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MULTIPLE SOLUTIONS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEMS VIA VARIATIONAL METHODS

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Abstract. In this paper, we prove the existence and multiplicity of (weak) solutions for the following fractional boundary value problem:

$$\begin{cases} -\frac{d}{dt}\Big(p(t)\Big(\frac{1}{2}{}_0D_t^{-\zeta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\zeta}(u'(t))\Big)\Big) \\ \qquad +r(t)\Big(\frac{1}{2}{}_0D_t^{-\zeta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\zeta}(u'(t))\Big) + q(t)u(t) = f(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where ${}_0D_t^{-\zeta}$ and ${}_tD_T^{-\zeta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \zeta < 1$ respectively, $L(t) := \int_0^t (r(s)/p(s))ds$, $0 < m \leq e^{-L(t)}p(t) \leq M$ and $q(t) - p(t) \geq 0$ where $t \in [0,T]$, $f \in C([0,T]\times\mathbb{R},\mathbb{R})$. Our approach is based on variational methods.

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1. Introduction

The aim of this paper is to establish the existence of infinitely many solutions for the following fractional boundary value problem

$$\begin{cases} -\frac{d}{dt}\Big(p(t)\Big(\frac{1}{2}{}_0D_t^{-\zeta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\zeta}(u'(t))\Big)\Big) \\ \quad +r(t)\Big(\frac{1}{2}{}_0D_t^{-\zeta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\zeta}(u'(t))\Big) + q(t)u(t) = f(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where ${}_0D_t^{-\zeta}$ and ${}_tD_T^{-\zeta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \zeta < 1$ respectively, $L(t) := \int_0^t (r(s)/p(s))ds$, $0 < m \leq e^{-L(t)}p(t) \leq M$ and $q(t) - p(t) \geq 0$ where $t \in [0,T]$, $f \in C([0,T]\times\mathbb{R},\mathbb{R})$.

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Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [3, 7, 10, 12, 13, 15, 17, 18, 20, 21] and the references therein.

For a thorough account on the subject, we refer to [1, 2, 4, 6, 8, 9, 22, 23] and the references therein.

In particular, if $\zeta = 0$ then problem (1.1) reduces to the standard second order boundary value problem of the following form

$$\begin{cases} (p(t)u'(t)) + r(t)u'(t) + q(t)u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \tag{1.2}$$

In the recent years, the existence and multiplicity of solutions for the similar second order boundary value problem (1.2) in the cases $p(t) \neq 1$ and $p(t) \equiv 1$ without or with impulses have been extensively studied via variational methods in many papers (e.g. [5, 16, 24, 25]).

In this paper we use critical point theory and variational methods to investigate the multiple solutions of (1.1).

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Section 3 is devoted to our results on existence and multiplicity of solutions.

2. Preliminaries and reminder about fractional calculus

In this section, we present some preliminaries and lemmas that are useful to the proof to the main results. For the convenience of the reader, we also present here the necessary definitions from fractional calculus theory. We refer the reader to [8, 11, 17] or other texts on basic fractional calculus.

Definition 2.1. (Left and Right Riemann-Liouville Fractional Integrals [11, 17]). *Let f be a function defined on $[a, b]$. The left and right Riemann-Liouville fractional integrals of order γ for function f denoted by ${}_aD_t^{-\gamma} f(t)$ and ${}_tD_b^{-\gamma} f(t)$, respectively, are defined by*

$$\begin{aligned} {}_aD_t^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, \quad t \in [a, b], \quad \gamma > 0, \\ {}_tD_b^{-\gamma} f(t) &= \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s) ds, \quad t \in [a, b], \quad \gamma > 0, \end{aligned}$$

provided the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma > 0$ is the Gamma function.

