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ESSENTIAL SETS OF FIXED POINTS FOR CORRESPONDENCES WITH APPLICATION TO NASH EQUILIBRIA

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Abstract. This paper studies essential stabilities of fixed points for correspondences. The existence of minimal essential sets of fixed points is proved under the perturbation of correspondences and domains. We show that a kind of minimal essential set is connected. As an application, the existence of minimal essential sets of Nash equilibria is deduced, and these sets can resist the dual perturbations of best responses and strategy sets for a noncooperative game.

Key Words and Phrases: Essential sets, fixed points, Nash equilibria, best responses.
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1. Introduction

The stability analysis of fixed points is an important focus in nonlinear analysis. In 1950, Fort investigated the essentially stable fixed points of correspondences, then essential stabilities is linked with generic stability, and some generic stability results were of important significance [1]. Nextly, Kinoshita studied essentially stable components of fixed points [2].

Yu considered essential sets of equilibrium points of correspondences in a metric space equipped with the uniform topology [3]. Interestingly, in [4], Xiang pointed out that there exist differences between the uniform topology and the graph topology for generic stability results of fixed points of correspondences, and this reveals that the stability of fixed points depends on topologies on spaces for correspondences.

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Essential stabilities have attracted many scholars to study deeply, e.g., [5, 6]. In fact, these are also closely related with the stability analysis of Nash equilibria in noncooperative games [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In 1962, Wu and Jiang raised the conception of essential equilibria in n-person noncooperative games, see [7] or [18]. Combining strategic stabilities in [8, 9] by Kohlberg, Mertens and Hillas, stabilities for Nash equilibria have gone into more detail about essentialities. Afterwards, many works in relation to essential stabilities in noncooperative games were contributed. And all these stability results, more or less, are related to the stability of fixed points under the perturbation of payoffs or strategies in noncooperative games.

In consideration of the bounded rationality of players in games, inspired by [9], the set of strategies of each player may shake in itself (e.g., by acknowledged facts, perfect equilibria can partly resist this kind of perturbation), meanwhile, the best response for each boundedly rational player may have perturbations. On basis of this consideration, in this paper, firstly, we study the essential stability of fixed points under the perturbation of both correspondences and domains. Next, as applications, we obtain essentially stable results for Nash equilibria in noncooperative games with infinite strategies, and the corresponding results include related results in [9] as special cases.

2. Preliminaries

Let X be a compact and convex set of a Euclidean space $(E, \|\cdot\|)$, $K(X)$ denotes the collection of nonempty compact convex subsets of X. $F : X \to K(X)$ is an upper semicontinuous correspondence (for the concept of usual semicontinuities of correspondences, see [19, p. 35]), and $C(X)$ represents the collection of all thus F. Let $u = (F, A) \in C(X) \times K(X)$, M be the set as follow

$$M = \{u \in C(X) \times K(X) : \forall x \in A,\ s.t.\ F(x) \subset A\}.$$

For any $u_1 = (F_1, A_1)$, $u_2 = (F_2, A_2) \in M$, using the Euclidean metric d defined by the norm on E, the Hausdorff metric between A_1 and A_2 is written as

$$h(A_1, A_2) = max\{h_l(A_1, A_2), h_u(A_1, A_2)\},$$

where

$$h_l(A_1, A_2) = \inf\{\lambda > 0 : A_1 \subset B(A_2, \lambda)\},$$
$$h_u(A_1, A_2) = \inf\{\lambda > 0 : A_2 \subset B(A_1, \lambda)\}$$
$$(B(A, \lambda) = \{x \in E, d(x, y) < \lambda \text{ for some } y \in A\}).$$

We define the metric between u_1 and u_2 as

$$\rho(u_1, u_2) = \sup_{x \in X} h(F_1(x), F_2(x)) + h(A_1, A_2).$$

Then (M, ρ) is a metric space.

Definition 2.1. For each $u \in M$, x is called a fixed point of u if $x \in A$ and $x \in F(x)$.

Many works focus on stabilities of fixed points on X under the perturbation of F in $C(X)$, that is, these cases consider the metric of $F_1, F_2 \in C(X)$ as

$$\rho_X(F_1, F_2) = \sup_{x \in X} h(F_1(x), F_2(x)).$$

It should be pointed that the stability of fixed points in the sense of ρ is different from that of ρ_X. This is reflected in their different convergences. Let $u = (F, A) \in M$ and $G_n \in C(X)$. Define $v = (G_n, X)$, then $v \in M$. If $\rho(u, v) \to 0$ $(n \to \infty)$, then $\rho_X(F, G_n) \to 0$. However, if $\rho_X(F, G_n) \to 0$, there may not exist $A_n \in X$ with $v_n = (G_n, A_n) \in M$ such that $\rho(u, v_n) \to 0$, see the following example.

