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ROTATIVE FIRMLY LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract. In 2005, J. Górnicki and K. Pupka gave conditions providing existence of fixed points for k-lipschitzian ($k > 1$) mappings in a Banach space which are n-rotative with $n \geqslant 3$. In the paper, using the same method, we study the existence of fixed points of rotative mappings in certain subclass of lipschitzian mappings, i.e. firmly lipschitzian mappings in order to obtain better estimates of Lipschitz constant k. We also show that $\mathrm{Fix}(T)$ is a Hölder continuous retract of C.
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1. Introduction

Let C be a nonempty closed convex subset of a Banach space X. A mapping $T : C \to C$ is called *k-lipschitzian* if for all x, y in C, $\|Tx - Ty\| \leqslant k\|x - y\|$. It is called *nonexpansive* if the same condition with $k = 1$ holds. In general, to assure the fixed point property for nonexpansive mappings some assumptions concerning the geometry of the spaces are added (see [13]). Another way is to put some additional restrictions on the mapping itself.

Such a condition imposed on the mapping may be rotativeness. A mapping $T\colon C \to C$ is called *(a, n)-rotative* if there exists an integer $n \geqslant 2$ and a real number $0 \leqslant a < n$ such that for any $x \in C$,

$$\|x - T^n x\| \leqslant a\|x - Tx\|. \tag{1.1}$$

In 1981, K. Goebel and M. Koter [3] showed that the condition of rotativness is actually quite strong; it assures the existence of fixed points of nonexpansive mappings even without weak compactness, or another special geometric structure of the set C. They obtained the following

Theorem 1.1. ([3]) *If C is a nonempty closed convex subset of a Banach space X, then any nonexpansive rotative mapping $T\colon C \to C$ has a fixed point.*

Rotativeness is independent of nonexpansiveness. Therefore if we consider k-lipschitzian mapping with $k > 1$, the condition of rotativeness (1.1) assures the existence of fixed points provided k is not too large.

Namely, we have the following

Theorem 1.2. ([4]) *If C is a nonempty closed convex subset of a Banach space X, then for any $n \geqslant 2$ and $a < n$ there exists $\gamma > 1$ such that any (a,n)-rotative and k-lipschitzian mapping $T\colon C \to C$ has a fixed point provided $k < \gamma$.*

Clearly, γ which appears in the above theorem depends on a, n and the space in which the set C is contained. Thus it is convenient to define the function $\gamma_n^X(a)$ as follows

$$\gamma_n^X(a) = \inf\{k\colon \text{there is a closed convex set } C \subset X \text{ and a fixed point free } k\text{-lipschitzian } (a,n)\text{-rotative selfmapping of } C\}.$$

In general, precise values of $\gamma_n^X(a)$ are unknown. If $n \geqslant 2$ is an arbitrary, from Kirk's theorem [12], it only estimates from below of the function $\gamma_n^X(a)$ at $a = 0$ are known. Namely,

$$\gamma_n^X(0) \geqslant \begin{cases} 2 & \text{for } n = 2, \\ \sqrt[n-1]{\frac{1}{n-2}\left(-1 + \sqrt{n(n-1) - \frac{1}{n-1}}\right)} & \text{for } n > 2. \end{cases}$$

It follows from the above that $\gamma_3^X(0) \geqslant 1.1598$; $\gamma_4^X(0) \geqslant 1.0649$; $\gamma_5^X(0) \geqslant 1.0351$; $\gamma_6^X(0) \geqslant 1.0219$. Some better results can be obtained for $n = 2$ (see [13], p.326).

Theorem 1.3. ([13]) *In an arbitrary Banach space X,*

$$\gamma_2^X(a) \geqslant \max\left\{\frac{1}{2}\left(2 - a + \sqrt{(2-a)^2 + a^2}\right),\right.$$
$$\left.\frac{1}{8}\left(a^2 + 4 + \sqrt{(a^2+4)^2 - 64a + 64}\right)\right\}, \quad a \in [0,2).$$

One can check that the first term gives better evaluation for $a \in [0, 2(\sqrt{2} - 1)]$, while the second one for $a \in [2(\sqrt{2}-1), 2)$.

In 1999, J. Górnicki [6] gave an evaluations of $\gamma_2^{l^p}(a)$ and $\gamma_2^{L^p}(a)$. In 2000, M. Koter-Mórgowska [15] gave an evaluation of $\gamma_n^H(a)$ in a Hilbert space for $n = 3, 4, 5, 6$. In 2005, J. Górnicki and K. Pupka [7] gave estimate for the function $\gamma_n^X(a)$ in all Banach spaces for $n \geqslant 3$.

In the paper, studying some three modifications of Halpern's iterative procedure [8], we give conditions providing existence of fixed points of rotative firmly k-lipschitzian mappings in a Banach space.

The notion of *firmly nonexpansive mapping* was introduced in 1973 by R.E. Bruck in [1]. The same class of mappings has been studied independently by K. Goebel and M. Koter in [2], where adopted a different name i.e. *regularly nonexpansive mapping.*

A mapping $T\colon C \to C$ is said to be *firmly k-lipschitzian* if for each $t \in [0,1]$ and for any $x, y \in C$,

$$\|Tx - Ty\| \leqslant \|k(1-t)(x-y) + t(Tx - Ty)\|. \tag{1.2}$$

Of course each firmly k-lipschitzian mappings is k-lipschitzian.

In 1986, M. Koter [14] received theorems for the existence of fixed point for firmly k-lipschitzian and 2-rotative mappings in a Banach space. In 1996, W. Kaczor and M. Koter - Mórgowska [11] obtained some results concerning asymptotic behaviour of firmly k-lipschitzian and rotative mappings. Some theorems on fixed points of λ-firmly nonexpansive mappings one can find in [9, 10, 17].

We now give an example of rotative firmly lipschitzian mapping.

