\textbf{\textsc{\textit{Fixed Point Theory,} 17(2016), No. 1, 151-158}}

http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

\Large{\textsc{\textbf{\textsc{\Large{\textit{\Delta-CONVERGENCE AND W-CONVERGENCE}}}}}}

\Large{\textit{OF THE MODIFIED MANN ITERATION FOR A FAMILY OF ASYMPTOTICALLY NONEXPANSIVE TYPE MAPPINGS}}

\Large{\textit{IN COMPLETE CAT(0) SPACES}}

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\textbf{Abstract.} In this paper, we show $\Delta$-convergence and $w$-convergence (in the sense of Ahmadi Kakavandi and Amini [2]) of modified Mann iteration

$$x_{n+1} = \alpha_n Py_n \oplus (1 - \alpha_n) T_n x_n, \quad d(y_n, x_n) \leq \epsilon_n, \quad x_0 \in C,$$

to a common fixed point of the sequence $(T_n)$ of asymptotically nonexpansive type self mappings on a closed and convex subset $C$ of a complete CAT(0) space $X$, where $(\alpha_n) \subset [0, 1]$, $(\epsilon_n) \subset \mathbb{R}^+$ and $P$ is the nearest point projection on $C$. Our results extend the results in [16, 21] in the setting of complete CAT(0) spaces.

\textbf{Key Words and Phrases:} $w$-convergence, $\Delta$-convergence, Asymptotically nonexpansive type self-mapping, Fixed point, CAT(0) space.

\textbf{2010 Mathematics Subject Classification:} 47H10, 47H09.

\section{1. Introduction}

Let $C$ be a nonempty subset of a metric space $(X, d)$ and $Y$ be a nonempty subset of $C$. A mapping $T : C \to C$ is called nonexpansive respect to $Y$ if for each $x \in C$ and $y \in Y$, $d(Tx, Ty) \leq d(x, y)$. If $Y = C$, $T$ is called nonexpansive and if $Y = F(T) := \{x \in C : T(x) = x\}$, $T$ is called quasi-nonexpansive. $T$ is said to be asymptotically nonexpansive respect to $Y$ if there exists a sequence $(k_n)$ of positive real numbers such that $k_n \to 1$ and for all $x \in C$ and $y \in Y$, $d(T^n x, T^n y) \leq k_n d(x, y)$. If $Y = C$, the mapping $T$ is called asymptotically nonexpansive and if $Y = F(T)$, $T$ is called asymptotically quasi-nonexpansive. The mapping $T$ is said to be asymptotically nonexpansive type respect to $Y$ if $\lim\sup_{n \to \infty} \sup_{y \in Y} (d(T^n x, T^n y) - d(x, y)) \leq 0$, for all $x \in C$. If $Y = C$, $T$ is called asymptotically nonexpansive type and if $Y = F(T)$, $T$ is called asymptotically quasi-nonexpansive type. It is clear that nonexpansive mappings (quasi-nonexpansive mappings) and asymptotically nonexpansive mappings (asymptotically quasi-nonexpansive mappings) are asymptotically nonexpansive type mappings (resp.
asymptotically quasi-nonexpansive type mappings). The sequence \((T_n)\) of selfmappings on \(C\) is called a family of asymptotically nonexpansive mappings with respect to \(Y\) if for each \(T_i\), there exists a sequence \((k_n, i)\) of positive real numbers such that \(k_n, i \to 1\) as \(n \to \infty\), and for all \(x \in C\) and \(y \in Y\), \(d(T_n^i x, T_n^i y) \leq k_n, i d(x, y)\). If \(Y = C\), the sequence \((T_n)\) is called a family of asymptotically nonexpansive mappings with respect to \(Y\) if each \(T_i\) satisfies \(\limsup_{n \to \infty} \sup_{y \in Y} (d(T_n^i x, T_n^i y) - d(x, y)) \leq 0\), for all \(x \in C\). If \(Y = C\), the sequence \((T_n)\) is called a family of asymptotically nonexpansive mappings and if \(Y = \bigcap_{n=1}^\infty F(T_n)\), the sequence \((T_n)\) is called a family of asymptotically quasi-nonexpansive mappings. The sequence \((T_n)\) of selfmappings on \(C\) is called a family of asymptotically nonexpansive type mappings respect to \(Y\) if \(Y\) is an asymptotically quasi-nonexpansive type mapping. The sequence \((T_n)\) is called a family of asymptotically quasi-nonexpansive type mappings and if \(Y = \bigcap_{n=1}^\infty F(T_n)\), the sequence \((T_n)\) is called a family of asymptotically quasi-nonexpansive type mappings.