Definition 2.2. *For $n \in \mathbb{N}$, if $\gamma = n$, Definition 2.1 coincides with nth integrals of the form* [11, 17]

$$
\begin{aligned}
{}_aD_t^{-n}f(t) &= \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}f(s)ds, \quad t\in[a,b], \ \ n\in\mathbb{N},\\
{}_tD_b^{-n}f(t) &= \frac{1}{(n-1)!}\int_t^b (s-t)^{n-1}f(s)ds, \quad t\in[a,b], \ \ n\in\mathbb{N}.
\end{aligned}
$$

Definition 2.3. (Left and Right Riemann-Liouville Fractional Derivatives [11, 17]). *Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order γ for function f denoted by ${}_aD_t^{\gamma}f(t)$ and ${}_tD_b^{\gamma}f(t)$, respectively, are defined by*

$$
\begin{aligned}
{}_aD_t^{\gamma}f(t) &= \frac{d^n}{dt^n}{}_aD_t^{\gamma-n}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\Big(\int_a^t (t-s)^{n-\gamma-1}f(s)ds\Big),\\
{}_tD_b^{\gamma}f(t) &= (-1)^n\frac{d^n}{dt^n}{}_tD_b^{\gamma-n}f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\Big(\int_t^b (s-t)^{n-\gamma-1}f(s)ds\Big),
\end{aligned}
$$

where $t\in[a,b]$, $n-1\le\gamma<n$ and $n\in\mathbb{N}$. In particular, if $0\le\gamma<1$, then

$$
{}_aD_t^{\gamma}f(t) = \frac{d}{dt}{}_aD_t^{\gamma-1}f(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\Big(\int_a^t (t-s)^{-\gamma}f(s)ds\Big), \quad t\in[a,b], \tag{2.1}
$$

$$
{}_tD_b^{\gamma}f(t) = -\frac{d}{dt}{}_tD_b^{\gamma-1}f(t) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\Big(\int_t^b (s-t)^{-\gamma}f(s)ds\Big), \quad t\in[a,b]. \tag{2.2}
$$

Remark 2.1. *If $f \in C([a,b],\mathbb{R}^N)$, it is obvious that Riemann-Liouville fractional integral of order $\gamma>0$ exists on $[a,b]$. On the other hand, following (see* [11]*, Lemma 2.2, pp. 73), we know that the Riemann-Liouville fractional derivative of order $\gamma\in[n-1,n)$ exists a.e. on $[a,b]$ if $f\in AC^n([a,b],\mathbb{R}^N)$, where $C^k([a,b],\mathbb{R}^N)(k=0,1,\dots)$ denotes the set of mappings having k times continuously differentiable on $[a,b]$, $AC([a,b],\mathbb{R}^N)$ is the space of functions which are absolutely continuous on $[a,b]$ and $AC^k([a,b],\mathbb{R}^N)(k=0,1,\dots)$ is the space of functions f such that $f\in C^{k-1}([a,b],\mathbb{R}^N)$ and $f^{k-1}\in AC([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N)=AC^1([a,b],\mathbb{R}^N)$. The left and right Caputo fractional derivatives are defined via the above Riemann-Liouville fractional derivatives (see* [11]*, pp. 91). In particular, they are defined for the function belonging to the space of absolutely continuous functions.*

Definition 2.4. (Left and Right Caputo Fractional Derivatives [11]). *Let $\gamma\ge 0$ and $n\in\mathbb{N}$.*
(i) If $\gamma\in(n-1,n)$ and $f\in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function f denoted by ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\gamma}f(t)$ and ${}_t^cD_b^{\gamma}f(t)$ are represented by

$$
\begin{aligned}
{}_a^cD_t^{\gamma}f(t) &= {}_aD_t^{\gamma-n}f^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)}\Big(\int_a^t (t-s)^{n-\gamma-1}f^{(n)}(s)ds\Big),\\
{}_t^cD_b^{\gamma}f(t) &= (-1)^n{}_tD_b^{\gamma-n}f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\Big(\int_t^b (s-t)^{n-\gamma-1}f^{(n)}(s)ds\Big),
\end{aligned}
$$

respectively, where $t \in [a,b]$. In particular, if $0 < \gamma < 1$, then

$$ {}_a^cD_t^{\gamma} f(t) = {}_aD_t^{\gamma-1} f'(t) = \frac{1}{\Gamma(1-\gamma)}\Big(\int_a^t (t-s)^{-\gamma} f'(s)ds\Big), \quad t \in [a,b], \tag{2.3}$$

$$ {}_t^cD_b^{\gamma} f(t) = -{}_tD_b^{\gamma-1} f'(t) = -\frac{1}{\Gamma(1-\gamma)}\Big(\int_t^b (s-t)^{-\gamma} f'(s)ds\Big), \quad t \in [a,b]. \tag{2.4}$$

(ii) If $\gamma = n-1$ and $f \in AC^n([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{n-1}f(t)$ and ${}_t^cD_b^{n-1}f(t)$ are represented by

$$\begin{aligned} {}_a^cD_t^{n-1}f(t) &= f^{(n-1)}(t), \quad t \in [a,b], \\ {}_t^cD_b^{n-1}f(t) &= (-1)^{(n-1)} f^{(n-1)}(t), \quad t \in [a,b]. \end{aligned}$$

In particular, ${}_a^cD_t^0 f(t) = {}_t^cD_b^0 f(t) = f(t)$, $t \in [a,b]$.

