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FIXED POINT PROPERTIES RELATED TO CHARACTER AMENABLE BANACH ALGEBRAS

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Abstract. For a Banach algebra $\mathfrak{A}$, we investigate several fixed point properties for $\mathfrak{A}$ with respect to a non-zero character ϕ on $\mathfrak{A}$. As the main results, we obtain some fixed point characterizations for ϕ-amenability of $\mathfrak{A}$. We also describe ϕ-amenability of $\mathfrak{A}$ in terms of the Hahn-Banach extension property. As a consequence, we offer some applications to the group algebra and the Fourier algebra of a locally compact group.
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1. Introduction

Let $\mathfrak{A}$ be a Banach algebra. We denote by $\Delta(\mathfrak{A})$ the set of all non-zero multiplicative linear functionals on $\mathfrak{A}$. For $\phi \in \Delta(\mathfrak{A})$, Kaniuth, Lau and Pym [7, 8] introduced and investigated a notion of amenability for Banach algebras called *ϕ-amenability*; see also Monfared [18]. In fact, $\mathfrak{A}$ is said to be *ϕ-amenable* if there exists $m \in \mathfrak{A}^{**}$ such that

$$m(\phi) = 1 \quad \text{and} \quad m(f \cdot a) = \phi(a)\, m(f)$$

for all $f \in \mathfrak{A}^*$ and $a \in \mathfrak{A}$, where $f \cdot a \in \mathfrak{A}^*$ is defined by

$$(f \cdot a)(b) = f(ab)$$

for all $b \in \mathfrak{A}$. Any such m is called a *ϕ-mean*; see also [1, 3, 20, 21]

The notion of ϕ-amenability is a generalization of *left amenability of Lau algebras* $\mathfrak{L}$ studied in Lau [11]; in fact, ϕ-amenability coincides with left amenability in the case where the character ϕ is taken to be the identity of the von Neumann algebra $\mathfrak{L}^*$. The class of left amenable Lau algebras includes the group algebra $L^1(G)$, and the measure algebra $M(G)$ of an amenable locally compact group G, as well as the quantum group algebra $L^1(Q)$ when Q is amenable; see Lau [11]. It also includes

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the Fourier algebra of a locally compact group and the Fourier-Stieljes algebra of a topological group; see Lau and Ludwig [14].

Several authors have investigated some common fixed point properties related to various notions of amenability; see for example Argabright [2], Lau [12], Lau and Takahashi [15, 16], Saeidi [24] and Wong [25, 26, 27]. For more recent results concerning relationships between left amenability of Lau algebras and fixed point properties, see Desaulniers and the authors [4].

In this paper, we characterize the existence of a ϕ-mean on certain topological left invariant and left introverted subspaces of $\mathfrak{A}^*$ in terms of some common fixed point properties. We also describe ϕ-amenability of $\mathfrak{A}$ in terms of the Hahn-Banach extension property.

2. Preliminaries

Let $\mathfrak{A}$ be a Banach algebra and $\phi \in \Delta(\mathfrak{A})$. An element a of $\mathfrak{A}$ is called *ϕ-maximal* if it satisfies

$$\|a\| = \phi(a) = 1.$$

Let $P_1(\mathfrak{A}, \phi)$ denote the collection of all ϕ-maximal elements of $\mathfrak{A}$; see in Kaniuth, Lau and Pym [8]. In general, $P_1(\mathfrak{A}, \phi)$ can be quite small; see Kaniuth, Lau and Pym [8], Examples 5.2 and 5.3. Let $X(\mathfrak{A}, \phi)$ denote the closed linear span of $P_1(\mathfrak{A}, \phi)$. Then $X(\mathfrak{A}, \phi)$ is a closed subalgebra of $\mathfrak{A}$. It is clear that $P_1(\mathfrak{A}, \phi)$ is a bounded subsemigroup of $\mathfrak{A}$.

When $\mathfrak{L}$ is a Lau algebra and u is the identity of the von Neumann algebra $\mathfrak{L}^*$, u-maximal elements are precisely elements of norm 1 in $\mathfrak{L}$ that induces a positive linear functional on $\mathfrak{L}^*$; hence $P_1(\mathfrak{L}, u)$ spans $\mathfrak{L}$. Moreover, $P_1(\mathfrak{L}, u)$ is weak* dense in $P_1(\mathfrak{L}^{**}, u^{**})$; this is because that the set of states in the predual of a von Neumann algebra is weak* dense in the set of states in its dual space; see Lau [13].

A linear subspace $\mathfrak{X}$ of $\mathfrak{A}^*$ is called *topological left* (resp. *right*) *invariant* if

$$\mathfrak{X} \cdot a \subseteq \mathfrak{X} \text{ (resp. } a \cdot \mathfrak{X} \subseteq \mathfrak{X})$$

for all $a \in \mathfrak{A}$ if, in addition, $\mathfrak{X}$ contains ϕ, then $m \in \mathfrak{X}^*$ is called a *ϕ-mean* on $\mathfrak{X}$ if

$$m(f \cdot a) = \phi(a)\, m(f)$$

for all $a \in \mathfrak{A}$ and $f \in \mathfrak{X}$. The linear subspace $\mathfrak{X}$ is called *topological left introverted* if for each $f \in \mathfrak{X}$ and $n \in \mathfrak{X}^*$ we have $n \cdot f \in \mathfrak{X}$, where $n \cdot f \in \mathfrak{A}^*$ is defined by

$$(n \cdot f)(a) = n(f \cdot a)$$

for all $a \in \mathfrak{A}$. If $\mathfrak{X}$ is a topological left introverted linear subspace of $\mathfrak{A}^*$, *the Arens product* on $\mathfrak{X}^*$ is defined for each $m, n \in \mathfrak{X}^*$ and each $f \in \mathfrak{X}$ by

$$(mn)(f) = m(n \cdot f).$$

Now, let $\mathfrak{X}$ be a topological left and right invariant linear subspace of $\mathfrak{A}^*$ containing ϕ. It is easy to check that $P_1(\mathfrak{A}, \phi)$ with the topology $\sigma(\mathfrak{A}, \mathfrak{X})$ is a semi-topological semigroup. In what follows, we shall always consider $P_1(\mathfrak{A}, \phi)$ in the topology $\sigma(\mathfrak{A}, \mathfrak{X})$. We denote by $\overline{P_1(\mathfrak{A}, \phi)}^{\sigma}$ the $\sigma(\mathfrak{X}^*, \mathfrak{X})$-closure of the set $P_1(\mathfrak{A}, \phi)$ in $\mathfrak{X}^*$.

