ-convergence. Ahmadi Kakavandi in [1] presented an equivalent definition of $w$-convergence in complete CAT(0) spaces without using of dual space, as follows:

**Definition 1.1.** [1] A sequence $(x_n)$ in a complete CAT(0) space $(X,d)$ *w-converges* to $x \in X$ iff $\lim_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle = 0$, for all $y \in X$.

$w$-convergence is equivalent to the weak convergence in Hilbert space $H$; because if $(.,.)$ is the inner product in Hilbert space $H$, then

$$2\langle \overrightarrow{xz}, \overrightarrow{xy} \rangle = d^2(x,y) + d^2(z,x) - d^2(z,y) = 2(x-z, x-y).$$

Also, Ahmadi Kakavandi [1] introduced a so-called $w$-topology such that convergence in this topology is equivalent to $w$-convergence for any sequence. It is obvious that metric convergence implies $w$-convergence, and in [2] it has been shown that $w$-convergence implies Δ-convergence but the converse is not valid (see [1]). However Ahmadi Kakavandi [1] proved that $(x_n)$ Δ-converges to $x \in X$ if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0,\ \forall y \in X$. In the sequel, we denote Δ-convergence by $\rightharpoonup$, $w$-convergence by $\rightsquigarrow$ and metric convergence by $\to$.

Nanjaras and Panyanak [16] extended the results of Schu [18, 19] to CAT(0) spaces. In fact, they proved Δ-convergence of the iteration $x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T^n x_n$ in CAT(0) spaces. Zhang and Cui [21] extended the results of [16] to asymptotically nonexpansive type mappings. In this paper, we extend the results of Nanjaras and Panyanak [16] and Zhang and Cui [21] to a family of asymptotically quasi-nonexpansive mappings in the setting of complete CAT(0) spaces. Consider the sequence given by the modified inexact Mann iteration

$$x_{n+1} = \alpha_n P y_n \oplus (1-\alpha_n) T_n^n P y_n, \qquad d(y_n, x_n) \leq e_n, \qquad x_0 \in C, \tag{1.1}$$

where $(T_n)$ is a family of asymptotically nonexpansive type selfmappings on a closed and convex subset $C$ of a complete CAT(0) space $X$, $(\alpha_n) \subset [0,1]$, $(e_n) \subset \mathbb{R}^+$ and $P$ is the nearest point projection on $C$. In fact, we prove Δ-convergence of the sequence given by (1.1) to a common fixed point of the sequence $(T_n)$ under appropriate assumptions on $(\alpha_n)$ and $(e_n)$ in complete CAT(0) spaces and also $w$-convergence of the sequence $(x_n)$ in CAT(0) spaces that are sequentially locally compact in $w$-topology. The following technical lemma is well-known in CAT(0) spaces.

**Lemma 1.2.** [7] Let $(X, d)$ be a CAT(0) space. Then, for all $x, y, z \in X$ and all $t \in [0, 1]$ :

(i) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$,

(ii) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$.

The following lemma is a generalization of Lemma 1 in [20] that has been proved in [17].

**Lemma 1.3.** [17] Let $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be non-negative sequences satisfying

$$\alpha_{n+1} \leq (1+\gamma_n)\alpha_n + \beta_n,\ n \geq 1,\ \sum_{n=1}^{\infty} \gamma_n < \infty,\ \sum_{n=1}^{\infty} \beta_n < \infty.$$

Then $\lim \alpha_n$ exists. Moreover, if $\liminf_{n\to\infty} \alpha_n = 0$, then $\lim \alpha_n = 0$.

## 2. Main results

In this section, we prove $\Delta$-convergence and $w$-convergence of the sequence $(x_n)$ generated by (1.1) such that the family $(T_n)$ of asymptotically quasi-nonexpansive selfmappings on subset $C$ of a CAT(0) space $(X, d)$ satisfies the following condition.

$$\begin{cases} \text{For subsequences } (T_{n_j})\ of\ (T_n),\ \text{and } (x_{n_j}) \subset C, \\ \text{such that } x_{n_j} \rightharpoonup x \text{ and } d(x_{n_j}, T_{n_j}^{n_j} x_{n_j}) \to 0 \\ \Rightarrow x \in \bigcap_{n=1}^{\infty} F(T_n). \end{cases} \tag{2.1}$$

The following lemma is a generalization of Opial lemma in CAT(0) spaces.