The first result yields the semigroup property of the Riemann-Liouville fractional integral operators.

Lemma 2.1. (See [11]). *The left and right Riemann-Liouville fractional integral operators have the property of a semigroup, i.e.*

$$\begin{aligned} &{}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2} f(t)) = {}_aD_t^{-\gamma_1-\gamma_2} f(t), \\ &{}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2} f(t)) = {}_tD_b^{-\gamma_1-\gamma_2} f(t), \quad \forall\, \gamma_1, \gamma_2 > 0, \end{aligned}$$

in any point $t \in [a,b]$ for continuous function f and for almost every point in $[a,b]$ if the function $f \in L^1([a,b],\mathbb{R}^N)$.

Let us recall that for any fixed $t \in [0,T]$ and $1 \le r < \infty$,

$$||u||_{L^r([0,t])} = \Big(\int_0^t |u(\xi)|^r d\xi\Big)^{\frac{1}{r}}, \quad ||u||_{L^r} = \Big(\int_0^T |u(\xi)|^r d\xi\Big)^{\frac{1}{r}},$$

and

$$||u||_\infty = \max_{t\in[0,T]} |u(t)|.$$

Lemma 2.2. (See [8]). *Let $0 < \alpha \le 1$ and $1 \le r < \infty$. For any $f \in L^r([a,b],\mathbb{R}^N)$, we have*

$$||{}_0D_\xi^{-\alpha} f||_{L^r([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)}||f||_{L^r([0,t])}, \quad \textit{for } \xi \in [0,t], \quad t \in [0,T].$$

Now, by Lemma 2.2, for any $h \in C_0^\infty([0,T],\mathbb{R}^N)$ and $1 < r < \infty$, we have $h \in L^r([0,T],\mathbb{R}^N)$ and ${}_0^cD_t^\alpha h \in L^r([0,T],\mathbb{R}^N)$. Thus, one can construct a set of space $E_0^{\alpha,p}$, which depend on L^r-integrability of the Caputo fractional derivative of a function.

Definition 2.5. *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm*

$$||u||_{\alpha,p} = \Big(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\Big)^{\frac{1}{p}}. \tag{2.5}$$

Remark 2.2. *(i) It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^p([0,T],\mathbb{R}^N)$ and $u(0)=u(T)=0$.*
(ii) For any $u \in E_0^{\alpha,p}$, noting the fact that $u(0)=0$, we have ${}_0^cD_t^\alpha u = {}_0D_t^\alpha u$, $t\in[0,T]$ according to (2.3).

Lemma 2.3. ([8]). *Let $0<\alpha\leq 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

The following lemma is well known, where we have employed the equivalent norm in $E_0^{\alpha,p}$(see [8]).

Lemma 2.4. ([8]). *Let $0<\alpha\leq 1$ and $1<p<\infty$. For all $u\in E_0^{\alpha,p}$, we have*

$$||u||_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}||{}_0^cD_t^\alpha u||_{L^p}. \tag{2.6}$$

Moreover, if $\alpha>\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$||u||_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}||{}_0^cD_t^\alpha u||_{L^p}. \tag{2.7}$$

Now, we will establish a variational structure which enables us to reduce the existence of solutions of problem (1.1) to the one of finding critical points of corresponding functional defined on the space $E_0^{\alpha,2}$ with $\frac{1}{2}<\alpha\leq 1$. Let $L(t):=\int_0^t(r(s)/p(s))ds$, $0<m\leq e^{-L(t)}p(t)\leq M$ and $q(t)-p(t)\geq 0$ where $t\in[0,T]$. One can transform the problem (1.1) into the following equivalent form:

$$\begin{cases} -\frac{d}{dt}\Big(e^{-L(t)}p(t)\Big(\frac{1}{2}{}_0D_t^{-\zeta}(u'(t))+\frac{1}{2}{}_tD_T^{-\zeta}(u'(t))\Big)\Big)+e^{-L(t)}q(t)u(t) \\ \qquad = e^{-L(t)}f(t,u(t)), \text{ a.e. } t\in[0,T], \\ u(0)=u(T)=0, \end{cases} \tag{2.8}$$