Example 1.4. Let $X = C[0,1]$ be a space of continuous functions on $[0,1]$ with norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$. Define a set $C \subset X$ as follows

$$C = \{x \in C[0,1]\colon x \text{ is nondecreasing and } x(0) = 0,\ x(1) = 1\}$$

and a mapping $T\colon C \to C$ by formula

$$Tx(t) = k \max\left\{x(t) - \left(1 - \frac{1}{k}\right), 0\right\},\ \text{ where } k > 1.$$

Note what the mapping T is k-lipschitzian and moves each point the same constat distance $1 - \frac{1}{k}$. Moreover

$$\begin{aligned}\|x - T^n x\| &= 1 - \frac{1}{k^n} = \left(1 + \frac{1}{k} + \dots + \frac{1}{k^{n-1}}\right)\left(1 - \frac{1}{k}\right)\\ &= \sum_{j=1}^{n} \frac{1}{k^{j-1}}\|x - Tx\|.\end{aligned}$$

Taking $a = \sum_{j=1}^{n} \frac{1}{k^{j-1}}$ for $k > 1$ we getting that $a \in (1, n)$. That implies that T is $(\sum_{j=1}^{n} \frac{1}{k^{j-1}}, n)$-rotative. Therefore for $a \in (1, n)$

$$\gamma_n^{C[0,1]}(a) \leqslant \sup\left\{s > 1\colon \sum_{j=1}^{n} \frac{1}{s^{j-1}} = a\right\}.$$

Now we show that T is also firmly k-lipschitzian, i.e. for every $\alpha \in [0,1]$ holds

$$\|k(1-\alpha)(x - y) + \alpha(Tx - Ty)\| \geqslant \|Tx - Ty\|. \tag{1.3}$$

Let us fix $t \in [0,1]$ and let be

$$A_\alpha(t) = |k(1-\alpha)[x(t) - y(t)] + \alpha[Tx(t) - Ty(t)]|\,.$$

If $x(t) \leqslant 1 - \frac{1}{k}$ and $y(t) \leqslant 1 - \frac{1}{k}$, then

$$A_\alpha(t) = k(1-\alpha)|x(t) - y(t)| \geqslant 0 = |Tx(t) - Ty(t)|\,.$$

If $y(t) \leqslant 1 - \frac{1}{k} \leqslant x(t)$, then

$$\begin{aligned}A_\alpha(t) &= k(1-\alpha)[x(t) - y(t)] + \alpha k\left[x(t) - \left(1 - \frac{1}{k}\right)\right]\\ &\geqslant k(1-\alpha)\left[x(t) - \left(1 - \frac{1}{k}\right)\right] + \alpha k\left[x(t) - \left(1 - \frac{1}{k}\right)\right]\\ &= k\left[x(t) - \left(1 - \frac{1}{k}\right)\right] = |Tx(t) - Ty(t)|\,.\end{aligned}$$

If $x(t) \geqslant 1 - \frac{1}{k}$ and $y(t) \geqslant 1 - \frac{1}{k}$, then

$$A_\alpha(t) = |k(1-\alpha)[x(t) - y(t)] + \alpha k[x(t) - y(t)]| = k|x(t) - y(t)| = |Tx(t) - Ty(t)|\,.$$

Thus, for any $t \in [0,1]$ we have:

$$A_\alpha(t) \geqslant |Tx(t) - Ty(t)|,$$

which implies (1.3). □

2. Main results

We will start with the following two lemmas:

Lemma 2.1. ([5]) *Let C be a nonempty closed convex subset of a Banach space X and let $T\colon C \to C$ be a k-lipschitzian. Assume that $A, B \in \mathbb{R}$ and $0 \leq A < 1$ and $0 < B$. If for an arbitrary $x \in C$ exists $u \in C$ such that*

$$\|Tu - u\| \leqslant A\|Tx - x\|$$

and

$$\|u - x\| \leqslant B\|Tx - x\|,$$

then T has a fixed point in C.

Lemma 2.2. *Let C be a nonempty subset of a Banach space X and mapping $T\colon C \to C$ be a firmly k-lipschitzian ($k > 1$) and (a,n)-rotative ($n > 2$), then for $x \in C$ we have*

$$\|T^{n-1}x - T^n x\| \leqslant \left(a\left(\frac{k}{k+1}\right)^{n-1} + \sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^j k^{n-j} \frac{1-k^{j-1}}{1-k} \right) \|x - Tx\|.$$

Proof. Let $n > 2$. Note at the beginning that for a firmly k-lipschitzian mapping $T : C \to C$ putting in (1.2), $t = \frac{k}{k+1}$ we obtain

$$\|Tx - Ty\| \leqslant \frac{k}{k+1}\|x - y + Tx - Ty\|. \tag{2.1}$$

Using the condition (2.1) two times, we obtain

$$\begin{aligned}
\|T^{n-1}x - T^n x\| &\leqslant \frac{k}{k+1}\|T^{n-2}x - T^{n-1}x + T^{n-1}x - T^n x\| \\
&= \frac{k}{k+1}\|T^{n-2}x - T^n x\| \\
&\leqslant \left(\frac{k}{k+1}\right)^2 \|T^{n-3}x - T^{n-1}x + T^{n-2}x - T^n x\| \\
&= \left(\frac{k}{k+1}\right)^2 \|T^{n-3}x - T^n x + T^{n-2}x - T^{n-1}x\| \\
&\leqslant \left(\frac{k}{k+1}\right)^2 \Big(\|T^{n-3}x - T^n x\| + \|T^{n-2}x - T^{n-1}x\|\Big).
\end{aligned}$$

Repeating this estimating operation, we get

$$\begin{aligned}\|T^{n-1}x-T^nx\| &\leqslant \left(\frac{k}{k+1}\right)^2\left(\frac{k}{k+1}\|T^{n-4}x-T^{n-1}x+T^{n-3}x-T^nx\| \right.\\ &\qquad \left. +\|T^{n-2}x-T^{n-1}x\|\right)\\ &\leqslant \left(\frac{k}{k+1}\right)^2\left(\frac{k}{k+1}\|T^{n-4}x-T^nx\| \right.\\ &\qquad \left. +\frac{k}{k+1}\|T^{n-3}x-T^{n-1}x\|+\|T^{n-2}x-T^{n-1}x\|\right)\\ &\leqslant \dots\\ &\leqslant \left(\frac{k}{k+1}\right)^2\left(\left(\frac{k}{k+1}\right)^{n-3}\|x-T^nx\| \right.\\ &\qquad +\left(\frac{k}{k+1}\right)^{n-3}\|Tx-T^{n-1}x\|+\dots\\ &\qquad \left. +\frac{k}{k+1}\|T^{n-3}x-T^{n-1}x\|+\|T^{n-2}x-T^{n-1}x\|\right).\end{aligned}$$