**Lemma 2.1.** Let $(X, d)$ be a CAT(0) space and $(x_n)$ a sequence in $X$. If there exists a nonempty subset $F$ of $X$ verifying:

(i) For every $z \in F$, $\lim_n d(x_n, z)$ exists.

(ii) If a subsequence $(x_{n_j})$ of $(x_n)$ is $\Delta$-convergent to $x \in X$, then $x \in F$.

Then, there exists $p \in F$ such that $(x_n)$ $\Delta$-converges to $p$ in $X$.

*Proof.* Suppose there exist subsequences $(x_{n_j})$ and $(x_{n_k})$ of $(x_n)$ such that $x_{n_j} \rightharpoonup x$ and $x_{n_k} \rightharpoonup y$. So, $\limsup_j \langle \overrightarrow{xx_{n_j}}, xy\rangle \leq 0$ and $\limsup_k \langle \overrightarrow{yx_{n_k}}, \overrightarrow{yx}\rangle \leq 0$. By (ii), $x, y \in F$ and by (i), set

$$l_1 = \lim_n d(x_n, x) \qquad \text{and} \qquad l_2 = \lim_n d(x_n, y)$$

Moreover,

$$2\langle \overrightarrow{xx_{n_j}}, \overrightarrow{xy}\rangle = d^2(x, x_{n_j}) - d^2(y, x_{n_j}) + d^2(x, y),$$
$$2\langle \overrightarrow{yx_{n_k}}, \overrightarrow{yx}\rangle = d^2(y, x_{n_k}) - d^2(x, x_{n_k}) + d^2(x, y).$$

Taking limsup when $j \to \infty$ and $k \to \infty$, we have $d^2(x, y) \leq l_1 - l_2 \leq -d^2(x, y)$. Thus, $x = y$ and $l_1 = l_2$. It is enough that we show every subsequence of $(x_n)$ has the unique asymptotic center $x$. Suppose that $(x_{n_i})$ be an arbitrary subsequence of $(x_n)$ and $z$ an element of $X$ that $z \neq x$.

$$2\langle \overrightarrow{xx_{n_i}}, \overrightarrow{xz}\rangle = d^2(x, x_{n_i}) - d^2(z, x_{n_i}) + d^2(x, z).$$

By taking lim sup we get

$$2\limsup_i \langle \overrightarrow{xx_{n_i}}, \overrightarrow{xz}\rangle + \limsup_i d^2(z, x_{n_i}) \geq \limsup_i d^2(x, x_{n_i}) + d^2(x, z). \tag{2.2}$$

Suppose $(x_{n_{i_j}})$ is a subsequence of $(x_{n_i})$ such that

$$\limsup_i \langle \overrightarrow{xx_{n_i}}, \overrightarrow{xz}\rangle = \lim_j \langle \overrightarrow{xx_{n_{i_j}}}, \overrightarrow{xz}\rangle \tag{2.3}$$

Since $(x_{n_{i_j}})$ is bounded, therefore it has a Δ-convergent subsequence. We denote it again by $(x_{n_{i_j}})$. By the above materials $x_{n_{i_j}} \rightharpoonup x$. So $\lim_j \langle \overrightarrow{xx_{n_{i_j}}}, \overrightarrow{xz}\rangle \leq 0$. Now (2.2) and (2.3) implies that $\limsup_i d^2(x_{n_i}, z) > \limsup_i d^2(x_{n_i}, x)$. Thus, asymptotic center of any arbitrary subsequence $(x_{n_i})$ of $(x_n)$ is $x$. Hence, there exists $p = x \in F$ such that $(x_n)$ Δ-converges to $p$ in $X$. □