Then, by Lemma 2.1, for every $u\in AC([0,T],\mathbb{R})$, problem (2.8) transforms to

$$\begin{cases} -\frac{d}{dt}\Big(e^{-L(t)}p(t)\Big(\frac{1}{2}{}_0D_t^{-\frac{\zeta}{2}}\Big({}_0D_t^{-\frac{\zeta}{2}}u'(t)\Big)+\frac{1}{2}{}_tD_T^{-\frac{\zeta}{2}}\Big({}_tD_T^{-\frac{\zeta}{2}}u'(t)\Big)\Big)\Big) \\ \qquad +e^{-L(t)}q(t)u(t)=e^{-L(t)}f(t,u(t)), \\ u(0)=u(T)=0, \end{cases} \tag{2.9}$$

for almost every $t\in[0,T]$, where $\zeta\in[0,1)$.

Furthermore, in view of Definition 2.4, it is obvious that $u\in AC([0,T],\mathbb{R})$ is a solution of problem (2.9) if and only if u is a solution of the following problem

$$\begin{cases} -\frac{d}{dt}\Big(e^{-L(t)}p(t)\Big(\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t))-\frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big)\Big) \\ \qquad +e^{-L(t)}q(t)u(t)=e^{-L(t)}f(t,u(t)), \\ u(0)=u(T)=0, \end{cases} \tag{2.10}$$

for almost every $t\in[0,T]$, where $\alpha=1-\frac{\zeta}{2}\in(\frac{1}{2},1]$. Therefore, we seek a solution u of problem (2.10) which, of course, corresponds to the solutions u of problem (1.1) provided that $u\in AC([0,T],\mathbb{R})$.

Let us denote by

$$D^{\alpha}(u(t)) = \Big(e^{-L(t)}p(t)\Big(\frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big)\Big). \tag{2.11}$$

We are now in a position to give a definition of the solution of problem (2.10).

Definition 2.6. *A function $u \in AC([0,T],\mathbb{R})$ is called a solution of problem (2.10) if*
(i) $D^{\alpha}(u(t))$ is differentiable for almost every $t \in [0,T]$, and
(ii) u satisfies (2.10).

In the proof of our main results, we first present an important definition and some lemmas.

Definition 2.7. *An operator $A : X \to X^*$ is of type $(S)_+$ if, for any sequence $\{u_n\}$ in X, $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0$ imply $u_n \to u$.*

Lemma 2.5. *(Mountain Pass Theorem in* [14]*). Let $\varphi \in C^1(X,\mathbb{R})$. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 such that u_1 is not in Ω and $\inf\limits_{v\in\partial\Omega}\varphi(v) > \max\{\varphi(u_0), \varphi(u_1)\}$. Then there exists a critical point u of φ, i.e., $\varphi'(u) = 0$, with $\varphi(u) > \max\{\varphi(u_0), \varphi(u_1)\}$.*

Note that if either u_0 or u_1 is a critical point of φ then we obtain the existence of at least two critical points for φ.

Lemma 2.6. *(Theorem 38.A in* [26]*). For the functional $F : M \subseteq X \to \mathbb{R}$ with M not empty, $\min_{u\in M} F(u) = a$ has a solution in case the following hold:*
(i) X is a real reflexive Banach space;
(ii) M is bounded and weak sequentially closed;
(iii) F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_k\}$ in M such that $u_k \rightharpoonup u$ as $k \to \infty$, we have $F(u) \le \liminf\limits_{k\to\infty} F(u_k)$.

Lemma 2.7. *(Theorem 9.12 in* [19]*). Let E be an infinite dimensional real Banach space and $u \in C^1(E,\mathbb{R})$ be even, satisfying the Palais-smale condition and $\varphi(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and φ satisfies the following conditions:*
(i) There exist constants $\rho, \sigma > 0$ such that $\varphi|_{\partial B_\rho \bigcap X} \ge \sigma$;
(ii) For each finite dimensional subspace $V_1 \subset E$, there is an $R = R(V_1)$ such that $\varphi(u) \le 0$ for every $u \in V_1$ with $||u|| > R$.
Then φ has an unbounded sequence of critical values.