Finally, using the fact that mapping T is (a,n)-rotative and k-lipschitzian, we have

$$\begin{aligned}\|T^{n-1}x-T^nx\| \leqslant \left(\frac{k}{k+1}\right)^2 &\left(a\left(\frac{k}{k+1}\right)^{n-3}\right.\\ &\left.+\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^{j-2}k^{n-j}\frac{1-k^{j-1}}{1-k}\right)\|x-Tx\|,\end{aligned}$$

which completes the proof. □

Let us mark

$$\begin{aligned}\bar{\gamma}_n^1(a)=\sup_{\alpha\in(0,1)}&\left\{s>1\colon \alpha^2as\left(\frac{s}{s+1}\right)^{n-1}+\alpha^2\sum_{j=2}^{n-1}\left(\frac{s}{s+1}\right)^{j}s^{n-j+1}\frac{1-s^{j-1}}{1-s}\right.\\ &\left.+\alpha^2\sum_{j=2}^{n-1}(1-\alpha)^{j-1}s^j\frac{1-s^{n-j}}{1-s}+\alpha(1-\alpha)as+(1-\alpha)^ns^{n-1}-1=0\right\},\end{aligned}$$

$$\begin{aligned}\bar{\gamma}_n^2(a)=\sup_{\alpha\in(0,1)}&\left\{s>1\colon \alpha^2as\left(\frac{s}{s+1}\right)^{n-1}+\alpha^2\sum_{j=2}^{n-1}\left(\frac{s}{s+1}\right)^{j}s^{n-j+1}\frac{1-s^{j-1}}{1-s}\right.\\ &\left.+\alpha^2s^n\sum_{j=2}^{n-1}(1-\alpha)^{j-1}\frac{1-s^j}{1-s}+\alpha(1-\alpha)^{n-1}s^{n-1}a+(1-\alpha)^ns^{n-1}-1=0\right\},\end{aligned}$$

$$\bar{\gamma}_n^3(a)=\sup_{\alpha\in(0,1)}\left\{s>1\colon \alpha a+\alpha^2as\left(\left(\frac{s}{s+1}\right)^{n-1}+1\right)\right.$$

$$+\alpha^2 \sum_{j=2}^{n-1} \left(\frac{s}{s+1}\right)^j s^{n-j+1} \frac{1-s^{j-1}}{1-s}$$

$$\left. +\alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} s^j \frac{1-s^{n-j}}{1-s} + (1-\alpha)^n s^{n-1} - 1 = 0 \right\},$$

where $a \in [0, n)$, $n \in \mathbb{N}$ and $n > 2$.

Now we are ready to formulate the main theorem of the paper.

Theorem 2.3. *Given an integer $n > 2$ and let C be a nonempty convex closed subset of a Banach space X. If $T\colon C \to C$ is firmly k-lipschitzian ($k > 1$) and (a, n)-rotative mapping such that*

$$k < \max\left\{\bar{\gamma}_n^1(a), \bar{\gamma}_n^2(a), \bar{\gamma}_n^3(a)\right\},$$

then T has a fixed point in C.

Proof. We must take three cases into consideration.

Case I. We consider the following sequence: let x be an arbitrary point in C, i.e. $x_0 = x \in C$ and

$$\begin{aligned} x_1 &= \alpha x_0 + (1-\alpha) T x_0, \\ x_2 &= \alpha x_0 + (1-\alpha) T x_1, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-2} &= \alpha x_0 + (1-\alpha) T x_{n-3}, \\ x_{n-1} &= \alpha T^n x_0 + (1-\alpha) T x_{n-2}, \end{aligned}$$

where $\alpha \in (0, 1)$. Put $z = x_{n-1}$, then

$$\begin{aligned} \|z - Tz\| &= \|\alpha T^n x_0 + (1-\alpha) T x_{n-2} - Tz\| \\ &= \|\alpha (T^n x_0 - Tz) + (1-\alpha)(T x_{n-2} - Tz)\| \\ &\leqslant \alpha k \|T^{n-1} x_0 - z\| + (1-\alpha) k \|x_{n-2} - z\|. \end{aligned} \tag{2.2}$$

Now, we have evaluation

$$\begin{aligned} \|T^{n-1} x_0 - z\| &= \|T^{n-1} x_0 - \alpha T^n x_0 - (1-\alpha) T x_{n-2}\| \\ &= \|\alpha (T^{n-1} x_0 - T^n x_0) + (1-\alpha)(T^{n-1} x_0 - T x_{n-2})\| \\ &\leqslant \alpha \|T^{n-1} x_0 - T^n x_0\| + (1-\alpha) k \|T^{n-2} x_0 - x_{n-2}\|, \end{aligned} \tag{2.3}$$

where

$$(1-\alpha) k \|T^{n-2} x_0 - x_{n-2}\| = (1-\alpha) k \|T^{n-2} x_0 - \alpha x_0 - (1-\alpha) T x_{n-3}\|$$

$$= (1-\alpha) k \|\alpha (T^{n-2} x_0 - x_0) + (1-\alpha)(T^{n-2} x_0 - T x_{n-3})\|$$

$$\leqslant (1-\alpha) \alpha k \|T^{n-2} x_0 - x_0\| + (1-\alpha)^2 k^2 \|T^{n-3} x_0 - x_{n-3}\| \leqslant \dots$$

$$\leqslant \alpha (1-\alpha) k \|T^{n-2} x_0 - x_0\| + \alpha (1-\alpha)^2 k^2 \|T^{n-3} x_0 - x_0\|$$

$$+\alpha (1-\alpha)^3 k^3 \|T^{n-4} x_0 - x_0\| + \dots + \alpha (1-\alpha)^{n-2} k^{n-2} \|T x_0 - x_0\|.$$

Finally, using only the triangle inequality and the fact that T is also k-lipschitzian we get

$$(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \leqslant \alpha \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j-1}\frac{1-k^{n-j}}{1-k}\|Tx_0 - x_0\|, \tag{2.4}$$

and consequently from (2.3), (2.4) and Lemma 2.2, we obtain

$$\begin{aligned}\|T^{n-1}x_0 - z\| \leqslant \Bigg(&\alpha a\left(\frac{k}{k+1}\right)^{n-1} + \alpha\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^{j} k^{n-j}\frac{1-k^{j-1}}{1-k} \\ &+ \alpha\sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j-1}\frac{1-k^{n-j}}{1-k}\Bigg)\|Tx_0 - x_0\|,\end{aligned} \tag{2.5}$$

For the next expression in (2.2), using (1.1), we have the following evaluation

$$\begin{aligned}\|x_{n-2} - z\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-3} - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|\alpha(x_0 - T^n x_0) + (1-\alpha)(Tx_{n-3} - Tx_{n-2})\| \\ &\leqslant \alpha a\|x_0 - Tx_0\| + (1-\alpha)k\|x_{n-3} - x_{n-2}\| \leqslant \dots \\ &\leqslant \alpha a\|x_0 - Tx_0\| + (1-\alpha)^{n-2}k^{n-2}\|x_0 - x_1\| \\ &= \left(\alpha a + (1-\alpha)^{n-1}k^{n-2}\right)\|x_0 - Tx_0\|.\end{aligned} \tag{2.6}$$