Mann [15], for approximation fixed point of nonexpansive mapping \(T\), suggested the iterative sequence given by \(x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n\). He proved the weak convergence of this sequence under the appropriate conditions on \((\alpha_n) \subset [0, 1]\). Since then many authors worked on Mann iteration and extended the results in Hilbert and Banach spaces. Schu in [18, 19] proved weak and strong convergence of the modified Mann iteration \(x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n\) for the asymptotically nonexpansive mappings in Hilbert and Banach spaces. Nanjaras and Panyanak [16] extended the results of Schu [18, 19] to CAT(0) spaces. Let us to introduce the CAT(0) spaces.

Let \((X, d)\) be a metric space and \(x, y \in X\). A geodesic path joining \(x\) to \(y\) is an isometry \(c : [0, d(x, y)] \to X\) such that \(c(0) = x, c(d(x, y)) = y\). The image of a geodesic path joining \(x\) to \(y\) is called a geodesic segment between \(x\) and \(y\). The metric space \((X, d)\) is said to be a geodesic space if every two points of \(X\) are joined by a geodesic, and \(X\) is said to be uniquely geodesic if there is exactly one geodesic joining \(x\) and \(y\) for each \(x, y \in X\).

A geodesic space \((X, d)\) is a CAT(0) space if satisfies the following inequality:

\[
CN - inequality: \text{ If } x, y_0, y_1, y_2 \in X \text{ such that } d(y_0, y_1) = d(y_0, y_2) = \frac{1}{2}d(y_1, y_2), \text{ then } \\
d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).
\]

It is known that a CAT(0) space is a uniquely geodesic space. For other equivalent definitions and basic properties, we refer the reader to the standard texts such as [4, 6, 10, 11]. Some examples of CAT(0) spaces are pre-Hilbert spaces (see [4]), R-trees (see [12]), Euclidean buildings (see [5]), the complex Hilbert ball with a hyperbolic metric (see [9]), Hadamard manifolds and many others.

Let \(X\) be a CAT(0) space and \(x, y \in X\). We write \((1 - t)x \oplus ty\) for the unique point \(z\) in the geodesic segment joining from \(x\) to \(y\) such that \(d(z, x) = td(x, y)\) and \(d(z, y) = (1 - t)d(x, y)\). Set \([x, y]\) = \(\{(1 - t) x \oplus ty : t \in [0, 1]\}\), a subset \(C\) of \(X\) is called convex if \([x, y]\) \(\subseteq C\), for all \(x, y \in C\).

A notion of convergence in complete CAT(0) spaces was introduced by Lim [14] that is called \(\Delta\)-convergence as follows:

Let \((X, d)\) be a complete CAT(0) space, \((x_n)\) be a bounded sequence in \(X\) and \(x \in X\). Set \(r(x, (x_n)) = \limsup_{n \to \infty} d(x, x_n)\). The asymptotic radius of \((x_n)\) is given by \(r((x_n)) = \inf\{r(x, (x_n)) : x \in X\}\) and the asymptotic center of \((x_n)\) is the set...
$A((x_n)) = \{ x \in X : r(x, (x_n)) = r((x_n)) \}$. It is known that in the complete CAT(0) spaces, $A((x_n))$ consists exactly one point (see [13]). A sequence $(x_n)$ in the complete CAT(0) space $(X, d)$ is said $\Delta$-convergent to $x \in X$ if $A((x_n)) = \{ x \}$ for every subsequence $(x_{n_k})$ of $(x_n)$. It is well-known that in all CAT(0) spaces every bounded sequence has a $\Delta$-convergent subsequence. The concept of $\Delta$-convergence that has been studied by many authors (e.g. [8, 7]), extends the notion of weak convergence of Hilbert spaces to CAT(0) spaces.

Another approach for extension of weak convergence to CAT(0) spaces proposed by Ahmadi Kakavandi and Amini [2], based on the concept of quasilinearization of Hilbert spaces to CAT(0) spaces.

Ahmadi Kakavandi and Amini [2] have introduced the concept of dual space in complete CAT(0) spaces, based on the work of Berg and Nikolaev [3]. Introducing of a vector $d$-dual space which is named $w$-dual space for a CAT(0) space implies a concept of weak convergence respect to the complete CAT(0) spaces, based on the work of Berg and Nikolaev [3].