In what follows, we will treat problem (2.10) in the Hilbert space $E^{\alpha} = E_0^{\alpha,2}$ with the corresponding norm

$$||u|| = \Big(\int_0^T e^{-L(t)}p(t)\Big(|{}_0^cD_t^{\alpha}u(t)|^2 + |u(t)|^2\Big)dt\Big)^{\frac{1}{2}}. \tag{2.12}$$

Also, we define $||u||_{\alpha} = ||u||_{\alpha,2}$ which we defined in (2.5).
The following estimate is useful for our further discussion.

Observe that

$$m\int_0^T |{}_0^cD_t^\alpha u(t)|^2dt \le \int_0^T e^{-L(t)}p(t)|{}_0^cD_t^\alpha u(t)|^2dt \le M\int_0^T |{}_0^cD_t^\alpha u(t)|^2dt, \tag{2.13}$$

so,

$$\begin{aligned}||u||_{L^2} &\le \frac{T^\alpha}{\Gamma(\alpha+1)}||{}_0^cD_t^\alpha u||_{L^2} = \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2dt\Big)^{\frac{1}{2}}\\ &\le \frac{T^\alpha}{\sqrt{m}\Gamma(\alpha+1)}\Big(\int_0^T e^{-L(t)}p(t)|{}_0^cD_t^\alpha u(t)|^2dt\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.14}$$

Thus, for $p=2$, $\frac{1}{2}<\alpha\le 1$, by (2.7), (2.13) and by inequality $x^{1/p}\le (x+y)^{1/p}$ for all $x,y\ge 0$, we have

$$||u||_\infty \le \frac{\sqrt{2}T^{\frac{2\alpha-1}{2}}}{\sqrt{m}\Gamma(\alpha)(\alpha+1)^{\frac{1}{2}}}||u||. \tag{2.15}$$

The following estimate is useful for our further discussion.

Lemma 2.8. *([8]). If $\frac{1}{2}<\alpha\le 1$, then for every $u\in E^\alpha$, we have*

$$|\cos(\pi\alpha)|||u||_\alpha^2 \le -\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt \le \frac{1}{|\cos(\pi\alpha)|}||u||_\alpha^2. \tag{2.16}$$

Then, according to Lemma 2.8, (2.13) and similar to Proof of Proposition 4.1 in [8], we can get

Remark 2.3. *If $\frac{1}{2}<\alpha\le 1$, then for every $u\in E^\alpha$, we have*

$$\begin{aligned}|\cos(\pi\alpha)|||u||^2 &\le -\int_0^T e^{-L(t)}p(t)({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt + \int_0^T e^{-L(t)}p(t)(u(t),u(t))dt\\ &\le \max\left\{\frac{M}{m|\cos(\pi\alpha)|}, 1\right\}||u||^2.\end{aligned} \tag{2.17}$$

3. Main result

We mean by a (weak) solution of problem (2.10) which, of course, corresponds to the solution of problem (1.1), any $u\in E^\alpha$ such that

$$\int_0^T e^{-L(t)}\Big[-\frac{1}{2}p(t)\Big(({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t))\Big) + p(t)(u(t),v(t))$$

$$+(q(t)-p(t))(u(t),v(t)) - f(t,u(t))v(t)\Big]dt = 0,$$

for every $v\in E^\alpha$.

Let $J : E^\alpha \to \mathbb{R}$ be defined by

$$J(u) = \int_0^T e^{-L(t)} \Big[\frac{1}{2} p(t) \Big(-({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) + p(t)|u(t)|^2 \Big) + \frac{1}{2}(q(t) - p(t))|u(t)|^2 - F(t, u(t)) \Big] dt, \quad \text{for all } u \in E^\alpha, \tag{3.1}$$

where $F(t,u) = \int_0^u f(t,\xi)d\xi$. Clearly J is continuously differentiable on E^α, and for every $u, v \in E^\alpha$, we have

$$\langle J'(u), v\rangle = \int_0^T e^{-L(t)} \Big[-\frac{1}{2} p(t) \Big(({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t)) \Big) + p(t)(u(t), v(t)) + (q(t) - p(t))(u(t), v(t)) - f(t, u(t))v(t) \Big] dt. \tag{3.2}$$

Thus, a critical point of $J(u)$, defined by (3.1), gives us a weak solution of problem (2.10) which, of course, corresponds to the solution of problem (1.1).