Combining (2.2) with (2.5) and (2.6) yields

$$\begin{aligned}\|z - Tz\| \leqslant \Bigg(&\alpha^2 ak\left(\frac{k}{k+1}\right)^{n-1} + \alpha^2\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^{j} k^{n-j+1}\frac{1-k^{j-1}}{1-k} \\ &+ \alpha^2\sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j}\frac{1-k^{n-j}}{1-k} \\ &+ \alpha(1-\alpha)ak + (1-\alpha)^n k^{n-1}\Bigg)\|x_0 - Tx_0\|.\end{aligned} \tag{2.7}$$

Moreover, we have

$$\begin{aligned}\|z - x_0\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - x_0\| \\ &= \|\alpha(T^n x_0 - x_0) + (1-\alpha)(Tx_{n-2} - x_0)\| \\ &\leqslant \alpha\|T^n x_0 - x_0\| + (1-\alpha)\|Tx_{n-2} - T^n x_0 + T^n x_0 - x_0\| \\ &\leqslant \alpha\|T^n x_0 - x_0\| + (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| + (1-\alpha)\|T^n x_0 - x_0\| \\ &= \|x_0 - T^n x_0\| + (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\|.\end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned}(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| &= (1-\alpha)k\|\alpha(x_0 - T^{n-1}x_0) + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \leqslant \dots \\ &\leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + \alpha(1-\alpha)^2k^2\|x_0 - T^{n-2}x_0\|\end{aligned}$$

$$+\alpha(1-\alpha)^3k^3\|x_0 - T^{n-3}x_0\| + \dots$$
$$+\alpha(1-\alpha)^{n-2}k^{n-2}\|x_0 - T^2x_0\| + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\|.$$

Now, using only the triangle inequality and the fact that T is k-lipschitzian, we have

$$(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| \leqslant \left((1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k}\right)\|x_0 - Tx_0\|,$$

which together with (2.8) gives

$$\|z - x_0\| \leqslant \left(a + (1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{j=1}^{n-2}(1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k}\right)\|x_0 - Tx_0\|. \tag{2.9}$$

Since

$$\begin{aligned} \alpha^2 ak\left(\frac{k}{k+1}\right)^{n-1} &+ \alpha^2\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^j k^{n-j+1}\frac{1-k^{j-1}}{1-k} \\ &+ \alpha^2\sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^j\frac{1-k^{n-j}}{1-k} \\ &+ \alpha(1-\alpha)ak + (1-\alpha)^n k^{n-1} < 1 \end{aligned}$$

for all $\alpha \in (0,1)$ and $k < \bar{\gamma}_n^1(a)$, by inequalities (2.7), (2.9), the Lemma 2.1 implies the existence of fixed points of T in C.

Case II. Consider the following sequence generated as follows:

$$\begin{aligned} x_0 &= x \in C, \\ x_1 &= \alpha T^n x_0 + (1-\alpha)Tx_0, \\ x_2 &= \alpha T^n x_0 + (1-\alpha)Tx_1, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-2} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-3}, \\ x_{n-1} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. Then for $z = x_{n-1}$, we get

$$\begin{aligned} \|z - Tz\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - Tz\| \\ &= \|\alpha(T^n x_0 - Tz) + (1-\alpha)(Tx_{n-2} - Tz)\| \\ &\leqslant \alpha k\|T^{n-1}x_0 - z\| + (1-\alpha)k\|x_{n-2} - z\|. \end{aligned} \tag{2.10}$$

Now, we have evaluation

$$\begin{aligned} \|T^{n-1}x_0 - z\| &= \|T^{n-1}x_0 - \alpha T^n x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|\alpha(T^{n-1}x_0 - T^n x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\| \\ &\leqslant \alpha\|T^{n-1}x_0 - T^n x_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
(1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| &\leqslant \dots \\
&\leqslant \alpha(1-\alpha)k^{n-1}\|x_0 - T^2x_0\| + \alpha(1-\alpha)^2k^{n-1}\|x_0 - T^3x_0\| + \dots \\
&\quad + \alpha(1-\alpha)^{n-3}k^{n-1}\|x_0 - T^{n-2}x_0\| + \alpha(1-\alpha)^{n-2}k^{n-1}\|x_0 - T^{n-1}x_0\| \\
&\leqslant \alpha(1-\alpha)k^{n-1}(1+k)\|x_0 - Tx_0\| \\
&\quad + \alpha(1-\alpha)^2k^{n-1}(1+k+k^2)\|x_0 - Tx_0\| + \dots \\
&\quad + \alpha(1-\alpha)^{n-3}k^{n-1}(1+k+\dots+k^{n-3})\|x_0 - Tx_0\| \\
&\quad + \alpha(1-\alpha)^{n-2}k^{n-1}(1+k+\dots+k^{n-2})\|x_0 - Tx_0\| \\
&= \left(\alpha k^{n-1}\sum_{j=2}^{n-1}(1-\alpha)^{j-1}\frac{1-k^j}{1-k}\right)\|x_0 - Tx_0\|.
\end{aligned} \tag{2.12}$$

Finally from Lemma 2.2 and inequalities (2.11), (2.12) we get

$$\begin{aligned}
\|T^{n-1}x_0 - z\| &\leqslant \alpha\|T^{n-1}x_0 - T^nx_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \\
&\leqslant \left(\alpha a\left(\frac{k}{k+1}\right)^{n-1} + \alpha\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^j k^{n-j}\frac{1-k^{j-1}}{1-k}\right. \\
&\qquad \left. + \alpha k^{n-1}\sum_{j=2}^{n-1}(1-\alpha)^{j-1}\frac{1-k^j}{1-k}\right)\|x_0 - Tx_0\|.
\end{aligned} \tag{2.13}$$

For the next expression in (2.10) we have the following evaluation (using (1.1)):

$$\begin{aligned}
\|x_{n-2} - z\| &= \|\alpha T^nx_0 + (1-\alpha)Tx_{n-3} - \alpha T^nx_0 - (1-\alpha)Tx_{n-2}\| \\
&\leqslant (1-\alpha)k\|x_{n-3} - x_{n-2}\| \leqslant \dots \\
&\leqslant (1-\alpha)^{n-2}k^{n-2}\|x_0 - x_1\| \\
&= (1-\alpha)^{n-2}k^{n-2}\|\alpha(x_0 - T^nx_0) + (1-\alpha)(x_0 - Tx_0)\| \\
&\leqslant \left(\alpha(1-\alpha)^{n-2}k^{n-2}a + (1-\alpha)^{n-1}k^{n-2}\right)\|x_0 - Tx_0\|.
\end{aligned} \tag{2.14}$$