Let $CB(P_1(\mathfrak{A},\phi))$ denote the space of bounded continuous complex-valued functions on $P_1(\mathfrak{A},\phi)$ with the supremum norm. For each $a \in P_1(\mathfrak{A},\phi)$, and $h \in CB(P_1(\mathfrak{A},\phi))$ define the two functions in $CB(P_1(\mathfrak{A},\phi))$ by

$$_ah(b) = h(ab) \quad \text{and} \quad h_a(b) = h(ba)$$

for all $b \in P_1(\mathfrak{A},\phi)$. A linear subspace $\mathfrak{Y}$ of $CB(P_1(\mathfrak{A},\phi))$ is called *left* (resp., *right*) *invariant* if $_a\mathfrak{Y} \subseteq \mathfrak{Y}$ (resp., $\mathfrak{Y}_a \subseteq \mathfrak{Y}$) for all $a \in P_1(\mathfrak{A},\phi)$. Let $\mathfrak{Y}$ be a left invariant subspace of $CB(P_1(\mathfrak{A},\phi))$ containing constants. Then $M \in \mathfrak{Y}^*$ is called a *mean* on $\mathfrak{Y}$ if

$$\|M\| = M(1) = 1.$$

We denote by $\mathfrak{M}(\mathfrak{Y})$ the weak* compact set of all means on $\mathfrak{Y}$. The functional $M \in \mathfrak{M}(\mathfrak{Y})$ is called *left invariant mean* on $\mathfrak{Y}$ if

$$M(_ah) = M(h)$$

for all $h \in \mathfrak{Y}$ and $a \in P_1(\mathfrak{A},\phi)$. Also, $\mathfrak{Y}$ is called *left introverted* if $M \cdot h \in \mathfrak{Y}$ for all $M \in CB(P_1(\mathfrak{A},\phi))^*$ and $h \in \mathfrak{Y}$, where

$$(M \cdot h)(a) = M(_ah)$$

for all $a \in P_1(\mathfrak{A},\phi)$. In this case, *the Arens product* $\odot$ on $\mathfrak{Y}^*$ is defined as

$$(M \odot N)(h) = M(N \cdot h)$$

for all $h \in \mathfrak{Y}$ and $M, N \in \mathfrak{Y}^*$. It is known that $\mathfrak{Y}^*$ becomes a Banach algebra under the Arens product.

3. ϕ-AMENABILITY AND FIXED POINT PROPERTIES

Let $\mathfrak{A}$ be a Banach algebra, let $\phi \in \Delta(\mathfrak{A})$, and let $\mathfrak{X} \subseteq \mathfrak{A}^*$ be a topological left invariant linear subspace of $\mathfrak{A}^*$ containing ϕ. A function $h \in CB(P_1(\mathfrak{A},\phi))$, is called *weakly left additively uniformly continuous* on $P_1(\mathfrak{A},\phi)$ if the map

$$a \mapsto {_ah}$$

of $P_1(\mathfrak{A},\phi)$ into $CB(P_1(\mathfrak{A},\phi))$ is uniformly continuous with respect to the uniformity induced on $P_1(\mathfrak{A},\phi)$ by $\sigma(\mathfrak{A},\mathfrak{X})$ and the uniformity induced on $CB(P_1(\mathfrak{A},\phi))$ by $\sigma(CB(P_1(\mathfrak{A},\phi)), CB(P_1(\mathfrak{A},\phi))^*)$. More precisely, for each $\varepsilon > 0$ and $M \in CB(P_1(\mathfrak{A},\phi))^*$, there is a corresponding neighborhood V of the origin in $\sigma(\mathfrak{A},\mathfrak{X})$ such that if $a, b \in P_1(\mathfrak{A},\phi)$ and $a - b \in V$, then

$$|M(_ah) - M(_bh)| < \varepsilon.$$

We denote by $WLUC(P_1(\mathfrak{A},\phi))$ the space of all such functions; see Wong [26].

It is straightforward to show that $WLUC(P_1(\mathfrak{A},\phi))$ is a norm closed left and right invariant linear subspace of $CB(P_1(\mathfrak{A},\phi))$ containing the constants. It is also left introverted. Note that the space $WLUC(P_1(\mathfrak{A},\phi))$ depends on $\mathfrak{X}$ and is always left introverted even if $\mathfrak{X}$ is not topological left introverted.

We start with the following lemma whose proof is adapted from that of Theorem 2.1 of [26].

Lemma 3.1. *Let $\mathfrak{A}$ be a Banach algebra, let $\phi \in \Delta(\mathfrak{A})$, and let $\mathfrak{X}$ be a topological left invariant subspace of $\mathfrak{A}^*$ containing ϕ. Consider the following conditions.*

(a) *$\mathfrak{X}$ has a ϕ-mean in $\overline{P_1(\mathfrak{A}, \phi)}^{\sigma}$.*

(b) *$WLUC(P_1(\mathfrak{A}, \phi))$ has a left invariant mean.*

Then (a) *implies* (b). *Moreover, if $\mathfrak{X}$ is topological left introverted and $X(\mathfrak{A}, \phi) = \mathfrak{A}$, then they are equivalent.*

Proof. (a)⇒(b). Assume that m is a ϕ-mean in $\overline{P_1(\mathfrak{A}, \phi)}^{\sigma}$. Then there is a net $(a_\alpha) \subseteq P_1(\mathfrak{A}, \phi)$ such that $a_\alpha \to m$ in the topology $\sigma(\mathfrak{X}^*, \mathfrak{X})$ of $\mathfrak{X}^*$. It is easy to check that

$$aa_\alpha - a_\alpha \to 0$$

in the topology $\sigma(\mathfrak{A}, \mathfrak{X})$ for all $a \in P_1(\mathfrak{A}, \phi)$. It follows that

$$aa_\alpha - ba_\alpha \to 0$$

in $\sigma(\mathfrak{A}, \mathfrak{X})$ for all $a, b \in P_1(\mathfrak{A}, \phi)$. For each α, define the functional M_α on $WLUC(P_1(\mathfrak{A}, \phi))$ by

$$M_\alpha(h) := h(a_\alpha)$$

for all $h \in WLUC(P_1(\mathfrak{A}, \phi))$. Let M be a weak*-cluster point of (M_α) and set

$$N := M \odot M.$$

We show that $N({}_a h) = N({}_b h)$ for all $h \in WLUC(P_1(\mathfrak{A}, \phi))$ and $a, b \in P_1(\mathfrak{A}, \phi)$. To see this, first note that without loss of generality, we may assume that M is the weak* limit of the net (M_α) and hence

$$\begin{aligned} N({}_a h) &= (M \odot M)({}_a h) = M(M \cdot {}_a h) = \lim_\alpha (M \cdot {}_a h)(a_\alpha) \\ &= \lim_\alpha M({}_a h \cdot a_\alpha) = \lim_\alpha M({}_{aa_\alpha} h). \end{aligned}$$

Since $h \in WLUC(P_1(\mathfrak{A}, \phi))$, it follows that for $\varepsilon > 0$, there is some neighborhood V in $\sigma(\mathfrak{A}, \mathfrak{X})$ such that if $c, d \in P_1(\mathfrak{A}, \phi)$ and $c - d \in V$, then

$$|M({}_c h) - M({}_d h)| < \varepsilon.$$

Also, there is some α_0 such that for each $\alpha \geq \alpha_0$ we have $aa_\alpha - ba_\alpha \in V$. Thus

$$|M({}_{aa_\alpha} h) - M({}_{ba_\alpha} h)| < \varepsilon$$

for all $\alpha \geq \alpha_0$ and so

$$N({}_a h) = N({}_b h).$$

This shows that $N \cdot h$ is a constant function on $P_1(\mathfrak{A}, \phi)$. Consequently $N \odot N$ is a left invariant mean on $WLUC(P_1(\mathfrak{A}, \phi))$.

