BIFURCATION INDEX FOR ACYCLIC MAPPINGS
OF ANR-S

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Abstract. The notion of bifurcation index and some of its consequences were studied in [5]. There
was considered the case of compact mappings on arbitrary ANR-s. In this paper we generalize the
above mentioned results for compact acyclic mappings.

Key Words and Phrases: ANR-s, fixed point index, bifurcation index, acyclic mappings.

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1. Introduction

The bifurcation theory is very useful in the qualitative theory of differential equations
(compare [1]). The key to do it is the topological bifurcation index and its properties
(see [5]).

In the case of multivalued mappings above problems were considered only for maps
of open subsets of normed spaces (see [4], [6]).

We shall show that is also possible for multivalued mappings of arbitrary absolute
neighbourhood retracts (ANR-spaces). We are doing this by recalling preliminary
notions from geometrical topology. Then in section 3 we consider acyclic mappings
and the fixed point index for acyclic mappings. Finally in section 4 the notion of the
bifurcation index and its properties for acyclic mappings of ANR-s are presented.

2. Preliminaries

In this paper, we assumed that all topological spaces are metric and all single-
valued mappings are continuous. Let TOP be the category of topological spaces and
continuous mappings. Moreover we denote by V the category of graded vector spaces
over the field of rational numbers Q and linear maps.

By \( H : \text{TOP} \to V \) we understand the Čech homology functor with compact carries
and coefficients in \( Q \). Then for every metric spaces \( X, Y \) we have \( H(X) = \{ \text{H}_q(X) \}_{q \geq 0} \)
a graded vector space in \( V \), and for any \( f : X \to Y \) we have induced map \( f_* = \{ f_*^q \} : \)
\( H(X) \to H(Y) \), where \( f_*^q : \text{H}_q(X) \to \text{H}_q(Y) \) is a linear map for every \( q \geq 0 \). For the
properties of, \( H \) we recomended, [3, p. 27-35].

A compact nonempty set \( X \) is caled acyclic if:
Definition 2.1. A map $p : \Gamma \to X$ is called a Vietoris map if:

(i) $p$ is proper, i.e., $f^{-1}(K)$ is compact for every compact set $K \subset X$,
(ii) $p^{-1}(x)$ is acyclic for every $x \in X$.

Theorem 2.2. (Vietoris) (see e.g. [3]) If $p : \Gamma \to X$ is a Vietoris map, then the induced linear map $p_* : H(\Gamma) \to H(X)$ is an isomorphism, i.e., for every $q \geq 0$ the linear map $p_*^q : H_q(\Gamma) \to H_q(X)$ is a linear isomorphism. For more properties of Vietoris mappings, see e.g. [3].

Definition 2.3. Recall that a space $X$ is an absolute neighbourhood retract ($X \in \text{ANR}$) if for every continuous map $f : A \to X$ where $A$ is closed subset of metric space $Y$ there exists $\tilde{f} : U \to X$, to be a continuous extension of $f$ onto an open $U \subset Y$ such that $A \subset U$.

Remark 2.4. It is well known that $X \in \text{ANR}$ if and only if $X$ is homeomorphic to a retract of an open set $U$ in some normed space compare [3, p. 5-9].

Definition 2.5. A multivalued mapping $\varphi : X \to Y$ is called acyclic if:

(i) $\varphi(x)$ is an acyclic set, for every $x \in X$,
(ii) for every open subset $V \subset Y$, the set $\varphi^{-1}(V) = \{x \in X, \varphi(x) \subset V\}$ is open subset of $X$, i.e., $\varphi$ is u.s.c.
(iii) $\varphi$ is a compact map, i.e., the closure $\overline{\varphi(X)}$ of $\varphi(X)$ is compact.

If $\varphi : X \rightrightarrows Y$, then by $\text{Fix}(\varphi)$ we denote the set $\{x \in X, x \in \varphi(x)\}$. The set $\text{Fix}(\varphi)$ is called the set of all fixed points of $\varphi$.

For given $\varphi : X \rightrightarrows X$ by $\Gamma_{\varphi} = \{(x, \varphi(x)), x \in X\}$ we shall denote a graph of $\varphi$.

Then we have two natural projections:

$p_{\varphi}(x, y) = x, q_{\varphi}(x, y) = y.$

Observe that, if $\varphi$ is acyclic, then $p_{\varphi}$ is a Vietoris map.

Definition 2.6. Let $\varphi : X \rightrightarrows X$ be an acyclic map. We define the induced mapping $\varphi_* : H(X) \to H(Y)$, by putting: $\varphi_* = q_{\varphi_*} \circ p_{\varphi_*}^{-1}$. In what follows by $L(\varphi) = L(\varphi_*)$ we shall denote the Lefschetz number of $\varphi$, compare [5, p. 418-423].

3. Fixed point index

The fixed point index theory for multivalued mappings was studied by several authors (see [3] and references there in). Below we recall the notion of fixed point index for acyclic mappings following [2]. We shall formulate only these properties which are needed in bifurcation problems. For more details see ([2], [3]).