Let

$$\langle Au, v\rangle := \int_0^T e^{-L(t)} \Big[-\frac{1}{2} p(t) \Big(({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t)) \Big) + p(t)(u(t), v(t)) + (q(t) - p(t))(u(t), v(t)) \Big] dt.$$

Lemma 3.1. *There exist constants $\gamma_2 > \gamma_1 > 0$ such that*

$$\gamma_1 ||u||^2 \le \langle Au, u\rangle \le \gamma_2 ||u||^2, \quad u \in E^\alpha. \tag{3.3}$$

Proof. By (2.17) we have

$$\begin{aligned}\langle Au, u\rangle &= \int_0^T e^{-L(t)} \Big[-p(t) \Big(({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) \Big) + p(t)|u(t)|^2 + (q(t) - p(t))|u(t)|^2 \Big] dt \\ &\ge \int_0^T e^{-L(t)} \Big[-p(t) \Big(({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) \Big) + p(t)|u(t)|^2 \Big] dt \ge |\cos(\pi\alpha)|||u||^2.\end{aligned}$$

On the other hand, since $p(t), q(t)$ are continuous in $[0, T]$, then there exists a constant $c_0 > 0$ such that $q(t) - p(t) < c_0$. Thus by (2.15) and (2.17), one can get

$$\langle Au, u\rangle = \int_0^T e^{-L(t)} \Big[-p(t) \Big(({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) \Big) + p(t)|u(t)|^2 + (q(t) - p(t))|u(t)|^2 \Big] dt$$

$$\le \max \left\{ \frac{M}{m|\cos(\pi\alpha)|}, 1 \right\} ||u||^2 + c_0||u||^2 = \left(\max \left\{ \frac{M}{m|\cos(\pi\alpha)|}, 1 \right\} + c_0 \right) ||u||^2. \quad \square$$

Lemma 3.2. *The functional J defined by (3.1) is continuous and weakly lower semi-continuous. Moreover, it satisfies the Palais-Smale condition, if the following condition holds:*

(H1) for all $u \in E^\alpha$, $\mu F(t, u) \le uf(t, u)$, where $\mu > \frac{2\gamma_2}{\gamma_1}$.

Proof. With continuity of f, it is well known that J and J' are continuous functionals and J is differential functional. We claim that J is weakly lower semi-continuous. To this end, let u_n weakly convergent to $u \in E^\alpha$. Then $||u|| \le \liminf_{n\to\infty} ||u_n||$ and $u_n \to u$ uniformly in $C([0,T],\mathbb{R})$. Thus, when $n \to \infty$, we get

$$\int_0^T e^{-L(t)}\Big[\frac{1}{2}(q(t)-p(t))|u_n(t)|^2 - F(t,u_n(t))\Big]dt$$
$$\to \int_0^T e^{-L(t)}\Big[\frac{1}{2}(q(t)-p(t))|u(t)|^2 - F(t,u(t))\Big]dt.$$

So, we conclude that $J(u) \le \liminf_{n\to\infty} J(u_n)$. Then, J is weakly lower semi-continuous.

Now, we claim that J satisfies the Palais-Smale condition. To this end, let $\{J(u_n)\}$ be a bounded sequence such that $\lim_{n\to\infty} J'(u_n) = 0$. We show that $||u_n||$ is bounded. By (3.2), one can get

$$\begin{aligned}\int_0^T e^{-L(t)}p(t)f(t,u_n(t))u_n(t)dt = \int_0^T e^{-L(t)}\Big[&-p(t)\Big(({}_0^cD_t^\alpha u_n(t), {}_t^cD_T^\alpha u_n(t))\Big)\\ &+p(t)(u_n(t),u_n(t)) + (q(t)-p(t))(u_n(t),u_n(t))\Big]dt\\ &-\langle J'(u_n), u_n\rangle. \end{aligned}\tag{3.4}$$

From (3.3), (3.4), Lemma 3.1 and the condition (H1), we have

$$\begin{aligned} J(u_n) &\ge \frac{\gamma_1}{2}||u_n||^2 - \int_0^T e^{-L(t)}F(t,u_n(t))dt\\ &\ge \frac{\gamma_1}{2}||u_n||^2 - \frac{1}{\mu}\int_0^T e^{-L(t)}f(t,u_n(t))u_n(t)dt\\ &\ge \left(\frac{\gamma_1}{2}-\frac{\gamma_2}{\mu}\right)||u_n||^2 + \frac{1}{\mu}\langle J'(u_n),u_n\rangle\\ &\ge \left(\frac{\gamma_1}{2}-\frac{\gamma_2}{\mu}\right)||u_n||^2 - \frac{1}{\mu}||J'(u_n)||||u_n||. \end{aligned}\tag{3.5}$$

Since $J(u_n)$ is bounded, by (3.5) we get $||u_n||$ is bounded.