(b)⇒(a). Suppose that $\mathfrak{X}$ is also topological left introverted and $X(\mathfrak{A}, \phi) = \mathfrak{A}$. Then the map $\tau : \mathfrak{X} \to CB(P_1(\mathfrak{A}, \phi))$ defined by

$$\tau(f)(a) = f(a)$$

for $f \in \mathfrak{X}$ and $a \in P_1(\mathfrak{A}, \phi)$ is a linear isometry of $\mathfrak{X}$ into $CB(P_1(\mathfrak{A}, \phi))$ with

$$\tau(\phi) = 1 \quad \text{and} \quad {}_a\tau(f) = \tau(f \cdot a)$$

for all $f \in \mathfrak{X}$ and $a \in P_1(\mathfrak{A}, \phi)$. Now, we show that

$$\tau(\mathfrak{X}) \subseteq WLUC(P_1(\mathfrak{A}, \phi)).$$

To see this, given $f \in \mathfrak{X}$, $\varepsilon > 0$ and $M \in CB(P_1(\mathfrak{A},\phi))^*$, we have $\tau^*(M) \in \mathfrak{X}^*$. Since $\mathfrak{X}$ is topological left introverted, it follows that $\tau^*(M) \cdot f$ is in $\mathfrak{X}$ and so is a $\sigma(\mathfrak{A},\mathfrak{X})$-continuous linear functional on $\mathfrak{A}$. Therefore, there is a neighborhood V in $\sigma(\mathfrak{A},\mathfrak{X})$ such that

$$|(\tau^*(M) \cdot f)(a)| < \varepsilon$$

for all $a \in V$. Thus, if $a, b \in P_1(\mathfrak{A},\phi)$ and $a - b \in V$, then

$$\begin{aligned} |M({}_a\tau(f)) - M({}_b\tau(f))| &= |M(\tau(f \cdot a)) - M(\tau(f \cdot b))| \\ &= |\tau^*(M)(f \cdot a) - \tau^*(M)(f \cdot b)| \\ &= |(\tau^*(M) \cdot f)(a) - (\tau^*(M) \cdot f)(b)| < \varepsilon. \end{aligned}$$

Thus $\tau(\mathfrak{X}) \subseteq WLUC(P_1(\mathfrak{A},\phi))$ as claimed. Now, let M be a left invariant mean on $WLUC(P_1(\mathfrak{A},\phi))$ and set

$$\delta(P_1(\mathfrak{A},\phi)) := \{\delta(a) : \ a \in P_1(\mathfrak{A},\phi)\},$$

where $\delta(a)$ denotes the point evaluation at a. Then the convex hull $\mathrm{co}(\delta(P_1(\mathfrak{A},\phi)))$ is weak*-dense in $\mathfrak{M}(WLUC(P_1(\mathfrak{A},\phi)))$ and

$$\mathrm{co}(\delta(P_1(\mathfrak{A},\phi))) = \delta(P_1(\mathfrak{A},\phi))$$

Let $(a_\alpha) \subseteq P_1(\mathfrak{A},\phi)$ be a net such that

$$\delta(a_\alpha) \to M$$

in the weak* topology of $WLUC(P_1(\mathfrak{A},\phi))^*$. Since τ^* is weak*-weak* continuous, it follows that

$$a_\alpha = \tau^*(\delta(a_\alpha)) \to \tau^*(M).$$

Thus $\tau^*(M) \in \overline{P_1(\mathfrak{A},\phi)}^{\sigma}$; it is easy to see that $\tau^*(M)$ is a ϕ-mean on $\mathfrak{X}$. □

Let $\mathfrak{E}$ be a separated locally convex vector space; that is, a complex vector space equipped with a compatible Hausdorff topology, which is generated by a family of semi-norms on $\mathfrak{E}$, and let K be a compact convex subset of $\mathfrak{E}$. An affine action of the semigroup $P_1(\mathfrak{A},\phi)$ on K is a map $T : P_1(\mathfrak{A},\phi) \times K \to K$ denoted by

$$(a,\xi) \to T_a\xi$$

for all $a \in P_1(\mathfrak{A},\phi)$ and $\xi \in K$ such that $T_a : K \to K$ is continuous affine and

$$T_a \circ T_b = T_{ab}$$

for all $a, b \in P_1(\mathfrak{A},\phi)$. The map T is called $\sigma(\mathfrak{A},\mathfrak{X})$-*uniformly continuous* on $P_1(\mathfrak{A},\phi)$ at $\xi_0 \in K$ if the map

$$a \mapsto T_a\xi_0$$

of $P_1(\mathfrak{A},\phi)$ into K is uniformly continuous with respect to the uniformity induced on $P_1(\mathfrak{A},\phi)$ by $\sigma(\mathfrak{A},\mathfrak{X})$ and the unique uniformity induced on K by the space $\mathfrak{E}$; it is called $\sigma(\mathfrak{A},\mathfrak{X})$-uniformly continuous on $P_1(\mathfrak{A},\phi)$ if it is so at each point of K.

We say that $P_1(\mathfrak{A},\phi)$ has *the fixed point property* for $\sigma(\mathfrak{A},\mathfrak{X})$-*uniformly continuous affine actions on compact convex sets* if every such action of $P_1(\mathfrak{A},\phi)$ has a common fixed point.

Here, we state the following result whose proof is omitted, since it can be proved in the same direction of Theorem 2.2 of [26]; see also [25].