Combining (2.10) with (2.13) and (2.14) yields

$$\begin{aligned}
\|z - Tz\| &\leqslant \left(\alpha^2ak\left(\frac{k}{k+1}\right)^{n-1} + \alpha^2\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^j k^{n-j+1}\frac{1-k^{j-1}}{1-k}\right. \\
&\qquad + \alpha^2k^n\sum_{j=2}^{n-1}(1-\alpha)^{j-1}\frac{1-k^j}{1-k} \\
&\qquad \left. + \alpha(1-\alpha)^{n-1}k^{n-1}a + (1-\alpha)^nk^{n-1}\right)\|x_0 - Tx_0\|.
\end{aligned} \tag{2.15}$$

Moreover, we have

$$\begin{aligned}\|z-x_0\| &= \|\alpha T^n x_0 + (1-\alpha)Tx_{n-2} - x_0\| \\ &= \|\alpha(T^n x_0 - x_0) + (1-\alpha)(Tx_{n-2} - x_0)\| \\ &\leqslant \alpha\|T^n x_0 - x_0\| + (1-\alpha)\|Tx_{n-2} - T^n x_0 + T^n x_0 - x_0\| \\ &= \|x_0 - T^n x_0\| + (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\|.\end{aligned} \tag{2.16}$$

Observe that

$$\begin{aligned}&(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| = \\ &\quad = (1-\alpha)k\|\alpha(T^n x_0 - T^{n-1}x_0) + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\quad \leqslant \alpha(1-\alpha)k\|T^n x_0 - T^{n-1}x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \leqslant \dots \\ &\quad \leqslant \alpha(1-\alpha)k\|T^n x_0 - T^{n-1}x_0\| + \alpha(1-\alpha)^2k^2\|T^n x_0 - T^{n-2}x_0\| + \dots \\ &\qquad + \alpha(1-\alpha)^{n-2}k^{n-2}\|T^n x_0 - T^2 x_0\| + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\| \\ &\quad \leqslant \alpha(1-\alpha)kk^{n-1}\|Tx_0 - x_0\| \\ &\qquad + \alpha(1-\alpha)^2k^2(k^{n-1} + k^{n-2})\|Tx_0 - x_0\| + \dots \\ &\qquad + \alpha(1-\alpha)^{n-2}k^{n-2}(k^{n-1} + \dots + k^2)\|Tx_0 - x_0\| \\ &\qquad + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\| \\ &\quad = \alpha k^n \sum_{j=1}^{n-2}(1-\alpha)^j \frac{1-k^j}{1-k}\|x_0 - Tx_0\| + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\|,\end{aligned}$$

which together with (2.16) gives

$$\|z-x_0\| \leqslant \left(a + \alpha k^n \sum_{j=1}^{n-2}(1-\alpha)^j \frac{1-k^j}{1-k} + (1-\alpha)^{n-1}k^{n-1}\right)\|x_0 - Tx_0\|. \tag{2.17}$$

Since

$$\begin{aligned}\alpha^2 ak\left(\frac{k}{k+1}\right)^{n-1} &+ \alpha^2\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^j k^{n-j+1}\frac{1-k^{j-1}}{1-k} \\ &+ \alpha^2 k^n \sum_{j=2}^{n-1}(1-\alpha)^{j-1}\frac{1-k^j}{1-k} \\ &+ \alpha(1-\alpha)^{n-1}k^{n-1}a + (1-\alpha)^n k^{n-1} < 1\end{aligned}$$

for all $\alpha \in (0,1)$ and $k < \bar{\gamma}_n^2(a)$, by inequalities (2.15), (2.17), the Lemma 2.1 implies the existence of fixed points of T in C.

Case III. We consider a sequence generated as follows: let x be an arbitrary point in C, i.e. $x_0 = x \in C$ and

$$\begin{aligned} x_1 &= \alpha x_0 + (1-\alpha)Tx_0, \\ x_2 &= \alpha x_0 + (1-\alpha)Tx_1, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-2} &= \alpha x_0 + (1-\alpha)Tx_{n-3}, \\ x_{n-1} &= \alpha x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. Then for $z = x_{n-1}$, we have

$$\begin{aligned} \|z - Tz\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-2} - Tz\| \\ &= \|\alpha(x_0 - Tz) + (1-\alpha)(Tx_{n-2} - Tz)\| \\ &\leqslant \alpha\|x_0 - T^n x_0 + T^n x_0 - Tz\| + (1-\alpha)k\|x_{n-2} - z\| \\ &\leqslant \alpha\|x_0 - T^n x_0\| + \alpha k\|T^{n-1}x_0 - z\| + (1-\alpha)k\|x_{n-2} - z\|. \end{aligned} \tag{2.18}$$

Now, we have evaluation

$$\begin{aligned} \|T^{n-1}x_0 - z\| &= \|T^{n-1}x_0 - \alpha x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|\alpha(T^{n-1}x_0 - x_0) + (1-\alpha)(T^{n-1}x_0 - Tx_{n-2})\| \\ &\leqslant \alpha\|T^{n-1}x_0 - x_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\|. \end{aligned} \tag{2.19}$$

Note that using condition (1.1) and Lemma 2.2 we obtain

$$\begin{aligned} \alpha\|T^{n-1}x_0 - x_0\| &\leqslant \alpha(\|T^{n-1}x_0 - T^n x_0\| + \|T^n x_0 - x_0\|) \\ &\leqslant \left(\alpha a\left(\left(\frac{k}{k+1}\right)^{n-1} + 1\right)\right. \\ &\quad \left. + \alpha\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^j k^{n-j}\frac{1-k^{j-1}}{1-k}\right)\|x_0 - Tx_0\|, \end{aligned} \tag{2.20}$$

Now, by (2.4) and (2.20) we obtain

$$\begin{aligned} \|T^{n-1}x_0 - z\| &\leqslant \alpha\|T^{n-1}x_0 - x_0\| + (1-\alpha)k\|T^{n-2}x_0 - x_{n-2}\| \\ &\leqslant \left(\alpha a\left(\left(\frac{k}{k+1}\right)^{n-1} + 1\right) + \alpha\sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^j k^{n-j}\frac{1-k^{j-1}}{1-k}\right. \\ &\quad \left. + \alpha\sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j-1}\frac{1-k^{n-j}}{1-k}\right)\|x_0 - Tx_0\|. \end{aligned} \tag{2.21}$$

For the next expression in (2.18) we have the following evaluation

$$\begin{aligned} \|x_{n-2} - z\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-3} - \alpha x_0 - (1-\alpha)Tx_{n-2}\| \\ &= \|(1-\alpha)(Tx_{n-3} - Tx_{n-2})\| \leqslant (1-\alpha)k\|x_{n-3} - x_{n-2}\| \leqslant \dots \\ &\leqslant (1-\alpha)^{n-2}k^{n-2}\|x_0 - x_1\| = (1-\alpha)^{n-1}k^{n-2}\|x_0 - Tx_0\|. \end{aligned} \tag{2.22}$$