Example 2.1. Let $X = [0, 1]$, $A = [0, \frac{1}{2}]$. Let $F \in C(X)$ such that

$$F(x) = \begin{cases} \frac{1}{2}x + \frac{1}{4}, & [0, \frac{1}{2}), \\ x, & [\frac{1}{2}, 1]. \end{cases}$$

Clearly, we have $u = (F, A) \in M$. Let $G_n \in C(X)$ such that

$$G_n(x) = \begin{cases} \frac{1}{2}x + \frac{1}{4} + \frac{1}{n}, & [0, \frac{1}{2}), \\ (1 - \frac{2}{n})x + \frac{2}{n}, & [\frac{1}{2}, 1]. \end{cases}$$

It can be easily calculated that

$$\rho_X(F, G_n) = \frac{1}{n} \to 0.$$

Also we can check that $G_n(x) > x$ for each $x \in [0, 1)$, and $G_n(x) = x$ while $x = 1$. Thus, for any subset $B \subset X \backslash \{1\}$, there definitely exists at least a point x such that $G_n(x) \not\subset B$. Then only $v_n^1 = (G_n, X)$ and $v_n^2 = (G_n, \{1\})$ satisfy that $v_n^1, v_n^2 \in M$. Noting that $A = [0, \frac{1}{2}]$, then $h(A, X) \not\to 0$ and $h(A, \{1\}) \not\to 0$. Therefore, there is no $v_n = (G_n, B) \in M$ such that $\rho(u, v_n) \to 0$.

Let $u = (F, A) \in M$. Denote the collection of fixed points of u by $S(u)$. From the definition of M and Kakutani's fixed point theorem [20], we can obtain that $S(u) \neq \emptyset$, and $S(u)$ is compact [21, p. 550], thus, S defines a correspondence from M to $K(X)$.

Definition 2.2. For each $u \in M$, a set $e(u)$ is called an essential fixed point set of $S(u)$ with respect to M if it satisfies the following conditions:

(i) $e(u)$ is closed subset of $S(u)$.

(ii) For any open set U with $U \supset e(u)$, there exists an open neighborhood $O(u)$ of u in M such that $U \cap S(g) \neq \emptyset$, $\forall g \in O(u)$. That is, near u there exists a fixed point near the set $e(u)$.

If the essential set $e(u)$ is a singleton set $\{x^*\}$, then x^* is called an essential fixed point of $S(u)$.

Definition 2.3. A set $e^*(u)$ is called minimal essential set of $S(u)$ with respect to M if it is a minimal set of all essential sets ordered by set inclusion in $S(u)$.

Lemma 2.1. *(see* [15]*) Let A, A_1, A_2 be compact convex subsets in E, satisfying $A_1 \subset intA, A_2 \subset intA$, where $intA$ is the relative interior of A in E. If $h(A, A_2) < \min_{x \in \partial A, y \in A_1} d(x, y)$, then $A_1 \subset A_2$, where ∂A is the boundary of A.*

Lemma 2.2. *(see* [11]*) Let A, B be two nonempty convex compact subsets of a normed vector space P. Then for any $\alpha, \beta \geq 0, \alpha + \beta = 1$, $h(A, \alpha A + \beta B) \leq h(A, B)$, where h is the Hausdorff metric defined on P.*

3. Essential Sets of Fixed Points

The fixed point correspondence S presents some continuities, this leads to essential stability of some subsets of $S(u)$ for any u in M.

Theorem 3.1. *$S : M \to K(X)$ is an upper semicontinuous correspondence with compact values.*

Proof. Since X is compact, it suffices to prove that the graph of S,

$$Gr(S) = \{(u, x) \in M \times X, x \in S(u)\},$$

is closed in $M \times X$.