Theorem 3.2. *Let $\mathfrak{A}$ be a Banach algebra, let $\phi \in \Delta(\mathfrak{A})$, and let $\mathfrak{X}$ be a topological left invariant subspace of $\mathfrak{A}^*$ containing ϕ. Then the following statements are equivalent.*

(a) *$WLUC(P_1(\mathfrak{A}, \phi))$ has a left invariant mean.*

(b) *$P_1(\mathfrak{A}, \phi)$ has the fixed point property for $\sigma(\mathfrak{A}, \mathfrak{X})$-uniformly continuous affine actions on compact convex sets.*

Let $\mathfrak{E}$ be a separated locally convex vector space and let $\mathfrak{A}$ be a Banach algebra. An action of $\mathfrak{A}$ on $\mathfrak{E}$ is a bilinear map

$$T : \mathfrak{A} \times \mathfrak{E} \to \mathfrak{E}$$

denoted by $(a, \xi) \to T_a\xi$ for all $a \in \mathfrak{A}$ and $\xi \in \mathfrak{E}$ such that

$$T_a : \mathfrak{E} \to \mathfrak{E}$$

is continuous and $T_a \circ T_b = T_{ab}$ for all $a, b \in \mathfrak{A}$. It is called $\sigma(\mathfrak{A}, \mathfrak{X})$-*separately continuous* if $a \mapsto T_a\xi$ is continuous for all $\xi \in \mathfrak{E}$ when $\mathfrak{A}$ has the topology $\sigma(\mathfrak{A}, \mathfrak{X})$. Let K be a compact convex subset of $\mathfrak{E}$, we say that K is $P_1(\mathfrak{A}, \phi)$-invariant under T if

$$T_a(K) \subseteq K$$

for all $a \in \mathfrak{A}$. In this case, T induces an affine action

$$T : P_1(\mathfrak{A}, \phi) \times K \to K$$

of the semigroup $P_1(\mathfrak{A}, \phi)$ which is $\sigma(\mathfrak{A}, X)$-uniformly continuous on $P_1(\mathfrak{A}, \phi)$. However, not every such affine action of $P_1(\mathfrak{A}, \phi)$ comes from an action of the Banach algebra $\mathfrak{A}$.

We say that $P_1(\mathfrak{A}, \phi)$ has *the fixed point property for the $\sigma(\mathfrak{A}, \mathfrak{X})$-separately continuous action of* $\mathfrak{A}$ if the induced action on $P_1(\mathfrak{A}, \phi)$ has a fixed point in K.

We now present the following result whose idea is inspired by Wong in [25] and Lau in [10].

Theorem 3.3. *Let $\mathfrak{A}$ be a Banach algebra, let $\phi \in \Delta(\mathfrak{A})$, and let $\mathfrak{X}$ be a topological left introverted and topological left invariant subspace of $\mathfrak{A}^*$ containing ϕ. Consider the following conditions.*

(a) *$\mathfrak{X}$ has a ϕ-mean in $\overline{P_1(\mathfrak{A}, \phi)}^{\sigma}$.*

(b) *$WLUC(P_1(\mathfrak{A}, \phi))$ has a left invariant mean.*

(c) *$P_1(\mathfrak{A}, \phi)$ has the fixed point property for $\sigma(\mathfrak{A}, \mathfrak{X})$-uniformly continuous affine actions on compact convex sets.*

(d) *$P_1(\mathfrak{A}, \phi)$ has the fixed point property for the $\sigma(\mathfrak{A}, \mathfrak{X})$-separately continuous action of $\mathfrak{A}$.*

Then the implications (a)⇒(b)⇒(c)⇒(d) *hold. Moreover, if $X(\mathfrak{A}, \phi) = \mathfrak{A}$, then they are equivalent.*

Proof. The equivalence of (a), (b), and (c) follows from Theorem 3.2 and Lemma 3.1. The proof of (a) ⇔ (d) is similar to that of [25], Theorem 3.1. □

Definition 3.4. Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \Delta(\mathfrak{A})$. A function $h \in CB(P_1(\mathfrak{A}, \phi))$ is called *additively uniformly continuous* if for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each $a, b \in P_1(\mathfrak{A}, \phi)$,

$$\|a - b\| < \delta \text{ implies } |h(a) - h(b)| < \varepsilon.$$

We will denote the set of all additively uniformly continuous functions on $P_1(\mathfrak{A},\phi)$ by $AUC(P_1(\mathfrak{A},\phi))$.

It is straightforward to show that $AUC(P_1(\mathfrak{A},\phi))$ is a norm closed translation invariant subspace of $CB(P_1(\mathfrak{A},\phi))$ containing constants and restrictions of elements in $\mathfrak{A}^*$ to $P_1(\mathfrak{A},\phi)$; see [17]. A very interesting property about the space $AUC(P_1(\mathfrak{A},\phi))$ is that it is independent of the multiplication of $\mathfrak{A}$, thus it only depends on the Banach space structure of $\mathfrak{A}$. However, it always lies inside the space of uniformly continuous functions on the semigroup $P_1(\mathfrak{A},\phi)$, which is normally denoted by $UCB(P_1(\mathfrak{A},\phi))$.

Whenever Q is a family of seminorms which generates the topology of $\mathfrak{E}$, there is a natural notion of a Q-uniform action.

We will say that *the action T of $P_1(\mathfrak{A},\phi)$ on a convex subset K of $\mathfrak{E}$ is Q-uniform* if for each $\xi\in K$ and $\rho\in Q$, the map from $P_1(\mathfrak{A},\phi)$ into K, given by $a\mapsto T_a\xi$ is uniformly continuous with respect to ρ. More precisely, for each $\varepsilon>0$, there is a corresponding $\delta>0$ such that for each $a,b\in P_1(\mathfrak{A},\phi)$,

$$\|a-b\|<\delta \text{ implies } \rho(T_a\xi-T_b\xi)<\varepsilon.$$

Theorem 3.5. *Let $\mathfrak{A}$ be a Banach algebra and let $\phi\in\Delta(\mathfrak{A})$. Suppose that $\mathfrak{E}$ is a separated locally convex space, and Q is a family of seminorms which generates the topology of $\mathfrak{E}$. Consider the following conditions.*

(a) *$\mathfrak{A}$ has a ϕ-mean in $\overline{P_1(\mathfrak{A},\phi)}^{w^*}$.*

(b) *$AUC(P_1(\mathfrak{A},\phi))$ has a left invariant mean.*

(c) *$P_1(\mathfrak{A},\phi)$ has the fixed point property for Q-uniform and separately continuous affine actions on compact convex sets.*

(d) *$P_1(\mathfrak{A},\phi)$ has the fixed point property for Q-uniform and jointly continuous affine actions on compact convex sets.*

Then the implications (a)⇒(b)⇒(c) ⇒(d) *hold. Moreover, if $X(\mathfrak{A},\phi)=\mathfrak{A}$, then they are equivalent.*

Proof. (a)⇒(b). Suppose that $m\in\overline{P_1(\mathfrak{A},\phi)}^{w^*}$ is a ϕ-mean on $\mathfrak{A}^*$. By Theorem 1.4 of [7] and its proof, there is a net $(a_\alpha)\subseteq P_1(\mathfrak{A},\phi)$ such that

$$\|aa_\alpha-\phi(a)a_\alpha\|\to 0$$

for all $a\in\mathfrak{A}$. For each α, define

$$M_\alpha(h):=h(a_\alpha)$$

for all $h\in AUC(P_1(\mathfrak{A},\phi))$. Let M be a weak*-cluster point of (M_α). Then M is a left invariant mean on $AUC(P_1(\mathfrak{A},\phi))$.