**Definition 1.1.** [1] A sequence $(x_n)$ in a complete CAT(0) space $(X, d)$ $w$-converges to $x \in X$ iff $\lim_{n \to \infty} \langle x_n, x \rangle = 0$, for all $x \in X$.

Weak convergence is equivalent to the weak convergence in Hilbert space $H$, because if $(\langle \cdot, \cdot \rangle)$ is the inner product in Hilbert space $H$, then

$$2\langle x_n, x \rangle = d^2(x, y) + d^2(z, x) - d^2(z, y) = 2(x - z, x - y).$$

Also, Ahmadi Kakavandi [1] introduced a so-called $w$-topology such that convergence in this topology is equivalent to $w$-convergence for any sequence. It is obvious that metric convergence implies $w$-convergence, and in [2] it has been shown that $w$-convergence implies $\Delta$-convergence but the converse is not valid (see [1]). However Ahmadi Kakavandi [1] proved that $(x_n)$ $\Delta$-converges to $x \in X$ if and only if $\lim \sup_{n \to \infty} \langle x_n, x \rangle \leq 0$, $\forall y \in X$. In the sequel, we denote $\Delta$-convergence by $\Delta$, $w$-convergence by $\omega$ and metric convergence by $\rightarrow$.

Nanjaras and Panyanak [16] extended the results of Schu [18, 19] to CAT(0) spaces. In fact, they proved $\Delta$-convergence of the iteration $x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)T_n x_n$ in CAT(0) spaces. Zhang and Cui [21] extended the results of [16] to asymptotically nonexpansive type mappings. In this paper, we extend the results of Nanjaras and Panyanak [16] and Zhang and Cui [21] to a family of asymptotically quasi-nonexpansive mappings in the setting of complete CAT(0) spaces. Consider the sequence given by the modified inexact Mann iteration

$$x_{n+1} = \alpha_n P y_n \oplus (1 - \alpha_n)T_n^m y_n, \quad d(y_n, x_n) \leq \epsilon_n, \quad x_0 \in C,$$

where $(T_n)$ is a family of asymptotically nonexpansive type selfmappings on a closed and convex subset $C$ of a complete CAT(0) space $X$, $(\alpha_n) \subset [0, 1]$, $(\epsilon_n) \subset \mathbb{R}^+$ and $P$ is the nearest point projection on $C$. In fact, we prove $\Delta$-convergence of the sequence.
given by (1.1) to a common fixed point of the sequence \((T_n)\) under appropriate assumptions on \((\alpha_n)\) and \((\epsilon_n)\) in complete CAT(0) spaces and also \(w\)-convergence of the sequence \((x_n)\) in CAT(0) spaces that are sequentially locally compact in \(w\)-topology. The following technical lemma is well-known in CAT(0) spaces.

**Lemma 1.2.** [7] Let \((X,d)\) be a CAT(0) space. Then, for all \(x,y,z \in X\) and all \(t \in [0,1]\):

(i) \(d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y),\)

(ii) \(d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z).\)

The following lemma is a generalization of Lemma 1 in [20] that has been proved in [17].

**Lemma 1.3.** [17] Let \(\{\alpha_n\}_{n \geq 1}, \{\beta_n\}_{n \geq 1}\) and \(\{\gamma_n\}_{n \geq 1}\) be non-negative sequences satisfying

\[
\alpha_{n+1} \leq (1 + \gamma_n)\alpha_n + \beta_n, \quad n \geq 1, \quad \sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \beta_n < \infty.
\]

Then \(\lim \alpha_n\) exists. Moreover, if \(\liminf_{n \to \infty} \alpha_n = 0\), then \(\lim \alpha_n = 0\).

2. **Main results**

In this section, we prove \(\Delta\)-convergence and \(w\)-convergence of the sequence \((x_n)\) generated by (1.1) such that the family \((T_n)\) of asymptotically quasi-nonexpansive selfmappings on subset \(C\) of a CAT(0) space \((X,d)\) satisfies the following condition.

\[
\begin{align*}
&\text{For subsequences } (T_{n_j}) \text{ of } (T_n), \text{ and } (x_{n_j}) \subset C, \\
&\text{such that } x_{n_j} \rightharpoonup x \text{ and } d(x_{n_j}, T_{n_j} x_{n_j}) \to 0 \\
&\Rightarrow x \in \bigcap_{n=1}^{\infty} F(T_n).
\end{align*}
\]

The following lemma is a generalization of Opial lemma in CAT(0) spaces.