Given $(u_n, x_n) \in Gr(S)$ with $(u_n, x_n) \to (u_0, x_0) \in M \times X$, where $u_n = (F_n, A_n)$, $u_0 = (F_0, A_0)$, we have $x_n \in A_n$ and $x_n \in F(x_n)$. We need to show that $x_0 \in S(u_0)$. Because $x_n \to x_0, x_n \in A_n$, and $A_n \to A$, we know that the right hand of the following inequality gets close to zero as $n \to \infty$,

$$h(x_0, A) \leq h(x_0, x_n) + h(x_n, A_n) + h(A_n, A).$$

Then it holds that $x_0 \in A$. Additionally, we have

$$\begin{aligned} h_l(x_0, F_0(x_0)) \leq\ & h_l(x_0, x_n) + h_l(x_n, F_n(x_n)) \\ & + h_l(F_n(x_n), F_0(x_n)) + h_l(F_0(x_n), F_0(x_0)). \end{aligned}$$

Since $x_n \to x_0, x_n \in F_n(x_n), F_n \to F_0$, and by the upper semicontinuous property of F_0, the right hand of the above inequality gets also close to zero as $n \to \infty$. Thus, we have $x_0 \in F_0(x_0)$. Therefore, we have proved that $x_0 \in S(u_0)$. The proof is completed. □

Theorem 3.2. *For each $u = (F, A) \in M$, there exists a minimal essential set of $S(u)$.*

Proof. By Theorem 3.1, $S : M \to K(X)$ is an upper semicontinuous correspondence with compact values. Then for any open set U with $S(u) \subset U$, there exists a neighborhood $N(u)$ of u such that $S(u') \subset U, \forall u' \in N(u)$, that is, $S(u)$ itself is an essential set. Denote by Φ all essential subsets of $S(u)$. Since each essential subsets of $S(u)$ is compact, we have the intersection of each deceasing chain in Φ is nonempty, and this intersection is a lower bound for the chain. Therefore, there is a minimal element in Φ by Zorn's lemma, and the minimal element is really a minimal essential set of $S(u)$. □

Remark 3.1. In Theorem 3.2, if we consider the stability of fixed points on X using ρ_X, we can deduce the existence of minimal essential sets of fixed points for each $F \in C(X)$. If $F \in C(X)$ is a mapping on X, then the existence of minimal essential sets of fixed points holds also. If X is in a Hausdorff locally convex topological vector space, we know that $S(u) \neq \emptyset$ by Kakutani-Fan-Glicksberg fixed point theorem (see, [21, p. 550]), then Theorem 3.1 and 3.2 still hold.

From Theorem 3.2, if the fixed point set $S(u)$ is a single point set, we can deduce the following result.

Corollary 3.3. *For each $u = (F, A) \in M$, if $S(u) = \{x^*\}$, then x^* is an essential fixed point of $S(u)$.*

Remark 3.2. For a point $u = (F, A) \in M$, the essential set restrained in $A \subset X$ is different from it in X. In Example 2.1, the unique fixed point set of $u = (F, A)$ is $\{\frac{1}{2}\}$. Let $v = (F, X) \in M$, then the fixed point set of v is the interval $[\frac{1}{2}, 1]$. By Theorem 3.2 and Corollary 3.3, obviously, the set $\{\frac{1}{2}\}$ is a minimal essential set of $S(u)$. However, the set $\{\frac{1}{2}\}$ is not a minimal essential set of $S(v)$, because the unique minimal essential set of $S(v)$ is just $[\frac{1}{2}, 1]$.

For each $A \in K(X)$, let $K(intA)$ denote all closed subsets of $intA$. In order to restrain the perturbation of (F, A) in some range, we define

$$M' = \{u = (F, B) \in C(X) \times K(intA) : \forall x \in B,\ s.t.\ F(x) \subset B\}.$$

Definition 3.1. For each $u = (F, B) \in M'$, x is called a fixed point of u with respect to M' if $x \in B \subset intA$ and $x \in F(x)$.

For each $A \in K(X)$, denote by $S'(u)$ the set of fixed points of $u = (F, B) \in M'$, then, S' is a correspondence from M' to $K(intA)$. Similar with Definition 2.2 and 2.3, for each $u = (F, A) \in M$, we can also define essential fixed points, essential sets and minimal essential sets of $S(u)$ with respect to M'. The difference of these definitions is that we restrain the perturbation of fixed points of $u = (F, A)$ in the range of $intA$.

Definition 3.2. For each $u = (F, A) \in M$, a set $e(u)$ is called an essential set of $S(u)$ with respect to M' if it satisfies:

(i) $e(u)$ is closed subset of $S(u)$.

(ii) For any open set U with $U \supset e(u)$, there exists an open neighborhood $O(u)$ of u in M' such that $U \cap S'(g) \neq \emptyset$, $\forall g \in O(u)$.