(b)⇒(a). The map $\tau:\mathfrak{A}^*\to CB(P_1(\mathfrak{A},\phi))$ defined by

$$\tau(f)(a)=f(a)$$

for all $f\in\mathfrak{A}^*$, $a\in P_1(\mathfrak{A},\phi)$ is a linear isometry of $\mathfrak{A}^*$ into $CB(P_1(\mathfrak{A},\phi))$ with

$$\tau(\phi)=1 \quad\text{and}\quad {}_a\tau(f)=\tau(f\cdot a)$$

for all $f\in\mathfrak{A}^*$ and $a\in P_1(\mathfrak{A},\phi)$. Moreover, $\tau(f)\in AUC(P_1(\mathfrak{A},\phi))$ and by the same argument used in the proof of Lemma 3.1, we can show that if M is a left invariant mean on $AUC(P_1(\mathfrak{A},\phi))$, then

$$\tau^*(M) \in \overline{P_1(\mathfrak{A},\phi)}^{w^*}$$

is a ϕ-mean on $\mathfrak{A}^*$.

(a)⇒(c). Suppose that (a_α) be as in the first part of the proof, and let

$$T : P_1(\mathfrak{A},\phi) \times K \to K$$

be a Q-uniform and separately continuous affine actions on $P_1(\mathfrak{A},\phi)$. Fix $\xi \in K$ and consider the net $(T_{a_\alpha}\xi) \subseteq K$. Since K is compact, there is a subnet of $(T_{a_\alpha}\xi)$ which converges to an elements ξ_0 of K. So, without loss of generality we may assume that

$$T_{a_\alpha}\xi \to \xi_0.$$

Since the topology of $\mathfrak{E}$ is generated by Q and the action is Q-uniform, it follows that

$$T_{aa_\alpha}\xi - T_{a_\alpha}\xi \to 0$$

in $\mathfrak{E}$ for all $a \in P_1(\mathfrak{A},\phi)$. Hence, for each $a \in P_1(\mathfrak{A},\phi)$ we have

$$T_a\xi_0 = T_a \lim_\alpha T_{a_\alpha}\xi = \lim_\alpha T_{aa_\alpha}\xi = \lim_\alpha T_{a_\alpha}\xi = \xi_0.$$

(c)⇒(d). This is trivial.

(d)⇒(b). Consider $\mathfrak{E} := AUC(P_1(\mathfrak{A},\phi))^*$ with the weak* topology. Let Q be the set of seminorms induced by elements of $AUC(P_1(\mathfrak{A},\phi))$, and set

$$K := \mathfrak{M}(AUC(P_1(\mathfrak{A},\phi))).$$

Then K is compact and convex in $\mathfrak{E}$. Define the map $T : P_1(\mathfrak{A},\phi) \times K \to K$ by

$$T_aM(h) = M({}_ah)$$

for all $h \in AUC(P_1(\mathfrak{A},\phi))$. It is easy to check that T is a Q-uniform and affine action on K. We show that the action is jointly continuous. To see this, suppose that $a_\alpha \to a$ in $P_1(\mathfrak{A},\phi)$ and $M_\beta \to M$ in $AUC(P_1(\mathfrak{A},\phi))^*$. Then for each $h \in AUC(P_1(\mathfrak{A},\phi))$ we have

$$\begin{aligned}
\lim_{\alpha,\beta} |T_{a_\alpha}M_\beta(f) - T_aM(f)| &= \lim_{\alpha,\beta} |M_\beta({}_{a_\alpha}h) - M({}_ah)| \\
&= \lim_{\alpha,\beta} |M_\beta({}_{a_\alpha}h - {}_ah) + (M_\beta - M)({}_ah)| \\
&\le \lim_\alpha \|{}_{a_\alpha}h - {}_ah\|_\infty + \lim_\beta |(M_\beta - M)({}_ah)| = 0,
\end{aligned}$$

as required. By hypothesis, there exists $M \in K$ which is fixed under the affine action T of $P_1(\mathfrak{A},\phi)$; that is,

$$T_aM = M$$

for all $a \in P_1(\mathfrak{A},\phi)$. Hence, M is a left invariant mean on $AUC(P_1(\mathfrak{A},\phi))$. □

4. ϕ-amenability and the Hahn-Banach extension property

We begin this section by introducing an important extension property.

Definition 4.1. Let $\mathfrak{A}$ be a Banach algebra, let $\phi \in \Delta(\mathfrak{A})$, and let $\mathfrak{X}$ be a topological left invariant subspace of $\mathfrak{A}^*$ containing ϕ. We say that $\mathfrak{X}$ has *the Hahn-Banach ϕ-extension property* if for every topological left invariant subspace $\mathfrak{Y}$ of $\mathfrak{X}$ containing ϕ, every ϕ-mean m on $\mathfrak{Y}$ can be extended to a ϕ-mean $\widetilde{m}$ on $\mathfrak{X}$ such that $\|\widetilde{m}\| = \|m\|$.

The following result generalizes Theorem 1 of [10].

Theorem 4.2. *Let $\mathfrak{A}$ be a Banach algebra, let $\phi \in \Delta(\mathfrak{A})$, and let $\mathfrak{X}$ be a topological left invariant and topological left introverted linear subspace of $\mathfrak{A}^*$ containing ϕ. Then the following statements are equivalent.*

(a) *$\mathfrak{X}^*$ has a ϕ-mean of norm 1.*

(b) *$\mathfrak{X}$ has the Hahn-Banach ϕ-extension property.*

Proof. Let $\mathfrak{Y}$ and m be as in Definition 4.1. By the Hahn-Banach theorem, there exists a linear functional n on $\mathfrak{X}$ such that $\|n\| = \|m\|$ and $n|_{\mathfrak{Y}} = m$. Let $\widetilde{m}$ be defined on $\mathfrak{X}$ by

$$\widetilde{m} := m_0 n,$$

where $m_0 \in \mathfrak{X}^*$ is a ϕ-mean on $\mathfrak{X}$ with $\|m_0\| = 1$. Then it is easy to check that $\widetilde{m}$ is a continuous linear functional on $\mathfrak{X}$. For every $f \in \mathfrak{X}$ and $b \in \mathfrak{A}$ we have

$$n \cdot (f \cdot b) = (n \cdot f) \cdot b.$$

It follows that

$$\widetilde{m}(f \cdot b) = m_0((n \cdot f) \cdot b) = \phi(b) m_0(n \cdot f) = \phi(b)\widetilde{m}(f).$$

On the other hand, for each $f \in \mathfrak{Y}$ and $a \in \mathfrak{A}$,

$$(n \cdot f)(a) = n(f \cdot a) = m(f \cdot a) = \phi(a)m(f).$$

Therefore $n \cdot f = m(f)\phi$. Consequently

$$\widetilde{m}(f) = m_0(n \cdot f) = m(f)m_0(\phi) = m(f).$$

It is clear that $\|\widetilde{m}\| = \|m\|$ as required.