**Theorem 2.2.** Suppose $C$ is a closed and convex subset of a complete CAT(0) space $(X, d)$ and $(T_n)$ be a family of selfmappings on $C$ such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $(\alpha_n) \subset [0, 1]$, $(e_n) \subset [0, \infty[$ and $(y_n) \subset X$ be sequences such that the sequence $(x_n)$ is generated by

$$x_{n+1} = \alpha_n P y_n \oplus (1 - \alpha_n) T_n^n P y_n, \qquad d(y_n, x_n) \leq e_n, \qquad x_0 \in C,$$

where $P$ is the nearest point projection on $C$.
Also, suppose $\sum_{n=1}^{\infty} e_n < \infty$ and $(\alpha_n) \subset [a, b]$ with $a, b \in (0, 1)$. We have
(i) Let $(T_n)$ be a sequence of asymptotically nonexpansive type mappings such that the condition (2.1) is satisfied. Set

$$c_{ni} = \max\{0, \sup_{x,y\in C} \big(d(T_i^n x, T_i^n y) - d(x, y)\big)\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, then $(x_n)$ is Δ-convergent to $q \in F$.
(ii) Let $e_n \equiv 0$ and $(T_n)$ be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition (2.1) is satisfied. Set

$$c_{ni} = \max\{0, \sup\{\big(d(T_i^n x, T_i^n p) - d(x, p)\big) \ ; \ x \in C, \ p \in F\}\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, then $(x_n)$ is Δ-convergent to $q \in F$.
(iii) Let $e_n \equiv 0$ and $(T_n)$ is a sequence of asymptotically quasi-nonexpansive mappings such that the conditions (2.1) are satisfied.
If $\sum_{n=1}^{\infty}(k_{nn}^2 - 1) < \infty$ then $(x_n)$ Δ-converges to $q \in F$.

*Proof.* Let $(T_n)$ be a sequence of asymptotically quasi-nonexpansive type mappings. Suppose $q \in F \subset C$, then

$$d(x_{n+1}, q) \leq \alpha_n d(Py_n, q) + (1 - \alpha_n) d(T_n^n P y_n, q) \leq d(y_n, q) + (1 - \alpha_n) c_{nn}$$
$$\leq d(x_n, q) + e_n + c_{nn},$$

so, by the assumptions and Lemma 1.3, $\lim_n d(x_n, q)$ exists for all $q \in F$ and $(x_n)$, $(y_n)$ and $(Py_n)$ are bounded. Moreover,

$$d^2(x_{n+1}, q) \leq \alpha_n d^2(Py_n, q) + (1 - \alpha_n) d^2(T_n^n P y_n, q) - \alpha_n(1 - \alpha_n) d^2(Py_n, T_n^n P y_n)$$
$$\leq d^2(y_n, q) + (1 - \alpha_n) c_{nn}^2 + 2(1 - \alpha_n) c_{nn} d(y_n, q) - \alpha_n(1 - \alpha_n) d^2(Py_n, T_n^n P y_n)$$
$$\leq d^2(x_n, q) + e_n^2 + 2e_n d(x_n, q) + (1 - \alpha_n) c_{nn}^2$$
$$+2(1 - \alpha_n) c_{nn} d(y_n, q) - \alpha_n(1 - \alpha_n) d^2(Py_n, T_n^n P y_n),$$

which implies

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)d^2(Py_n, T_n^n Py_n) < \infty. \tag{2.4}$$

Now, we prove (i). Since obviously every asymptotically nonexpansive type sequence is asymptotically quasi-nonexpansive type sequence, by the assumption on the sequence $(\alpha_n)$ and (2.4), we get $\lim_n d(Py_n, T_n^n Py_n) = 0$. By $Px_n = x_n \quad \forall n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, T_n^n x_n) &\le d(x_n, Py_n) + d(Py_n, T_n^n Py_n) + d(T_n^n Py_n, T_n^n x_n) \\ &\le 2d(x_n, y_n) + c_{nn} + d(Py_n, T_n^n Py_n) \le 2e_n + c_{nn} + d(Py_n, T_n^n Py_n) \end{aligned}$$

which implies $\lim_n d(x_n, T_n^n x_n) = 0$. Therefore, the condition (2.1) guaranties that if $x_{n_j} \rightharpoonup x$, then $x \in F$. Hence, Lemma 2.1 completes the proof.

In (ii), $y_n = x_n, \quad \forall n \in \mathbb{N}$, because of $e_n \equiv 0$. By (2.4), we get $\lim_n d(x_n, T_n^n x_n) = 0$. Therefore, the condition (2.1) guaranties that if $x_{n_j} \rightharpoonup x$, then $x \in F$. Hence, Lemma 2.1 gives the desired result.

