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MULTIPLE SOLUTIONS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS VIA VARIATIONAL METHODS

NEMAT NYAMORADI* AND YONG ZHOU**

*Department of Mathematics, Faculty of Sciences
Razi University, 67149 Kermanshah, Iran
E-mail: nyamoradi@razi.ac.ir; neamat80@yahoo.com

**School of Mathematics and Computational Science
Xiangtan University, Hunan 411105, PR China
E-mail: yzhou@xtu.edu.cn

Abstract. In this paper, we prove the existence and multiplicity of (weak) solutions for the following fractional boundary value problem:

$$\begin{cases} -\frac{d}{dt}\Big(p(t)\Big(\frac{1}{2}{}_0D_t^{-\zeta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\zeta}(u'(t))\Big)\Big) \\ \qquad +r(t)\Big(\frac{1}{2}{}_0D_t^{-\zeta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\zeta}(u'(t))\Big) + q(t)u(t) = f(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where ${}_0D_t^{-\zeta}$ and ${}_tD_T^{-\zeta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \zeta < 1$ respectively, $L(t) := \int_0^t (r(s)/p(s))ds$, $0 < m \leq e^{-L(t)}p(t) \leq M$ and $q(t) - p(t) \geq 0$ where $t \in [0,T]$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$. Our approach is based on variational methods.

Key Words and Phrases: Variational methods, fractional differential equations, Palais-Smale condition, Riemann-Liouville fractional integrals.
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**Corresponding author.

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