Since E^α is a reflexive Banach space and so by passing to a subsequence (for simplicity denoted gain by $\{u_n\}$) if necessary, by (2.17) of Lemma 2.4, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } E^\alpha,\\ u_n \to u, & \text{a.e. in } C([0,T],\mathbb{R}). \end{cases}\tag{3.6}$$

Therefore

$$\langle J'(u_n), u_n - u\rangle \to 0 \text{ and } \int_0^T f(t,u_n(t))(u_n(t)-u(t))dt \to 0,$$

so we get

$$\varepsilon_n||u_n - u|| \ge \langle J'(u_n), u_n-u\rangle = \langle Au_n, u_n-u\rangle - \int_0^T f(t,u_n(t))(u_n(t)-u(t))dt$$

with $\varepsilon_n \to 0$. Thus $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$. By (3.6), it is easy to get $\lim_{n\to\infty}\langle Au, u_n - u\rangle = 0$. Therefore

$$\limsup_{n\to\infty}\langle Au_n - Au, u_n - u\rangle \le \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle - \liminf_{n\to\infty}\langle Au, u_n - u\rangle \le 0.$$

Since A is of type $(S)_+$ (The proof is similar to Theorem 5.1 in [23] and is omitted), so we obtain $u_n \to u$ in E^a. Then, J satisfies the Palais-Smale condition. □

For convenience an simplicity in the following discussion, we use the notations:

$$F^0(t,u) = \limsup_{u\to 0}\frac{e^{-L(t)}F(t,u)}{|u|^\mu}, \qquad F_\infty(t,u) = \liminf_{u\to\infty}\frac{e^{-L(t)}F(t,u)}{|u|^\mu}.$$

Theorem 3.1. *Assume that $\frac{1}{2} < \alpha \le 1$ and (H1) hold. Then the Problem (1.1) has at least two solutions if the following condition hold:*

(H2) There exist $\varrho, \xi > 0$ such that $F^0(t,u) \le \varrho$ and $F_\infty(t,u) \ge \xi$.

Proof. In order to apply Lemma 2.6 to our problem, let $X := E^\alpha$ and we apply this Lemma to show that there exists T_0 such that the functional J has a local minimum $u_0 \in B_{T_0} = \{u \in E^\alpha :\ ||u|| < T_0\}$.

For every $T_0 > 0$, since E^α is a Hilbert space, it is easy to deduce that $\overline{B}_{T_0}$ is a bounded and weak sequentially closed. Lemma 3.2 has shown that J is weak sequentially lower semi-continuous on $\overline{B}_{T_0}$. Besides, E^α is a reflexive Banach space, so by Lemma 2.6 we can have this u_0 such that $J(u_0) = \min_{u\in\overline{B}_{T_0}} J(u)$. Now we will show that $J(u_0) < \inf_{u\in\partial B_{T_0}} J(u)$ for some $T_0 = T_1$. Now, by (H2) we can choose $T_1, \epsilon > 0$ satisfying

$$\begin{aligned} &e^{-L(t)}F(t,u) \le \varrho|u|^\mu, \quad \text{for all } ||u|| \le T_1,\\ &\frac{\gamma_1}{2}T_1^2 - \varrho T\left(\frac{T_1\sqrt{2}T^{\frac{2\alpha-1}{2}}}{\sqrt{m}\Gamma(\alpha)(\alpha+1)^{\frac{1}{2}}}\right)^\mu > \epsilon. \end{aligned}$$