Combining (2.18) with (2.21) and (2.22) yields

$$\begin{aligned}\|z - Tz\| \leqslant \Bigg(\alpha a + \alpha^2 ak \left(\left(\frac{k}{k+1}\right)^{n-1} + 1\right) \\ + \alpha^2 \sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^j k^{n-j+1} \frac{1-k^{j-1}}{1-k} \\ + \alpha^2 \sum_{j=2}^{n-1} (1-\alpha)^{j-1} k^j \frac{1-k^{n-j}}{1-k} + (1-\alpha)^n k^{n-1}\Bigg) \|x_0 - Tx_0\|.\end{aligned} \tag{2.23}$$

Moreover, we have

$$\begin{aligned}\|z - x_0\| &= \|\alpha x_0 + (1-\alpha)Tx_{n-2} - x_0\| = (1-\alpha)\|Tx_{n-2} - x_0\| \\ &\leqslant (1-\alpha)\left(\|Tx_{n-2} - T^n x_0\| + \|T^n x_0 - x_0\|\right) \\ &\leqslant (1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| + (1-\alpha)a\|Tx_0 - x_0\|.\end{aligned} \tag{2.24}$$

Observe that

$$\begin{aligned}(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| &= \\ &= (1-\alpha)k\|\alpha x_0 + (1-\alpha)Tx_{n-3} - T^{n-1}x_0\| \\ &= (1-\alpha)k\|\alpha(x_0 - T^{n-1}x_0) + (1-\alpha)(Tx_{n-3} - T^{n-1}x_0)\| \\ &\leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + (1-\alpha)^2k^2\|x_{n-3} - T^{n-2}x_0\| \leqslant \dots \\ &\leqslant \alpha(1-\alpha)k\|x_0 - T^{n-1}x_0\| + \alpha(1-\alpha)^2k^2\|x_0 - T^{n-2}x_0\| \\ &\quad + \alpha(1-\alpha)^3k^3\|x_0 - T^{n-3}x_0\| + \dots \\ &\qquad + (1-\alpha)^{n-1}k^{n-1}\|x_0 - Tx_0\|.\end{aligned}$$

Now, using only the triangle inequality and the fact that T is k-lipschitzian, we have

$$\begin{aligned}(1-\alpha)k\|x_{n-2} - T^{n-1}x_0\| \\ \leqslant \left((1-\alpha)^{n-1}k^{n-1} + \alpha \sum_{j=1}^{n-2} (1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k}\right) \|x_0 - Tx_0\|,\end{aligned}$$

which together with (2.24) gives

$$\begin{aligned}\|z - x_0\| \leqslant \Bigg((1-\alpha)a + (1-\alpha)^{n-1}k^{n-1} \\ + \alpha \sum_{j=1}^{n-2} (1-\alpha)^j k^j \frac{1-k^{n-j}}{1-k}\Bigg) \|x_0 - Tx_0\|.\end{aligned} \tag{2.25}$$

Since

$$\begin{aligned}\alpha a + \alpha^2 ak\left(\left(\frac{k}{k+1}\right)^{n-1} + 1\right) \\ + \alpha^2 \sum_{j=2}^{n-1}\left(\frac{k}{k+1}\right)^{j} k^{n-j+1}\frac{1-k^{j-1}}{1-k} \\ + \alpha^2 \sum_{j=2}^{n-1}(1-\alpha)^{j-1}k^{j}\frac{1-k^{n-j}}{1-k} + (1-\alpha)^n k^{n-1} < 1\end{aligned}$$

for all $\alpha \in (0,1)$ and $k < \bar{\gamma}_n^3(a)$, by inequalities (2.23) and (2.25), the Lemma 2.1 implies the existence of fixed points of T in C. $\square$

Define function $\bar{\gamma}_n^X(a)$ in the class of firmly k-lipschitzian mappings as follows

$$\begin{aligned}\bar{\gamma}_n^X(a) = \inf\{k\colon & \text{ there is a closed convex set } C \subset X \text{ and a fixed point free} \\ & \text{firmly } k\text{-lipschitzian } (a,n)\text{-rotative selfmapping of } C\}.\end{aligned}$$

Remark 2.4. Theorem 2.3 follows that

$$\bar{\gamma}_n^X(a) \geqslant \max\left\{\bar{\gamma}_n^1(a), \bar{\gamma}_n^2(a), \bar{\gamma}_n^3(a)\right\}$$

for $n > 2$ and $a \in [0,n)$. The figure 2.1 illustrates the lower bound of $\bar{\gamma}_3^X(a)$ (bold line) in a Banach space X.

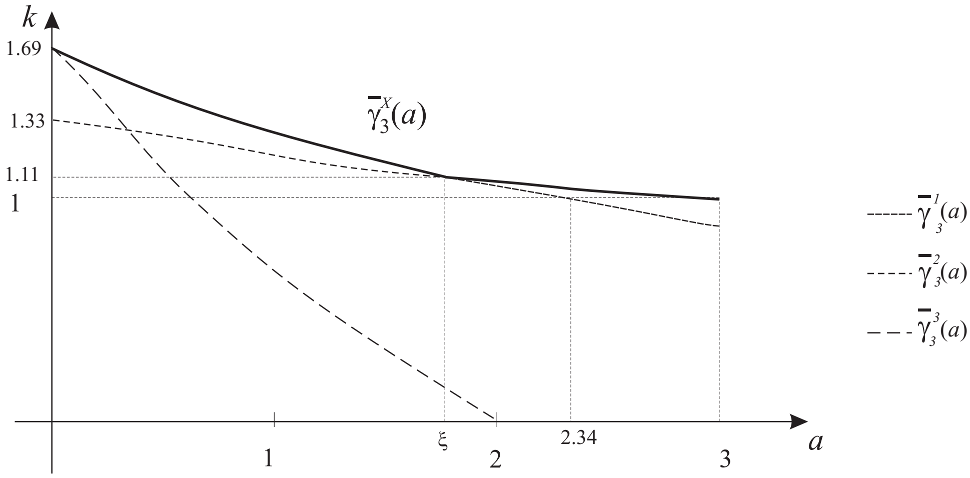

Figure 2.1

Let fix $\alpha = \frac{1}{2}$. Then from $\bar{\gamma}_3^1(a)$ we get the equation

$$2k^4 + (2-3a)k^3 + (2a-3)k^2 + (a-8)k - 4 = 0,$$

and from $\bar{\gamma}_3^2(a)$ the another one

$$k^6 + 3k^5 + (6+a)k^4 + (3+4a)k^3 + (a-7)k^2 - 16k - 8 = 0.$$

The solution of the first equation gives better estimate for $a \in [0, \xi]$, while the solution of the second equations for $a \in [\xi, 3)$, where $\xi \approx 1.7$.