Conversely, consider the topological left invariant linear subspace $\mathfrak{Y} = \mathbb{C}\phi$ of $\mathfrak{X}$. Now, if we define $m \in \mathfrak{Y}^*$ by

$$m(\lambda\phi) = \lambda$$

for all $\lambda \in \mathbb{C}$, then it is easy to check that m is a ϕ-mean on $\mathfrak{Y}$ with $\|m\| = 1$. Consequently, it is clear that any invariant extension $\widetilde{m}$ of m on $\mathfrak{X}$ is a ϕ-mean on $\mathfrak{X}$ with norm 1. □

Corollary 4.3. *Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \Delta(\mathfrak{A})$ with $X(\mathfrak{A}, \phi) = \mathfrak{A}$. Then $\mathfrak{A}$ is ϕ-amenable if one of the following conditions holds.*

(a) *$P_1(\mathfrak{A}, \phi)$ has the fixed point property for $\sigma(\mathfrak{A}, \mathfrak{A}^*)$-uniformly continuous affine actions on compact convex sets.*

(b) *$P_1(\mathfrak{A}, \phi)$ has the fixed point property for the $\sigma(\mathfrak{A}, \mathfrak{A}^*)$-separately continuous action of $\mathfrak{A}$.*

(c) $P_1(\mathfrak{A}, \phi)$ *has the fixed point property for Q-uniform and separately continuous affine actions on compact convex sets.*

(d) $P_1(\mathfrak{A}, \phi)$ *has the fixed point property for Q-uniform and jointly continuous affine actions on compact convex sets.*

(e) $\mathfrak{A}^*$ *has the Hahn-Banach ϕ-extension property.*

We close this section with some two applications of our results.

Example 1. For a locally compact group G with a left Haar measure λ_G, let $L^1(G)$ denote the usual Lebesgue space; see [6]. Suppose $\widehat{G}$ denote the set of all continuous homomorphisms ρ from G into the circle group $\mathbb{T}$, and define $\phi_\rho \in \Delta(L^1(G))$ to be the character induced by ρ on $L^1(G)$; that is,

$$\phi_\rho(f) = \int_G \overline{\rho(x)} f(x)\, d\lambda_G(x) \quad (f \in L^1(G)).$$

It is known that there are no other characters on $L^1(G)$; that is,

$$\Delta(L^1(G)) = \left\{ \phi_\rho : \rho \in \widehat{G} \right\};$$

see for example Kaniuth [6], Theorem 2.7.2 and Rudin [23]. Let us recall that, G is called *amenable* if $L^1(G)^{**}$ has a ϕ_1-mean of norm one; see [22] for details. It is clear that

$$P_1(L^1(G), \phi_\rho) = \left\{ f \in L^1(G) : \quad \overline{\rho} f \geq 0, \int_G \overline{\rho} f d\lambda = 1 \right\}.$$

For every $h \in L^1(G)$ and $\rho \in \widehat{G}$, the function $\overline{\rho}h$ can be written as a linear combination

$$\overline{\rho}h = \sum_{j=1}^{4} c_j h_j,$$

where $c_j \in \mathbb{C}$, $h_j \geq 0$ and $\|h_j\|_1 = 1$, $1 \leq j \leq 4$. Hence

$$h = \sum_{j=1}^{4} c_j \rho h_j$$

and $\rho h_j \in P_1(L^1(G), \phi_\rho)$. So $P_1(L^1(G), \phi_\rho)$ spans $L^1(G)$; that is

$$X(L^1(G), \phi_\rho) = L^1(G).$$

It is known that $P_1(L^1(G), \phi_1)$ is dense in

$$P_1(L^1(G)^{**}, {\phi_1}^{**}) = \left\{ m \in L^1(G)^{**} : m(\phi_1) = \|m\| = 1 \right\}$$

in the weak* topology of $L^1(G)^{**}$; see [13], Lemma 2.1. It follows easily that $P_1(L^1(G), \phi_\rho)$ is dense in $P_1(L^1(G)^{**}, {\phi_\rho}^{**})$ in the weak* topology of $L^1(G)^{**}$.

Now, an application of Corollary 4.3 together with the fact that G is amenable if and only if $L^1(G)^{**}$ has a ϕ_ρ-mean of norm one, show that the following statements are equivalent.

(a) G is amenable.

(b) There exists $\rho \in \widehat{G}$ such that the semigroup $\{f \in L^1(G) : \overline{\rho} f \geq 0, \int_G \overline{\rho} f d\lambda = 1\}$ has the fixed point property for the $\sigma(L^1(G), L^1(G)^*)$-separately continuous action of $L^1(G)$.

(c) There exists $\rho \in \widehat{G}$ such that the semigroup $\{f \in L^1(G) : \overline{\rho} f \geq 0, \int_G \overline{\rho} f d\lambda = 1\}$ has the fixed point property for Q-uniform and separately continuous affine actions on compact convex sets.

(d) There exists $\rho \in \widehat{G}$ such that the semigroup $\{f \in L^1(G) : \overline{\rho} f \geq 0, \int_G \overline{\rho} f d\lambda = 1\}$ has the fixed point property for Q-uniform and jointly continuous affine actions on compact convex sets.

(e) There exists $\rho \in \widehat{G}$ such that $L^1(G)^*$ has the Hahn-Banach ϕ_ρ-extension property.

Example 2. Let G be a locally compact group with identity e and let $A(G)$ be the Fourier algebra of G which is spanned by functions with compact support in $P(G)$, the set of all continuous positive definite functions on G; see [5] for details. Then the spectrum of $A(G)$ can be canonically identified with G. More precisely, for each $x \in G$, the map $x \to \phi_x$, where $\phi_x(v) = v(x)$ for $v \in A(G)$, is a homeomorphism from G onto $\Delta(A(G))$. It is clear that

$$P_1(A(G), \phi_x) = \{v \in A(G) : \ l_x v \in P(G), v(x) = 1 \},$$

where $l_x v(y) = v(xy)$ for all $x, y \in G$. Moreover, in [19], Lemma 3.1, it is proved that $A(G)$ has a ϕ_x-mean of norm one for all $x \in G$. Thus, the following statements hold.

(a) The semigroup $\{v \in A(G) : \ l_x v \in P(G), v(x) = 1 \}$ has the fixed point property for the $\sigma(A(G), A(G)^*)$-separately continuous action of $L^1(G)$.

(b) The semigroup $\{v \in A(G) : \ l_x v \in P(G), v(x) = 1 \}$ has the fixed point property for Q-uniform and separately continuous affine actions on compact convex sets.