A set $e^*(u)$ is called minimal essential set of $S(u)$ with respect to M' if it is a minimal element of all essential sets ordered by set inclusion in $S(u)$.

Noting the fact that $M' \subset M$, using similar methods in the proof of Theorem 3.1 and 3.2, we have the following result.

Theorem 3.4. *For each $u = (F, A) \in M$, there exists a minimal essential set of $S'(u)$ with respect to M'.*

The following result shows the connectedness of minimal essential sets of $S(u)$ with respect to M' under some certain conditions.

Theorem 3.5. *For each $u = (F, A) \in M$, if $m(u)$ is a minimal essential set of $S(u)$ with respect to M' such that $m(u) \subset intA$, then $m(u)$ is connected.*

Proof. Assume that $m(u)$ is disconnected. Then there are two nonempty sets $C_1(u)$ and $C_2(u)$ such that $m(u) = C_1(u) \cup C_2(u)$, as well as two open sets Q_1 and Q_2, such that $C_1(u) \subset Q_1$ and $C_2(u) \subset Q_2$ with $Q_1 \cap Q_2 = \emptyset$.

Since $m(u)$ is minimal essential, both $C_1(u)$ and $C_2(u)$ are not essential sets. Then there exist two open sets W_1 and W_2 with $W_1 \supset C_1(u)$ and $W_2 \supset C_2(u)$ such that for each $\delta' > 0$, there are $u_1, u_2 \in M'$ satisfying $\rho(u, u_1) < \delta'$ and $\rho(u, u_2) < \delta'$ such that $S'(u_1) \cap W_1 = \emptyset$ and $S'(u_2) \cap W_2 = \emptyset$. Let $V_1 = Q_1 \cap W_1$ and $V_2 = Q_2 \cap W_2$. Clearly, it is true that $C_1(u) \subset V_1$ and $C_2(u) \subset V_2$. Since $C_1(u), C_2(u) \subset intA$ are compact sets, there exist two open sets $U_1, U_2 \subset intA$ such that $C_1(u) \subset U_1 \subset \bar{U}_1 \subset V_1$ and $C_2(u) \subset U_2 \subset \bar{U}_2 \subset V_2$. It follows that $U_1 \cup U_2 \supset m(u)$.

Since $m(u)$ is essential, there is $\delta > 0$ such that for any $u' \in M'$ with $\rho(u, u') < \delta$ satisfies $S'(u') \cap (U_1 \cup U_2) \neq \emptyset$. Let

$$\delta_1 = \min_{x \in \partial A, y \in \bar{U}_1} d(x, y) \text{ and } \delta_2 = \min_{x \in \partial A, y \in \bar{U}_2} d(x, y).$$

Furthermore, since $U_1 \subset W_1$, for $\delta'' > 0$ with $\delta'' < \min\{\delta_1, \delta_2, \frac{\delta}{3}\}$ there is $v_1 \in M'$ such that $\rho(u, v_1) < \delta''$ but $S(v_1) \cap U_1 = \emptyset$, where $v_1 = (F_1, A_1)$. Because $\rho(u, v_1) < \delta''$, we have $h(A, A_1) < \delta_1$ and $h(A, A_1) < \delta_2$. Furthermore, we can obtain that $\bar{U}_1 \subset A_1$ and $\bar{U}_2 \subset A_1$ by Lemma 2.1. Let

$$\delta_3 = \min_{x \in \partial A, y \in A_1} d(x, y)$$

(clearly, $\delta_3 < \min\{\delta_1, \delta_2\}$). Since $U_2 \subset W_2$, for any $\delta''' > 0$ with $\delta''' < \min\{\delta_3, \frac{\delta}{3}\}$, there is $v_2 \in M'$ ($v_2 = (F_2, A_2)$) such that $\rho(u, v_2) < \delta'''$, nevertheless, it holds that $S'(v_2) \cap U_2 = \emptyset$. Since $\rho(u, v_2) < \delta'''$, we have $h(A, A_2) < \delta_3$, it follows that $A_1 \subset A_2$ by Lemma 2.1.