In (iii), $y_n = x_n, \quad \forall n \in \mathbb{N}$, because of $e_n \equiv 0$. Therefore, we have

$$\begin{aligned} d^2(x_{n+1}, q) &\le \alpha_n d^2(x_n, q) + (1-\alpha_n)d^2(T_n^n x_n, q) - \alpha_n(1-\alpha_n)d^2(x_n, T_n^n x_n) \\ &\le \alpha_n d^2(x_n, q) + (1-\alpha_n)k_{nn}^2 d^2(x_n, q) - \alpha_n(1-\alpha_n)d^2(x_n, T_n^n x_n) \\ &\le (1 + (1-\alpha_n)(k_{nn}^2 - 1))d^2(x_n, q) - \alpha_n(1-\alpha_n)d^2(x_n, T_n^n x_n), \end{aligned}$$

which, by the assumptions and Lemma 1.3, $\lim_n d(x_n, q)$ exists for all $q \in F$ and

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)d^2(x_n, T_n^n x_n) < \infty. \tag{2.5}$$

So, by the assumptions on the sequence $(\alpha_n)$, we get $d(x_n, T_n^n x_n) \to 0$, which by the condition (2.1), implies if $x_{n_j} \rightharpoonup x$, then $x \in F$. Hence, by Lemma 2.1, the proof is complete. □

**Theorem 2.3.** Suppose $C$ be a closed and convex subset of a complete CAT(0) space $(X, d)$ and $(T_n)$ be a family of self mappings on $C$ such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and satisfies the following conditions

$$\begin{cases} \text{for any subsequence } (T_{n_j}) \ of \ (T_n), \text{ for } (x_{n_j}) \subset C, \\ \text{such that } x_{n_j} \rightharpoonup x \text{ and } d(x_{n_j}, T_{n_j} x_{n_j}) \to 0 \\ \Rightarrow x \in \bigcap_{n=1}^{\infty} F(T_n). \end{cases} \tag{2.6}$$

and

$$\text{for any bounded sequence } (z_n) \subset C, \quad d(T_n^{n-1} z_n, T_{n-1}^{n-1} z_n) \to 0. \tag{2.7}$$

Let $(\alpha_n) \subset [0,1]$, $(e_n) \subset [0, \infty[$ and $(y_n) \subset X$ be sequences such that the sequence $(x_n)$ is generated by

$$x_{n+1} = \alpha_n Py_n \oplus (1-\alpha_n)T_n^n Py_n, \qquad d(y_n, x_n) \le e_n, \qquad x_0 \in C,$$

where $P$ is the nearest point projection on $C$. Also, suppose $\sum_{n=1}^{\infty} e_n < \infty$ and $(\alpha_n) \subset [a, b]$ with $a, b \in (0, 1)$. We have

(i) Let $(T_n)$ be a sequence of asymptotically nonexpansive type mappings that are uniformly continuous. Set $c_{ni} = \max\{0, \sup_{x,y \in C} \big(d(T_i^n x, T_i^n y) - d(x, y)\big)\}$.

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, then $(x_n)$ is $\Delta$-convergent to $q \in F$.
(ii) If $e_n \equiv 0$ and $(T_n)$ is a sequence of asymptotically nonexpansive mappings such that $\sum_{n=1}^{\infty}(k_{nn}^2 - 1) < \infty$, then $(x_n)$ $\Delta$-converges to $q \in F$.

*Proof.* (i) By the proof of Theorem 2.2, $\lim_n d(x_n, q)$ exists for all $q \in F$ and $(x_n)$, $(y_n)$ and $(Py_n)$ are bounded. By (2.4) and the assumptions on the sequence $(\alpha_n)$, we get $\lim_n d(Py_n, T_n^n Py_n) = 0$. On the other hand