**Lemma 2.1.** Let \((X,d)\) be a CAT(0) space and \((x_n)\) a sequence in \(X\). If there exists a nonempty subset \(F\) of \(X\) verifying:

(i) For every \(z \in F\), \(\liminf d(x_n, z)\) exists.

(ii) If a subsequence \((x_{n_j})\) of \((x_n)\) is \(\Delta\)-convergent to \(x \in X\), then \(x \in F\).

Then, there exists \(p \in F\) such that \((x_n) \Delta\)-converges to \(p\) in \(X\).

**Proof.** Suppose there exist subsequences \((x_{n_j})\) and \((x_{n_k})\) of \((x_n)\) such that \(x_{n_j} \rightharpoonup x\) and \(x_{n_k} \rightharpoonup y\). So, \(\limsup \langle \bar{x}_n, xy \rangle \leq 0\) and \(\limsup \langle \bar{y}_n, xy \rangle \leq 0\). By (ii), \(x, y \in F\) and by (i), set

\[
l_1 = \lim_n d(x_n, x) \quad \text{and} \quad l_2 = \lim_n d(x_n, y)
\]

Moreover,

\[
2(\bar{x}_n, \bar{x}_n) = d^2(x, x_n) - d^2(y, x_n) + d^2(x, y),
\]

\[
2(\bar{y}_n, \bar{y}_n) = d^2(y, x_n) - d^2(x, x_n) + d^2(x, y).
\]

Taking \(\limsup\) when \(j \to \infty\) and \(k \to \infty\), we have \(d^2(x, y) \leq l_1 - l_2 \leq -d^2(x, y)\). Thus, \(x = y\) and \(l_1 = l_2\). It is enough that we show every subsequence of \((x_n)\) has the unique asymptotic center \(x\). Suppose that \((x_{n_i})\) is an arbitrary subsequence of \((x_n)\) and \(z\) an element of \(X\) that \(z \neq x\).

\[
2(\bar{x}_{n_i}, \bar{z}) = d^2(x, x_{n_i}) - d^2(z, x_{n_i}) + d^2(x, z).
\]
By taking \( \limsup \) we get
\[
2 \limsup_i \langle \overrightarrow{x_{ni}}, \overrightarrow{x^2} \rangle + \limsup_i d^2(z, x_{ni}) \geq \limsup_i d^2(x, x_{ni}) + d^2(x, z). \tag{2.2}
\]

Suppose \( \{x_{ni}\} \) is a subsequence of \( \{x_n\} \) such that
\[
\limsup_i \langle \overrightarrow{x_{ni}}, \overrightarrow{x^2} \rangle = \lim \langle \overrightarrow{x_{ni}}, \overrightarrow{x^2} \rangle \tag{2.3}
\]
Since \( \{x_{ni}\} \) is bounded, therefore it has a \( \Delta \)-convergent subsequence. We denote it again by \( \{x_{ni}\} \). By the above materials \( x_{ni} \rightarrow x \). So \( \lim_i \langle \overrightarrow{x_{ni}}, \overrightarrow{x^2} \rangle \leq 0 \). Now (2.2) and (2.3) implies that \( \limsup_i d^2(x_{ni}, z) \geq \limsup_i d^2(x_{ni}, x) \). Thus, asymptotic center of any arbitrary subsequence \( \{x_{ni}\} \) of \( \{x_n\} \) is \( x \). Hence, there exists \( p = x \in F \) such that \( \{x_n\} \Delta \)-converges to \( p \) in \( X \).

**Theorem 2.2.** Suppose \( C \) is a closed and convex subset of a complete CAT(0) space \( (X, d) \) and \( \{T_n\} \) is a family of self-mappings on \( C \) such that \( F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset \).

Let \( \{\alpha_n\} \subset [0, 1] \), \( \{\alpha_n\} \subset [0, \infty] \) and \( \{y_n\} \subset X \) be sequences such that the sequence \( \{x_n\} \) is generated by
\[
x_{n+1} = \alpha_n P y_n + (1 - \alpha_n)T_n^a P y_n, \quad d(y_n, x_n) \leq e_n, \quad x_0 \in C,
\]
where \( P \) is the nearest point projection on \( C \).