For every $u \in \partial B_{T_1}$, $||u|| = T_1$, by (2.15), we get

$$\begin{aligned} J(u) &= \int_0^T e^{-L(t)}\Big[\frac{1}{2}p(t)\Big(-({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) + p(t)|u(t)|^2\Big)\\ &\qquad + \frac{1}{2}(q(t)-p(t))|u(t)|^2 - F(t,u(t))\Big]dt\\ &\ge \frac{\gamma_1}{2}||u||^2 - \int_0^T e^{-L(t)}F(t,u(t))dt \ge \frac{\gamma_1}{2}||u||^2 - \varrho\int_0^T |u(t)|^\mu dt\\ &\ge \frac{\gamma_1}{2}||u||^2 - \varrho T||u||_\infty^\mu \ge \frac{\gamma_1}{2}||u||^2 - \varrho T\left(\frac{\sqrt{2}T^{\frac{2\alpha-1}{2}}}{\sqrt{m}\Gamma(\alpha)(\alpha+1)^{\frac{1}{2}}}\right)^\mu ||u||^\mu\\ &\ge \frac{\gamma_1}{2}T_1^2 - \varrho T\left(\frac{T_1\sqrt{2}T^{\frac{2\alpha-1}{2}}}{\sqrt{m}\Gamma(\alpha)(\alpha+1)^{\frac{1}{2}}}\right)^\mu > \epsilon. \end{aligned}$$

Thus $J(u) > \epsilon$ for every $u \in \partial B_{T_1}$. Moreover $J(u_0) \le J(0) = 0$. Then $J(u) > \epsilon > J(0) \ge J(u_0)$ for every $u \in \partial B_{T_1}$. So $J(u_0) < \inf_{u\in\partial B_{T_1}} J(u)$. Therefore, J has a local minimum $u_0 \in B_{T_1}$.

Next, we will show that there exists u_1 with $||u_1|| > T_1$ such that

$$J(u_1) < \inf_{u\in\partial B_{T_1}} J(u).$$

For the above T_1, by (H2), we can choose a sufficiently large T_2 such that For all $||u|| \geq T_2 > T_1$ satisfying $e^{-L(t)}F(t,u) \geq \xi|u|^{\mu}$.

Then, for every $||u|| \geq T_2 > T_1$ and $k > 0$, by (2.15), one can get

$$\begin{aligned}J(ku) = &\int_0^T e^{-L(t)}\Big[\frac{1}{2}p(t)\Big(-({}_0^cD_t^{\alpha}ku(t), {}_t^cD_T^{\alpha}ku(t)) + p(t)|ku(t)|^2\Big)\\ &+ \frac{1}{2}(q(t)-p(t))|ku(t)|^2 - F(t,ku(t))\Big]dt \leq \frac{\gamma_2}{2}k^2||u||^2 - \int_0^T e^{-L(t)}F(t,ku(t))dt\\ &\leq \frac{\gamma_1}{2}k^2||u||^2 - \xi\int_0^T |ku(t)|^{\mu}dt \leq \frac{\gamma_1}{2}k^2||u||^2 - \varrho k^{\mu}||u||_{L^{\mu}}^{\mu} \to -\infty\end{aligned}$$

as $k \to +\infty$. Then there exists a sufficiently large k_0 such that $J(k_0u) < 0$. Thus, we can choose $u_1 = k_0u$ with $||u_1|| \geq T_2$ sufficiently large such that $J(u_1) < 0$. Then we have $\max\{J(u_0), J(u_1)\} < \inf_{u\in\partial B_{T_1}} J(u)$. Also Lemma 3.2 has shown that J satisfies Palais-smale condition. So, by Lemma 2.5 there exists a critical point u^*. Therefore, u_0 and u^* are two critical points of J, and they are solutions of (2.10) which, of course, corresponds to the solution of problem (1.1). □

Theorem 3.2. *Assume that $\frac{1}{2} < \alpha \leq 1$, (H1) and (H2) hold. Moreover, $f(t,u)$ is odd about u. Then the problem (1.1) has infinitely many solutions.*

Proof. Our aim is to apply Lemma 2.7 to our problem. Firstly, J is even, since $f(t,u)$ is odd about u. Moreover, by Lemma 3.2, we know that $J \in C^1(E^{\alpha}, \mathbb{R})$, $J(0) = 0$, and J satisfies the Palais-smale condition. In the same way as in Theorem 3.1, we can easily verify that the conditions (i) and (ii) of Lemma 2.7 are satisfied. According to Lemma 2.7, J possesses infinitely many critical points, i.e. the problem (2.10) has infinitely many classical solutions which, of course, corresponds to the solution of problem (1.1). □

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