Remark 2.5. Theorem 2.3 implies that

$$\begin{aligned}
\bar{\gamma}_3^X(0) &\geqslant 1.6977 \quad \text{(we get them from } \bar{\gamma}_3^1(0) \text{ for } \alpha = 0.54);\\
\bar{\gamma}_4^X(0) &\geqslant 1.3059 \quad \text{(we get them from } \bar{\gamma}_4^1(0) \text{ for } \alpha = 0.34);\\
\bar{\gamma}_5^X(0) &\geqslant 1.1902 \quad \text{(we get them from } \bar{\gamma}_5^1(0) \text{ for } \alpha = 0.24);\\
\bar{\gamma}_6^X(0) &\geqslant 1.1364 \quad \text{(we get them from } \bar{\gamma}_6^1(0) \text{ for } \alpha = 0.19).
\end{aligned}$$

All these estimates are better then respective, previously obtained estimates in [7] for the class of lipschitzian and rotative mappings in Banach space. The estimate $\bar{\gamma}_3^X(0)$ is even better than the estimate obtained in [16] for Hilbert space.

Remark 2.6. The reasoning carried out in the proof of Theorem 2.3 allows to obtain another proof of well known fact ([4]) that if C is a nonempty, closed, convex subset of a Banach space X, then for each $n \geqslant 3$ and $0 \leqslant a < n$ exists $\bar{\gamma}_n^X(a) > 1$ such that every (a, n)-rotative and firmly k-lipschitzian mapping $T\colon C \to C$ with $k < \bar{\gamma}_n^X(a)$ has a fixed point. Consider function obtained from the inequality (2.15)

$$\begin{aligned}
g(\alpha, k) = {}& \alpha^2 a k \left(\frac{k}{k+1}\right)^{n-1} + \alpha^2 \sum_{j=2}^{n-1} \left(\frac{k}{k+1}\right)^j k^{n-j+1} \frac{1-k^{j-1}}{1-k}\\
&+ \alpha^2 k^n \sum_{j=2}^{n-1} (1-\alpha)^{j-1} \frac{1-k^j}{1-k}\\
&+ \alpha(1-\alpha)^{n-1} k^{n-1} a + (1-\alpha)^n k^{n-1},
\end{aligned}$$

which is continuous for $\alpha \in (0, 1)$, $k > 1$ and $n > 2$. For $\alpha = \frac{1}{n}$ we have

$$\lim_{k \to 1^+} g\left(\frac{1}{n}, k\right) = g\left(\frac{1}{n}\right) = 1 + (a - n) \frac{[2(n-1)]^{n-1} + n^{n-2}}{2^{n-1} n^n}.$$

Since $g(\frac{1}{n}) < 1$ for $a < n$, then $g(\frac{1}{n}, k) < 1$ if k is enough close to 1 (but $k > 1$) and $0 \leqslant a < n$. For this k, $a \in [0, n)$ and $\alpha = \frac{1}{n}$, the sequence generating by iterative procedure, considered in Case II of the proof of Theorem 2.3, converges to a fixed point of T for each $x_0 \in C$. This guarantees that $\bar{\gamma}_n^X(a) > 1$ for arbitrary Banach space X, $0 \leqslant a < n$ and $n > 2$.

3. Hölder continuous retractions

In this chapter, we will show that for narrowing considering mapping T to bounded, closed and convex set C, the limits of the iterative processes discussed in cases I, II and III of the proof of the Theorem 2.3 are Hölder continuous retraction from C to $\mathrm{Fix}(T)$.

Let C be a nonempty, closed, convex and bounded subset of a Banach space X. Recall that a set $D \subset C$ is a *retract* of C if there is a continuous mapping $R : C \to D$ (*retraction*) with $\mathrm{Fix}(R) = D$. We say that a mapping $R : C \to C$ is *Hölder* *continuous* if there are constants $L \geqslant 0$ and $0 < \beta < 1$ such that for any $x, y \in C$ holds:

$$\|Rx - Ry\| \leqslant L\|x - y\|^{\beta}. \tag{3.1}$$

An example of a real function (with $x \geqslant 0$) satisfying the Hölder condition but not satisfying the Lipschitz condition is a function $f(x) = x^{\beta}$.

The following lemma gives condition for existence of Hölder continuous retraction on the fixed point set

Lemma 3.1. ([16]) *Let X be a complete metric space and $T : X \to X$ a continuous mapping. Suppose there are $u : X \to X$, $0 < A < 1$ and $B > 0$, such that for every $x \in X$:*

$$(i) \quad d(Tu(x), u(x)) \leqslant Ad(Tx, x),$$
$$(ii) \quad d(u(x), x) \leqslant Bd(Tx, x),$$

then $\mathrm{Fix}(T) \neq \emptyset$. *If we define $R(x) = \lim_{n\to\infty} u^n(x)$ and u is a continuous mapping, then R is a retraction from X to* $\mathrm{Fix}(T)$. *If additionally u satisfies the Lipschitz condition with constant $k > 1$ and* $\mathrm{diam}(X) < \infty$, *then R is a Hölder continuous retraction from X to* $\mathrm{Fix}(T)$.

We shall now show that the transformations, defined by iterative processes used in proof of Theorem 2.3 satisfies the Lipschitz condition with constat $s > 1$.

Lemma 3.2. *Let $n > 2$ be integer and let C be a nonempty, closed, convex subset of a Banach space X. Let a mapping $T\colon C \to C$ be k-lipschitzian with $k > 1$. For $x \in C$ we generate the sequence:*

$$\begin{aligned}
x_0 &= x,\\
x_1 &= \alpha x_0 + (1-\alpha)Tx_0,\\
x_2 &= \alpha x_0 + (1-\alpha)Tx_1,\\
&\dots\dots\dots\dots\dots\dots\\
x_{n-2} &= \alpha x_0 + (1-\alpha)Tx_{n-3},\\
x_{n-1}(x) := x_{n-1} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-2},
\end{aligned}$$

where $\alpha \in (0, 1)$. If we define mapping $F : C \to C$ such that $Fx = x_{n-1}(x)$, then F is s-lipschitzian with constant $s > 1$.