$$\begin{aligned}
d(Py_n, T_n^{n-1}Py_n) &\leq d(Py_n, x_n) + d(x_n, T_{n-1}^{n-1}Py_{n-1}) \\
&+ d(T_{n-1}^{n-1}Py_{n-1}, T_{n-1}^{n-1}Py_n) + d(T_{n-1}^{n-1}Py_n, T_n^{n-1}Py_n) \\
&\leq e_n + \alpha_{n-1}d(Py_{n-1}, T_{n-1}^{n-1}Py_{n-1}) + d(Py_{n-1}, Py_n) + c_{n-1,n-1} \\
&+ d(T_{n-1}^{n-1}Py_n, T_n^{n-1}Py_n) \\
&\leq e_n + \alpha_{n-1}d(Py_{n-1}, T_{n-1}^{n-1}Py_{n-1}) + d(Py_{n-1}, x_n) + d(x_n, Py_n) + c_{n-1,n-1} \\
&+ d(T_{n-1}^{n-1}Py_n, T_n^{n-1}Py_n) \\
&\leq 2e_n + \alpha_{n-1}d(Py_{n-1}, T_{n-1}^{n-1}Py_{n-1}) + (1-\alpha_{n-1})d(Py_{n-1}, T_{n-1}^{n-1}Py_{n-1}) + c_{n-1,n-1} \\
&+ d(T_{n-1}^{n-1}Py_n, T_n^{n-1}Py_n) \\
&\leq 2e_n + d(Py_{n-1}, T_{n-1}^{n-1}Py_{n-1}) + c_{n-1,n-1} + d(T_{n-1}^{n-1}Py_n, T_n^{n-1}Py_n),
\end{aligned}$$

which, by condition (2.7), implies $d(Py_n, T_n^{n-1}Py_n) \to 0$. Thus, by uniform continuity

$$\begin{aligned}
d(Py_n, T_nPy_n) &\leq d(Py_n, T_n^nPy_n) + d(T_n^nPy_n, T_nPy_n) \\
&\leq d(Py_n, T_n^nPy_n) + d(T_n(T_n^{n-1}Py_n), T_nPy_n)
\end{aligned}$$

implies $d(Py_n, T_nPy_n) \to 0$. Moreover, by $d(Py_n, x_n) \to 0$ and uniform continuity,

$$d(x_n, T_nx_n) \leq d(x_n, Py_n) + (Py_n, T_nPy_n) + d(T_nPy_n, T_nx_n)$$

implies $d(x_n, T_nx_n) \to 0$. Therefore, the condition (2.6) guaranties that if $x_{n_j} \rightharpoonup x$, then $x \in F$. Hence, Lemma 2.1 gives the desired result.
(ii) By the proof of part (iii) of Theorem 2.2, $\lim_n d(x_n, q)$ exists for all $q \in F$ and $(x_n)$ is bounded. By (2.5) and the assumption on the sequence $(\alpha_n)$, we get $\lim_n d(x_n, T_n^n x_n) = 0$. By using the same proof of part (i), we get $d(x_n, T_nx_n) \to 0$. Therefore, the condition (2.6) guaranties that if $x_{n_j} \rightharpoonup x$, then $x \in F$. Hence, Lemma 2.1 gives the desired result. □

**Remark 2.4.** If $T$ is a continuous asymptotically nonexpansive type self mapping on the closed and convex subset $C$ of a complete CAT(0) space $X$, then by [21, Corollary 3.4], $T$ is demiclosed (i.e. if a sequence $(x_n)$ $\Delta$-converges to $x$ and $d(x_n, Tx_n) \to 0$, then $x \in F(T)$). Thus, if in Theorem 2.3, $T_n \equiv T$, then the conditions (2.6) and (2.7) are satisfied. Hence, Theorem 2.3 extends the results of Nanjaras and Panyanak [16] and Zhang and Cui [21] in complete CAT(0) spaces.

**Remark 2.5.** The main results of the paper and Lemma 2.1 remain true if we replace $\Delta$-convergence with $w$-convergence provided we impose locally sequentially compactness of $w$-topology in CAT(0) space. Because in this case every bounded sequence has a $w$-convergent subsequence, therefore the set of all cluster points of a bounded sequence is nonempty. This condition is satisfied for example in every CAT(0) space with (S) property (see [1]), like symmetric Hadamard manifolds and Hilbert spaces. Because in such spaces every $w$-convergent sequence is $\Delta$-convergent, and on the other hand it is well-known that in all CAT(0) spaces every bounded sequence has a $\Delta$-convergent subsequence. Indeed, we do not know whether the main theorems of the paper for $w$-convergence are satisfied in general CAT(0) spaces.

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