(i) Let \( \{T_n\} \) be a sequence of asymptotically nonexpansive mappings such that the condition (2.1) is satisfied. Set
\[
c_{ni} = \max \{0, \sup_{x,y \in C} \{d(T^n_i x, T^n_i y) - d(x, y)\}\}.
\]

If \( \sum_{n=1}^{\infty} c_{nn} < \infty \), then \( \{x_n\} \) is \( \Delta \)-convergent to \( q \in F \).

(ii) Let \( e_n \equiv 0 \) and \( \{T_n\} \) be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition (2.1) is satisfied. Set
\[
c_{ni} = \max \{0, \sup \{\langle d(T^n_i x, T^n_i p) - d(x, p) \rangle : x \in C, \ p \in F\}\}.
\]

If \( \sum_{n=1}^{\infty} c_{nn} < \infty \), then \( \{x_n\} \) is \( \Delta \)-convergent to \( q \in F \).

(iii) Let \( e_n \equiv 0 \) and \( \{T_n\} \) is a sequence of asymptotically quasi-nonexpansive mappings such that the conditions (2.1) are satisfied.

If \( \sum_{n=1}^{\infty} (k_{nn}^2 - 1) < \infty \) then \( \{x_n\} \Delta \)-converges to \( q \in F \).

**Proof.** Let \( \{T_n\} \) be a sequence of asymptotically quasi-nonexpansive type mappings. Suppose \( q \in F \subset C \), then
\[
d(x_{n+1}, q) \leq \alpha_n d(P y_n, q) + (1 - \alpha_n)d(T^n_n P y_n, q) \leq d(y_n, q) + (1 - \alpha_n)c_{nn}
\]
so, by the assumptions and Lemma 1.3, \( \lim_n d(x_n, q) \) exists for all \( q \in F \) and \( (x_n) \), \( (y_n) \) and \( (P y_n) \) are bounded. Moreover,\[
\
\]
\[
d^2(x_{n+1}, q) \leq \alpha_n d^2(P y_n, q) + (1 - \alpha_n)d^2(T^n_n P y_n, q) - \alpha_n(1 - \alpha_n)d^2(P y_n, T^n_n P y_n)
\]
\[
\leq d^2(y_n, q) + (1 - \alpha_n)c_{nn}^2 + 2(1 - \alpha_n)c_{nn}d(y_n, q) - \alpha_n(1 - \alpha_n)d^2(P y_n, T^n_n P y_n)
\]
\[
\leq d^2(x_n, q) + e_n^2 + 2e_n d(x_n, q) + (1 - \alpha_n)c_{nn}^2
\]
\[
+ 2(1 - \alpha_n)c_{nn}d(y_n, q) - \alpha_n(1 - \alpha_n)d^2(P y_n, T^n_n P y_n),
\]
\[
\frac{d^2(x_{n+1}, q) - d^2(x_n, q)}{2} \leq \alpha_n d^2(P y_n, q) + (1 - \alpha_n)d^2(T^n_n P y_n, q) - \alpha_n(1 - \alpha_n)d^2(P y_n, T^n_n P y_n).
\]

\[
\frac{d^2(x_{n+1}, q) - d^2(x_n, q)}{2} \leq \alpha_n d^2(P y_n, q) + (1 - \alpha_n)d^2(T^n_n P y_n, q) - \alpha_n(1 - \alpha_n)d^2(P y_n, T^n_n P y_n).
\]

which implies
\[ \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) d^2(Py_n, T_n^n Py_n) < \infty. \]  
(2.4)

Now, we prove (i). Since obviously every asymptotically nonexpansive type sequence is asymptotically quasi-nonexpansive type sequence, by the assumption on the sequence \((\alpha_n)\) and (2.4), we get \(\lim_{n} d(Py_n, T_n^n Py_n) = 0\). By \(Px_n = x_n\ \forall n \in \mathbb{N}\), we have
\[ d(x_n, T_n^n x_n) \leq d(x_n, Py_n) + d(Py_n, T_n^n Py_n) + d(T_n^n Py_n, T_n^n x_n) \]
\[ \leq 2d(x_n, y_n) + c_n + d(Py_n, T_n^n Py_n) \leq 2e_n + c_n + d(Py_n, T_n^n Py_n) \]
which implies \(\lim_{n} d(x_n, T_n^n x_n) = 0\). Therefore, the condition (2.1) guarantees that if \(x_{n_j} \to x\), then \(x \in F\). Hence, Lemma 2.1 completes the proof.