Proof. For $x, y \in C$ we have

$$\|Fx - Fy\| \leqslant \alpha k^n \|x - y\| + (1-\alpha)k\|x_{n-2} - y_{n-2}\|. \tag{3.2}$$

For the second component of the sum we have evaluation

$$\begin{aligned}
\|x_{n-2} - y_{n-2}\| &\leqslant \alpha\|x - y\| + (1-\alpha)k\|x_{n-3} - y_{n-3}\|\\
&\leqslant \big(\alpha + \alpha(1-\alpha)k + \alpha(1-\alpha)^2k^2 + \dots\\
&\quad + \alpha(1-\alpha)^{n-3}k^{n-3} + (1-\alpha)^{n-2}k^{n-2}\big)\|x - y\|.
\end{aligned} \tag{3.3}$$

Form inequalities (3.2) and (3.3) we obtain

$$\|Fx - Fy\| \leqslant \left(\alpha k^n + (1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{i=1}^{n-2}(1-\alpha)^i k^i\right)\|x - y\|.$$

Since $k > 1$, then for expression in brackets we get

$$\begin{aligned} s &= \alpha k^n + (1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{i=1}^{n-2}(1-\alpha)^i k^i \\ &> \alpha + (1-\alpha)^{n-1} + \alpha\sum_{i=1}^{n-2}(1-\alpha)^i \\ &= (1-\alpha)^{n-1} + \alpha\left(1 + (1-\alpha) + (1-\alpha)^2 + \cdots + (1-\alpha)^{n-2}\right) \\ &= (1-\alpha)^{n-1} + \alpha\frac{1-(1-\alpha)^{n-1}}{1-(1-\alpha)} = 1 \end{aligned} \tag{3.4}$$

which completes the proof. □

Lemma 3.3. *Let $n > 2$ be integer and let C be a nonempty, closed, convex subset of a Banach space X. Let a mapping $T\colon C \to C$ be k-lipschitzian with $k > 1$. For $x \in C$ we generate the sequence:*

$$\begin{aligned} x_0 &= x, \\ x_1 &= \alpha T^n x_0 + (1-\alpha)Tx_0, \\ x_2 &= \alpha T^n x_0 + (1-\alpha)Tx_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-2} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-3}, \\ x_{n-1}(x) := x_{n-1} &= \alpha T^n x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. If we define mapping $F : C \to C$ such that $Fx = x_{n-1}(x)$, then F is s-lipschitzian with constant $s > 1$.

Proof. For $x, y \in C$ we have

$$\|Fx - Fy\| \leqslant \alpha k^n\|x-y\| + (1-\alpha)k\|x_{n-2} - y_{n-2}\|. \tag{3.5}$$

For the second component of the sum we have evaluation

$$\begin{aligned} \|x_{n-2} - y_{n-2}\| &\leqslant \alpha k^n\|x-y\| + (1-\alpha)k\|x_{n-3} - y_{n-3}\| \\ &\leqslant \big(\alpha k^n + \alpha(1-\alpha)k^{n+1} + \alpha(1-\alpha)^2k^{n+2} + \dots \\ &\quad + \alpha(1-\alpha)^{n-3}k^{n+n-3} + (1-\alpha)^{n-2}k^{n-2}\big)\|x-y\|. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6) we obtain

$$\|Fx - Fy\| \leqslant \left(\alpha k^n + (1-\alpha)^{n-1}k^{n-1} + \alpha k^n\sum_{i=1}^{n-2}(1-\alpha)^i k^i\right)\|x-y\|,$$

that after estimating the expression in brackets in the same manner as in (3.4) completes the proof. □

Lemma 3.4. *Let $n > 2$ be integer and let C be a nonempty, closed, convex subset of a Banach space X. Let a mapping $T\colon C \to C$ be k-lipschitzian with $k > 1$. For $x \in C$*

we generate the sequence:

$$\begin{aligned} x_0 &= x, \\ x_1 &= \alpha x_0 + (1-\alpha)Tx_0, \\ x_2 &= \alpha x_0 + (1-\alpha)Tx_1, \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-2} &= \alpha x_0 + (1-\alpha)Tx_{n-3}, \\ x_{n-1}(x) := x_{n-1} &= \alpha x_0 + (1-\alpha)Tx_{n-2}, \end{aligned}$$

where $\alpha \in (0,1)$. If we define mapping $F : C \to C$ such that $Fx = x_{n-1}(x)$, then F is s-lipschitzian with constant $s > 1$.

Proof. For $x, y \in C$ we have

$$\|Fx - Fy\| \leqslant \alpha\|x-y\| + (1-\alpha)k\|x_{n-2} - y_{n-2}\|. \tag{3.7}$$

By expressions (3.7) and (3.3) we get

$$\|Fx - Fy\| \leqslant \left((1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{i=0}^{n-2}(1-\alpha)^i k^i\right)\|x-y\|.$$

Since $k > 1$, then for expression in brackets we get evaluation

$$\begin{aligned} s &= (1-\alpha)^{n-1}k^{n-1} + \alpha\sum_{i=0}^{n-2}(1-\alpha)^i k^i \\ &> (1-\alpha)^{n-1} + \alpha\sum_{i=0}^{n-2}(1-\alpha)^i \\ &= (1-\alpha)^{n-1} + \alpha\left(1 + (1-\alpha) + (1-\alpha)^2 + \cdots + (1-\alpha)^{n-2}\right) \\ &= (1-\alpha)^{n-1} + \alpha\frac{1-(1-\alpha)^{n-1}}{1-(1-\alpha)} = 1 \end{aligned} \tag{3.8}$$

which completes the proof. □

Now, based on the Lemma 3.1, using the Theorem 2.3, Lemma 3.2, Lemma 3.3, Lemma 3.4 and inequalities (2.7) and (2.9), (2.15) and (2.17), (2.23) and (2.25) we get the following conclusion:

Corollary 3.5. *Let $n > 2$ be integer and let C be a nonempty, closed, convex and bounded subset of a Banach space X. Consider the sequence $\{z_p\}$ generated as follows*

$$\begin{aligned} z_1(x) &= x_{n-1}(x), \\ z_2(x) &= x_{n-1}(z_1(x)), \\ &\dots\dots\dots \\ z_p(x) &= x_{n-1}(z_{p-1}(x)), \dots, \end{aligned}$$

where $x_{n-1}(x) := x_{n-1}$ is defined in the same manner like in respect cases of the proof of the Theorem 2.3. If the mapping $T\colon C \to C$ is firmly k-lipschitzian ($k > 1$) and (a,n)-rotative with

$$k < \max\left\{\bar{\gamma}_n^1(a), \bar{\gamma}_n^2(a), \bar{\gamma}_n^3(a)\right\},$$

then a mapping $R\colon C \to C$ *defined by*

$$R(x) = \lim_{p\to\infty} z_p(x)$$

is Hölder continuous retraction from C *to* $\mathrm{Fix}(T)$.

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