(c) The semigroup $\{v \in A(G) : \ l_x v \in P(G), v(x) = 1 \}$ has the fixed point property for Q-uniform and jointly continuous affine actions on compact convex sets.

(d) $A(G)^*$ has the Hahn-Banach ϕ_x-extension property.

5. ϕ-means on almost periodic functionals

Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \Delta(\mathfrak{A})$. We denote by $AP(P_1(\mathfrak{A}, \phi))$ the Banach space of *almost periodic functions on* $P_1(\mathfrak{A}, \phi)$; that is, the subspace of the Banach space $CB(P_1(\mathfrak{A}, \phi))$ consisting of those $h \in CB(P_1(\mathfrak{A}, \phi))$ such that the (left) translates of h form a relatively compact subset of $CB(P_1(\mathfrak{A}, \phi))$. Also, we define the space of *almost periodic functionals* on $\mathfrak{A}$, denoted by $AP(\mathfrak{A})$, to be the collection of all $f \in \mathfrak{A}^*$ such that the set

$$\{f \cdot a : \ \|a\| \leq 1\}$$

form a relatively compact subset of the Banach space $\mathfrak{A}^*$; this is equivalent to the set

$$\{a \cdot f : \ \|a\| \leq 1\}$$

being relatively compact. It is easy to check that $AP(\mathfrak{A})$ is a closed, topological left introverted and topological left invariant subspace of $\mathfrak{A}^*$ containing ϕ.

Definition 5.1. Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \Delta(\mathfrak{A})$. A function $h \in CB(P_1(\mathfrak{A}, \phi))$, is called *left additively uniformly continuous* on $P_1(\mathfrak{A}, \phi)$ if the map

$$a \mapsto {}_a f$$

of $P_1(\mathfrak{A}, \phi)$ into $CB(P_1(\mathfrak{A}, \phi))$ is uniformly continuous with respect to the uniformity induced on $P_1(\mathfrak{A}, \phi)$ by $\sigma(\mathfrak{A}, AP(\mathfrak{A}))$ and the uniformity induced on $CB(P_1(\mathfrak{A}, \phi))$ by the supremum norm. We will denote by $LUC(P_1(\mathfrak{A}, \phi))$ the space of all such functions on $P_1(\mathfrak{A}, \phi)$; see Wong [27].

It is easy to check that $LUC(P_1(\mathfrak{A}, \phi))$ is a norm closed left and right invariant linear subspace of $CB(P_1(\mathfrak{A}, \phi))$ containing the constants. It is also left introverted and

$$LUC(P_1(\mathfrak{A}, \phi)) \subseteq WLUC(P_1(\mathfrak{A}, \phi)).$$

Let $\mathfrak{E}$ be a separated locally convex vector space and let K be a compact convex subset of $\mathfrak{E}$. An affine action T of $P_1(\mathfrak{A}, \phi)$ on K is $\sigma(\mathfrak{A}, AP(\mathfrak{A}))$*-equiuniformly continuous* if for each $W \in \mathfrak{U}$, there is some neighborhood U in $\sigma(\mathfrak{A}, AP(\mathfrak{A}))$ such that

$$(T_a\xi, T_b\xi) \in W$$

for all $\xi \in K$ and $a, b \in P_1(\mathfrak{A}, \phi)$ with $a - b \in U$, where $\mathfrak{U}$ is the unique uniformity which determines the topology of K; see [9], page 197.

Moreover, T is called *equicontinuous* on K if for each $\xi \in K$ and $W \in \mathfrak{U}$, there is some $Z \in \mathfrak{U}$ such that $(T_a\eta, T_a\xi) \in W$ for all $a \in P_1(\mathfrak{A}, \phi)$ and $\eta \in K$ with $(\eta, \xi) \in Z$. It is interesting to ask when $AP(\mathfrak{A})$ has a topological left invariant mean.

Theorem 5.2. *Let $\mathfrak{A}$ be a Banach algebra and let $\phi \in \Delta(\mathfrak{A})$. Consider the following statements.*

(a) $AP(\mathfrak{A})$ *has a ϕ-mean in* $\overline{P_1(\mathfrak{A}, \phi)}^{\sigma}$.

(b) $LUC(P_1(\mathfrak{A}, \phi))$ *has a left invariant mean.*

(c) $P_1(\mathfrak{A}, \phi)$ *has the fixed point property for* $\sigma(\mathfrak{A}, AP(\mathfrak{A}))$*-equiuniformly continuous affine actions of* $P_1(\mathfrak{A}, \phi)$ *on compact convex sets.*

(d) $P_1(\mathfrak{A}, \phi)$ *has the fixed point property for* $\sigma(\mathfrak{A}, AP(\mathfrak{A}))$*- equiuniformly continuous and equicontinuous affine actions of* $P_1(\mathfrak{A}, \phi)$ *on compact convex sets.*

Then the implications (a)⇒(b)⇒(c)⇒(d) *hold. Moreover, if* $X(\mathfrak{A}, \phi) = \mathfrak{A}$*, then they are equivalent.*

Proof. (a) ⇒ (b). If $AP(\mathfrak{A})$ has a ϕ-mean in $\overline{P_1(\mathfrak{A}, \phi)}^{\sigma}$, then $WLUC(P_1(\mathfrak{A}, \phi))$ has a left invariant mean by Theorem 3.3. Since

$$LUC(P_1(\mathfrak{A}, \phi)) \subseteq WLUC(P_1(\mathfrak{A}, \phi)),$$

it follows that $LUC(P_1(\mathfrak{A}, \phi))$ also has a left invariant mean.

(b) ⇒ (c). For each $b \in P_1(\mathfrak{A}, \phi)$, consider the map

$$R_b : P_1(\mathfrak{A}, \phi) \to P_1(\mathfrak{A}, \phi)$$

defined by $R_b(a) = ab$ for all $a \in P_1(\mathfrak{A}, \phi)$. By the same direction of Theorem 4.2 of [4], R_b is equiuniformly continuous for all $b \in P_1(\mathfrak{A}, \phi)$ This together with equi-uniform continuity of the affine action

$$T : P_1(\mathfrak{A}, \phi) \times K \to K$$

implies that T is an A-representation for the pair $P_1(\mathfrak{A},\phi)$ and $LUC(P_1(\mathfrak{A},\phi))$ on K in the sense of Argabright [2]. Hence, T has a common fixed point by [2], Theorem 1.

(c) $\Rightarrow$ (d). This is trivial.