In (ii), \(y_n = x_n\), \(\forall n \in \mathbb{N}\), because of \(e_n \equiv 0\). By (2.4), we get \(\lim_{n} d(x_n, T_n^n x_n) = 0\). Therefore, the condition (2.1) guarantees that if \(x_{n_j} \to x\), then \(x \in F\). Hence, Lemma 2.1 gives the desired result.

In (iii), \(y_n = x_n\), \(\forall n \in \mathbb{N}\), because of \(e_n \equiv 0\). Therefore, we have
\[ d^2(x_{n+1}, q) \leq \alpha_n d^2(x_n, q) + (1 - \alpha_n)d^2(T_n^n x_n, q) - \alpha_n (1 - \alpha_n)d^2(x_n, T_n^n x_n) \]
\[ \leq \alpha_n d^2(x_n, q) + (1 - \alpha_n)k_n d^2(x_n, q) - \alpha_n (1 - \alpha_n)d^2(x_n, T_n^n x_n) \]
\[ \leq (1 + (1 - \alpha_n)(k_n^2 - 1))d^2(x_n, q) - \alpha_n (1 - \alpha_n)d^2(x_n, T_n^n x_n), \]
which, by the assumptions and Lemma 1.3, \(\lim_{n} d(x_n, q) \exists\) for all \(q \in F\) and
\[ \sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) d^2(x_n, T_n^n x_n) < \infty. \]  
(2.5)

So, by the assumptions on the sequence \((\alpha_n)\), we get \(d(x_n, T_n^n x_n) \to 0\), which by the condition (2.1), implies if \(x_{n_j} \to x\), then \(x \in F\). Hence, by Lemma 2.1, the proof is complete.

**Theorem 2.3.** Suppose \(C\) be a closed and convex subset of a complete CAT(0) space \((X, d)\) and \((T_n)\) be a family of self mappings on \(C\) such that \(F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset\) and satisfies the following conditions

\[
\begin{align*}
\text{for any subsequence } (T_{n_j}) \text{ of } (T_n), \text{ for } (x_{n_j}) \subset C, \\
\text{such that } x_{n_j} \to x \text{ and } d(x_{n_j}, T_{n_j} x_{n_j}) \to 0 \\
\Rightarrow x \in \bigcap_{n=1}^{\infty} F(T_n).
\end{align*}
\]
(2.6)

and

for any bounded sequence \((z_n) \subset C\), \(d(T_n^{n-1} z_n, T_{n-1}^{n-1} z_n) \to 0\).  
(2.7)

Let \((\alpha_n) \subset [0, 1]\), \((e_n) \subset [0, \infty]\) and \((y_n) \subset X\) be sequences such that the sequence \((x_n)\) is generated by
\[ x_{n+1} = \alpha_n Py_n \oplus (1 - \alpha_n)T_n^n Py_n, \quad d(y_n, x_n) \leq e_n, \quad x_0 \in C, \]
where \(P\) is the nearest point projection on \(C\). Also, suppose \(\sum_{n=1}^{\infty} e_n < \infty\) and \((\alpha_n) \subset [a, b]\) with \(a, b \in (0, 1)\). We have

(i) Let \((T_n)\) be a sequence of asymptotically nonexpansive type mappings that are uniformly continuous. Set \(c_{n} = \max\{0, \sup_{x,y \in C} (d(T_n^n x, T_n^n y) - d(x, y))\}\).
If \( \sum_{n=1}^{\infty} c_{nn} < \infty \), then \((x_n)\) is \(\Delta\)-convergent to \(q \in F\).
(ii) If \( c_n \equiv 0 \) and \((T_n)\) is a sequence of asymptotically nonexpansive mappings such that \( \sum_{n=1}^{\infty} (k_{nn}^2 - 1) < \infty \), then \((x_n)\) \(\Delta\)-converges to \(q \in F\).