In order to complete the proof, we define a correspondence T on X as follows

$$T(x) = \begin{cases} F_1(x), & x \in U_1, \\ \overline{B(F(x), \frac{\delta}{3})} \cap A_2, & x \in \partial U_1, \\ F_2(x), & x \in X \backslash \bar{U}_1. \end{cases}$$

Since $\bar{U}_1 \subset A_1$, we have $\partial U_1 \subset A_1$, consequently, for any $x \in \partial U_1$ it is true that $F_1(x) \subset A_1 \subset A_2$. From the fact that $\rho(u, v_1) < \frac{\delta}{3}$, we have $h(F(x), F_1(x)) < \frac{\delta}{3}$, it follows that $F_1(x) \subset B(F(x), \frac{\delta}{3})$. Therefore, for any $x \in \partial U_1$, we assert that

$$T(x) = \overline{B\left(F(x), \frac{\delta}{3}\right)} \cap A_2 \neq \emptyset.$$

Combining the upper semicontinuous property of F_1 and F_2, we obtain that T is also an upper semicontinuous correspondence on X. Additionally, it is obvious that for any $x \in X$, $T(x)$ is compact and convex, and for any $x \in A_2$, we have $T(x) \subset A_2$. Let $v = (T, A_2)$. Then $v \in M'$.

Using the correspondence T, we construct a special correspondence G on X such that

$$G(x) = \alpha(x)T(x) + \beta(x)F_2(x),$$

where

$$\alpha(x) = d(x, \bar{U}_2)/(d(x, \bar{U}_1) + d(x, \bar{U}_2)),$$
$$\beta(x) = d(x, \bar{U}_1)/(d(x, \bar{U}_1) + d(x, \bar{U}_2)).$$

Clearly, $\alpha(x)$ and $\beta(x)$ are nonnegative and continuous at each $x \in X$ with $\alpha(x) + \beta(x) = 1$. It can be routinely checked that G is an upper semicontinuous correspondence with compact and convex values. Note that $G(x) \subset A_2$ for each $x \in A_2$ because $T(x), F_2(x) \subset A_2$ and A_2 is convex. Therefore, we have $v' = (G, A_2) \in M'$.

Next, we investigate $\rho(v', u_2)$, that is, the distance of u_2 and v'. By Lemma 2.2, for each $x \in X$, we have

$$h(G(x), F_2(x)) = h(\alpha(x)T(x) + \beta(x)F_2(x), F_2(x)) \le h(T(x), F_2(x)).$$

Hence, if $x \in U_1$,

$$h(T(x), F_2(x)) = h(F_1(x), F_2(x)) \le h(F_1(x), F(x)) + h(F(x), F_2(x)) < \frac{2\delta}{3};$$

if $x \in X \backslash \bar{U}_1$, then

$$h(T(x), F_2(x)) = h(F_2(x), F_2(x)) \equiv 0;$$

and if $x \in \partial U_1$, then

$$h(T(x), F_2(x)) = h\left(\overline{B\left(F(x), \frac{\delta}{3}\right)} \cap A_2, F_2(x)\right).$$

Since $\rho(u, v_2) < \frac{\delta}{3}$, we have

$$h(F(x), F_2(x)) < \frac{\delta}{3},$$

consequently, $F_2(x) \subset B(F(x), \frac{\delta}{3})$. Then we have

$$h(T(x), F_2(x)) \le \frac{\delta}{3}, \forall x \in \partial U_1.$$

Therefore, we assert that $h(T(x), F_2(x)) < \frac{2\delta}{3}$ for each $x \in X$. It follows that

$$\sup_{x \in X} h(G(x), F_2(x)) < \frac{2\delta}{3}.$$

Then it holds that

$$\rho(v', u_2) = \sup_{x \in X} h(G(x), F_2(x)) + h(A_2, A_2) < \frac{2\delta}{3}.$$

Therefore, we have

$$\rho(v', u) \le \rho(v', u_2) + \rho(u_2, u) < \frac{2\delta}{3} + \frac{\delta}{3} = \delta.$$

Thus, $S'(v') \cap (U_1 \cup U_2) \neq \emptyset$.

However, for each $x \in U_1$, we know that $G(x) = T(x) = F_1(x)$. Since

$$S'(v_1) \cap U_1 = \emptyset,$$

there must have $x \notin F_1(x)$ because it is impossible that $x \notin A_1$ noting that $U_1 \subset A_1$. That is, $x \notin S'(v')$. On the other hand, if $x \in U_2$, similarly, since $S'(v_2) \cap U_2 = \emptyset$, it should be that $x \notin A_2$ or $x \in A_2$ but $x \notin F_2(x)$. Observing that $U_2 \subset A_1 \subset A_2$, it holds that $x \notin F_2(x)$. Hence, we have $x \notin S'(v')$ because $G(x) = F_2(x)$. Therefore, there is a contradiction with the fact that $S'(v') \cap (U_1 \cup U_2) \neq \emptyset$. Then, $m(u)$ is definitely connected. $\square$

Remark 3.3. Noting that the connectedness of essential sets of fixed points in [3, 5, 14] only considers the perturbation of correspondences, Theorem 3.5 shows the connectedness of a minimal essential set (in the relative interior of A for each $u = (F, A) \in M$) with respect to M' which considers the dual perturbations of correspondences and domains. However, it is still a problem for study in the future whether each minimal essential set of fixed points with respect to M is connected.