(c) $\Rightarrow$ (a). Let $\mathfrak{Y} := AP(P_1(\mathfrak{A},\phi)) \cap LUC(P_1(\mathfrak{A},\phi))$. Then $\mathfrak{Y}$ is a left invariant and left introverted linear subspace of $CB(P_1(\mathfrak{A},\phi))$ containing the constants. Let

$$K := \mathfrak{M}(\mathfrak{Y}).$$

Then K is convex and compact with respect to the weak* topology of $\mathfrak{Y}^*$. Define the map $T : P_1(\mathfrak{A},\phi) \times K \to K$ by

$$T_a M(h) = M({}_a h)$$

for all $a \in P_1(\mathfrak{A},\phi)$, $h \in \mathfrak{Y}$ and $M \in K$. It is easy to check that T is an affine action on K. Since

$$\mathfrak{Y} \subseteq LUC(P_1(\mathfrak{A},\phi)),$$

it follows from the definition of $LUC(P_1(\mathfrak{A},\phi))$ that this action is $\sigma(\mathfrak{A}, AP(\mathfrak{A}))$-equiuniformly continuous. Moreover, since

$$\mathfrak{Y} \subseteq AP(P_1(\mathfrak{A},\phi)),$$

the arguments used in Lau [12], Theorem 3.2, applied to $P_1(\mathfrak{A},\phi)$ show that this action is equicontinuous. Thus, this action has a fixed point, M which is left invariant on $\mathfrak{Y}$. We know that the map

$$\tau : AP(\mathfrak{A}) \to CB(P_1(\mathfrak{A},\phi))$$

defined by

$$\tau(f)(a) = f(a)$$

for $f \in AP(\mathfrak{A})$ and $a \in P_1(\mathfrak{A},\phi)$ is a linear isometry of $AP(\mathfrak{A})$ into $CB(P_1(\mathfrak{A},\phi))$ such that

$$\tau(\phi) = 1 \quad \text{and} \quad {}_a\tau(f) = \tau(f \cdot a)$$

for all $f \in AP(\mathfrak{A})$ and $a \in P_1(\mathfrak{A},\phi)$. Now, we show that

$$\tau(AP(\mathfrak{A})) \subseteq \mathfrak{Y}.$$

To prove this, fix $f \in AP(\mathfrak{A})$ and set

$$h := \tau(f).$$

It is easy to see that $h \in AP(P_1(\mathfrak{A},\phi))$. Given $\varepsilon > 0$. Since $P_1(\mathfrak{A},\phi) \cdot f$ is totally bounded, there is some neighborhood U in $\sigma(\mathfrak{A}, AP(\mathfrak{A}))$ such that if $a, b \in P_1(\mathfrak{A},\phi)$ and $a - b \in U$, then

$$\begin{aligned} \|{}_a h - {}_b h\| &= \sup\{|h(ac) - h(bc)| : \ c \in P_1(\mathfrak{A},\phi)\} \\ &= \sup\{|(c \cdot f)(a - b)| : \ c \in P_1(\mathfrak{A},\phi)\} < \varepsilon. \end{aligned}$$

Hence, $h \in LUC(P_1(\mathfrak{A},\phi))$. By the same argument used in the proof of Lemma 3.1, we can show that $\tau^*(M) \in \overline{P_1(\mathfrak{A},\phi)}^{w^*}$ is a ϕ-mean on $AP(\mathfrak{A})^*$. □

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References

[1] F. Akhtari, R. Nasr-Isfahani, *Strongly amenable representations of involutive Banach algebras*, Positivity, **18**(2014), 395-412.

[2] L. Argabright, *Invariant means and fixed points, A sequel to Mitchells paper*, Trans. Amer. Math. Soc., **130**(1968), 127-130.

[3] M. Dashti, R. Nasr-Isfahani, S. Soltani Renani, *Character amenability of Lipschitz algebras*, Canad. Math. Bull., **57**(2014), 37-41.

[4] S. Desaulniers, R. Nasr-Isfahani, M. Nemati, *Common fixed point properties and amenability of a class of Banach algebras*, J. Math. Anal. Appl., **402**(2013), 536-544.

[5] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, **92**(1964), 181-236.

[6] E. Kaniuth, *A Course in Commutative Banach Algebras.* Springer-Verlag, New York, 2009.

[7] E. Kaniuth, A.T. Lau, J. Pym, *On ϕ-amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc., **144**(2008), 85-96.

[8] E. Kaniuth, A.T. Lau, J. Pym, *On character amenability of Banach algebras*, J. Math. Anal. Appl., **344**(2008), 942-955.

[9] J.L. Kelley, *General Topology*, Van Nostrand, Princeton, 1955.

[10] A.T. Lau, *Amenability and invariant subspaces*, J. Austral. Math. Soc., **18**(1974), 200-204.

[11] A.T. Lau, *Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups*, Fund. Math., **118**(1983), 161-175.

[12] A.T. Lau, *Invariant means on almost periodic functions and fixed point properties*, Rocky Mountain J. Math., **3**(1973), 69-76.

[13] A.T. Lau, *Uniformly continuous functionals on Banach algebras*, Colloq. Math., **51**(1987), 195-205.

[14] A.T. Lau, J. Ludwig, *Fourier-Stieltjes algebra of a topological group*, Adv. Math., **229**(2012), 2000-2023.

[15] A.T. Lau, W. Takahashi, *Invariant submeans and semigroups of nonexpansive mappings on Banach spaces with normal structure* , J. Funct. Anal., **142**(1996), 79-88.

[16] A.T. Lau, W. Takahashi, *Fixed point properties for semigroup of nonexpansive mappings on Fréchet spaces*, Nonlinear Anal., **70**(2009), 3837-3841.

[17] A.T. Lau, J.C.S. Wong, *Invariant subspaces for algebras of linear operators and amenable locally compact groups*, Proc. Amer. Math. Soc., **102**(1988), 581-586.

[18] M.S. Monfared, *Character amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc., **144**(2008), no. 3, 697-706.

[19] M.S. Monfared, *Extensions and isomorphims for the generalized Fourier algebras of a locally compact group*, J. Funct. Anal., **198**(2003), 413-444.

[20] R. Nasr-Isfahani, M. Nemati, *Character pseudo-amenability of Banach algebras*, Colloq. Math., **132**(2013), 177-193.

[21] R. Nasr-Isfahani, S. Soltani Renani, *Character contractibility of Banach algebras and homological properties of Banach modules,* Studia Math., **202**(2011), no. 3, 205-225.

[22] J.P. Pier, *Amenable Banach Algebras*, Pitman Research Notes in Mathematics Series, Vol. 172, Longman Scientific and Technical, Harlow, 1988.

[23] W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, Inc., New York, 1962.

[24] S. Saeidi, *Ergodic properties of a particular amenable semigroup of mappings in a Banach space*, Fixed Point Theory, **10**(2009), 321-328.

[25] J.C.S. Wong, *Topological invariant means on locally compact groups and fixed points*, Proc. Amer. Math. Soc., **27**(1971), 572-578.

[26] J.C.S. Wong, *On the semigroup of probability measures of a locally compact semigroup*, Canad. Math. Bull., **30**(1987), 142-146.

[27] J.C.S. Wong, *On the semigroup of probability measures of a locally compact semigroup II*, Canad. Math. Bull., **30**(1987), 273-281.

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