**Proof.** (i) By the proof of Theorem 2.2, \(\lim_n d(x_n, q)\) exists for all \(q \in F\) and \((x_n), (y_n)\) are bounded. By (2.4) and the assumptions on the sequence \((\alpha_n)\), we get \(\lim_n d(Py_n, T_n^n Py_n) = 0\). On the other hand

\[
\begin{align*}
  d(Py_n, T_n^n Py_n) &\leq d(Py_n, x_n) + d(x_n, T_n^{n-1} Py_n) \\
  &+ d(T_n^{n-1} Py_n, T_n^{n-1} Py_n) + d(T_n^{n-1} Py_n, T_n^{n-1} Py_n) \\
  \leq e_n + \alpha_n - 1 d(Py_n, T_n^{n-1} Py_n) + d(x_n, Py_n) + c_{n-1, n-1} \\
  &+ d(T_n^{n-1} Py_n, T_n^{n-1} Py_n)
\end{align*}
\]

\[
\begin{align*}
  \leq 2e_n + \alpha_n - 1 d(Py_n, T_n^{n-1} Py_n) + (1 - \alpha_n - 1) d(Py_n, T_n^{n-1} Py_n) + d(T_n^{n-1} Py_n, T_n^{n-1} Py_n) \\
  \leq 2e_n + d(Py_n, T_n^{n-1} Py_n) + c_{n-1, n-1} + d(T_n^{n-1} Py_n, T_n^{n-1} Py_n),
\end{align*}
\]

which, by condition (2.7), implies \(d(Py_n, T_n^{n-1} Py_n) \to 0\). Thus, by uniform continuity

\[
\begin{align*}
  d(Py_n, T_n Py_n) &\leq d(Py_n, T_n^n Py_n) + d(T_n^n Py_n, T_n Py_n) \\
  \leq d(Py_n, T_n^n Py_n) + d(T_n^n Py_n, T_n Py_n)
\end{align*}
\]

implies \(d(Py_n, T_n Py_n) \to 0\). Moreover, by \(d(Py_n, x_n) \to 0\) and uniform continuity,

\[
\begin{align*}
  d(x_n, T_n Py_n) \leq d(x_n, Py_n) + (Py_n, T_n Py_n) + d(T_n Py_n, T_n x_n)
\end{align*}
\]

implies \(d(x_n, T_n x_n) \to 0\). Therefore, the condition (2.6) guarantees that if \(x_{n_j} \to x\), then \(x \in F\). Hence, Lemma 2.1 gives the desired result.

(ii) By the proof of part (iii) of Theorem 2.2, \(\lim_n d(x_n, q)\) exists for all \(q \in F\) and \((x_n)\) is bounded. By (2.5) and the assumption on the sequence \((\alpha_n)\), we get \(\lim_n d(x_n, T_n^n x_n) = 0\). By using the same proof of part (i), we get \(d(x_n, T_n x_n) \to 0\). Therefore, the condition (2.6) guarantees that if \(x_{n_j} \to x\), then \(x \in F\). Hence, Lemma 2.1 gives the desired result.

**Remark 2.4.** If \(T\) is a continuous asymptotically nonexpansive type self mapping on the closed and convex subset \(C\) of a complete CAT(0) space \(X\), then by [21, Corollary 3.4], \(T\) is demiclosed (i.e. if a sequence \((x_n)\) \(\Delta\)-converges to \(x\) and \(d(x_n, T_n x_n) \to 0\), then \(x \in F(T)\)). Thus, if in Theorem 2.3, \(T_n \equiv T\), then the conditions (2.6) and (2.7) are satisfied. Hence, Theorem 2.3 extends the results of Nanjaras and Panyanak [16] and Zhang and Cui [21] in complete CAT(0) spaces.

**Remark 2.5.** The main results of the paper and Lemma 2.1 remain true if we replace \(\Delta\)-convergence with \(w\)-convergence provided we impose locally sequentially compactness of \(w\)-topology in CAT(0) space. Because in this case every bounded sequence has a \(w\)-convergent subsequence, therefore the set of all cluster points of a bounded sequence is nonempty. This condition is satisfied for example in every CAT(0) space with \((S)\) property (see [1]), like symmetric Hadamard manifolds and Hilbert spaces. Because in such spaces every \(w\)-convergent sequence is \(\Delta\)-convergent,
and on the other hand it is well-known that in all CAT(0) spaces every bounded sequence has a $\Delta$-convergent subsequence. Indeed, we do not know whether the main theorems of the paper for $w$-convergence are satisfied in general CAT(0) spaces.

**Acknowledgements.** The authors would like to thank the referees for valuable comments.

**References**


Received: May 28, 2013; Accepted: November 15, 2013.