4. An application

The results in Section 3 can be applied to the stability analysis of Nash equilibria. Here, let us recall some notions in relation to noncooperative game theory. Let $I = \{1, 2, \cdots, n\}$ be the set of players. For each $i \in I$, X_i be a compact and convex subset of a Euclidean space E_i. Let $A_i = X_i$ denotes the i player's strategy set, f_i be the i player's utility function with concave property. Then $A = \times_{i\in I} A_i$ is a compact and convex subset of the space $E = \times_{i\in I} E_i$. For each profile of strategy $x \in A$, the set

$$\beta_i(x) = \{x_i \in A_i : f_i(x_i, x_{-i}) = max_{z_i \in A_i} f_i(z_i, x_{-i})\}$$

is the i player's best response set for x. Let $\beta(x) = \times_{i\in I}\beta_i(x)$, then β defines a correspondence on A with nonempty, compact and convex values. It is known that the best response correspondence β is upper semicontinuous on A.

Straightforwardly, by Theorem 3.4, we have the following result in concerning the perturbation of the best response and strategic set for a noncooperative game.

Theorem 4.1. *For each $u = (\beta, A)$, there exists a minimal essential set of $S(u)$ with respect to M' .*

The following example shows a minimal essential set with respect to the perturbation of strategies and payoffs for a game.

Example 4.1. Let $I = \{1, 2\}$, S be the payoff matrix of a finite game such that

$$S = \begin{array}{ccc} & L & R \\ U & (1,2) & (-1,-1) \\ D & (0,2) & (2,2). \end{array}$$

Let $A = [0, 1] \times [0, 1]$. For any $(x, y) \in A$, x (y) denotes the strategy (probability) of play 1 (2) for choosing U (L). Let $\beta(x, y) = (\beta_1(y), \beta_2(x)) \in A$ denotes the best response of $(x, y) \in A$ for the game, where β_1 and β_2 indicate the best response of player 1 and 2 respectively.

Then for this game we know that there exist two connected components of Nash equilibria, and they are

$$C_1 = \{(x, y) \in A : (x, y) \in 0 \times [0, \frac{3}{4}]\}$$

and

$$C_2 = \{(x, y) \in A : (x, y) = (1, 1)\}.$$

For any small enough $\varepsilon > 0$, firstly, we restrict the strategy in

$$[1 - \varepsilon, 1 - \varepsilon] \times [1 - \varepsilon, 1 - \varepsilon] \subset intA.$$

Secondly, we define the perturbation of the best response $\beta^\varepsilon = (\beta_1^\varepsilon, \beta_2^\varepsilon)$ as follows:

$$\beta_1^\varepsilon(y) = \begin{cases} 1-\varepsilon, & \frac{3}{4} < y < 1-\varepsilon, \\ [\varepsilon, 1-\varepsilon], & y = \frac{3}{4}, \\ \varepsilon, & \varepsilon < y < \frac{3}{4}, \end{cases}$$

and

$$\beta_2^\varepsilon(x) = 1-\varepsilon,\ x > \varepsilon.$$

Clearly, we can check that the unique Nash equilibrium is $(1-\varepsilon, 1-\varepsilon)$ for the game under the perturbation, which is near the point $(1,1)$. Strictly, in this game, the point $(1,1) \in A$ is an essential solution and a minimal essential set also with respect to Definition 3.2.

Follow Theorem 3.5, we can obtain the following connectedness of a minimal essential set of Nash equilibria.

Theorem 4.2. *For each $u = (\beta, A)$, if $m(u) \subset intA$ is a minimal essential set of $S(u)$ with respect to M', then $m(u)$ is connected.*

Remark 4.1. Theorem 4.1 includes the corresponding result in [9] as a special case, that is, each noncooperative game with finite strategies has at least a quasistable set of Nash equilibria, where a quasistable set can resist the perturbation of the best response for a finite game. There exist many games which their minimal essential sets of Nash equilibria are included in the interior of strategic sets, e.g., Rock-Paper-Scissor games.

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