Let $X \in \text{ANR}, U \subset X$ open and $\varphi : X \rightrightarrows X$ an acyclic map such that $\text{Fix}(\varphi) \cap \partial U = \emptyset$ ($\partial U$ is the boundary of $U$ in $X$). Then the fixed point index $\text{Ind}(\varphi, U)$ of $\varphi$ with respect to $U$ is well defined (see [2] and [3]).

Below, we shall list some properties of the index needed in what follows:

(3.1) (Existence). If $\text{Ind}(\varphi, U) \neq 0$ then $\text{Fix}(\varphi) \cap U \neq \emptyset$.

(3.2) (Additivity). If $V_1$, $V_2$ are open subsets of $X$ such that $V_1 \cap V_2 = \emptyset$ and $\text{Fix}(\varphi) \cap V_1$, $\text{Fix}(\varphi) \cap V_2$ are compact sets, then

$$\text{Ind}(\varphi, V_1 \cup V_2) = \text{Ind}(\varphi, V_1) + \text{Ind}(\varphi, V_2).$$
(3.3) (Homotopy). If $\chi : [0, 1] \times X \to X$, $\chi(t, \cdot) : X \to X$ is an acyclic mapping such that $\text{Fix}(\chi(t, \cdot)) \cap \partial U = \emptyset$ for every $t \in [0, 1]$. Then
\[
\text{Ind}(\chi(0, \cdot), U) = \text{Ind}(\chi(1, \cdot), U).
\]

(3.4) (Excision) If $\text{Fix}(\varphi) \cap W \subset V \subset W$ is compact, then $\text{Ind}(\varphi, V) = \text{Ind}(\varphi, W)$.

(3.5) (Normalization) If $U = X$, then $\text{Ind}(\varphi, U) = L(\varphi)$.

Remark 3.6. If $X$ is a contractible space then for every acyclic mapping $\varphi : X \to X$ we have $L(\varphi) = 1$. 

4. Bifurcation index

Numerous questions, ranging from nonlinear Sturm-Liouville problems in ordinary differential equations to eigenvalue problems for elliptical partial differential equations, reduce in a natural way to the study of solutions of equations of the form
\[
x = F(x, \lambda)
\]
where $x$ is an element of a Banach space $E$, $\lambda$ is a real parameter, and $F : E \times \mathbb{R} \to E$ is a completely continuous mapping with $F(0, \lambda) = 0$ for each $\lambda \in \mathbb{R}$.

The above problem was considered in [5, p. 338-350].

In this paper we are considering the following inclusion
\[
x \in \varphi(x, \lambda),
\]
where $\varphi : X \times J \to X$ is an acyclic mapping, $J = (\alpha, \beta)$ is an open interval, $X$ is an ANR with some based point $x_0$ such that $x_0 \in \varphi(x_0, \lambda)$ for every $\lambda \in J$.

The set $T = \{p_0\} \times J$ is called the set of trivial solutions. If $t \in J$ then we let $\varphi_t = \varphi(\cdot, t) : X \to X$. Moreover we denote $B_r = B(p_0, r)$ and $K_r = K(p_0, r)$ to be open and closed balls in $X$ respectively. If $U \subset X \times J$, then $U_t = \{x \in X, (x, t) \in U\}$.

With above assumptions we can formulate the definition of a singular set.

Definition 4.1. A set $\Lambda \subset J$ is called a singular set associated with $\varphi$ if:

(i) $\Lambda$ is finite or countable,
(ii) for any closed $J_0 = [a, b] \subset J$, the intersection $\Lambda \cap J_0$ is finite,
(iii) for any closed $J_0 = [a, b] \subset J \setminus \Lambda$, there is a ball $B_{\epsilon} = B(p_0, \epsilon)$, with a sufficiently small radius $\epsilon = \epsilon(J_0) > 0$, for which
\[
\text{Fix}(\varphi_t) \cap B_{\epsilon} = \text{Fix}(\varphi_t|_{B_{\epsilon}}) = \{p_0\} \text{ for all } t \in J_0.
\]

A map $\varphi$ is said to be allowable if it is equipped with a singular set $\Lambda = \Lambda_{\varphi}$.

Proposition 4.2. Let $\lambda_0 \in \Lambda_{\varphi}$, and $J_0 = (a, b) \subset J$ be such that $\Lambda_{\varphi} \cap J_0 = \{\lambda_0\}$. Then:

(4.2.1) for each $t \in J_0 \setminus \{\lambda_0\}$, there exists an $r(t) > 0$ such that $\text{Fix}(\varphi_t|_{B_{r(t)}}) = \{p_0\}$,

(4.2.2) $\text{Ind}(F_t, B_{r(t)})$ is constant in two open intervals $(a, \lambda_0)$, $(\lambda_0, b)$.

See (3.3).

Definition 4.3. A point $(p_0, \lambda_0)$ is called a bifurcation point of following inclusion:
\[
x \in \varphi_t(x) = \varphi(x, t),
\]
if for every open neighbourhood $U$ of $(p_0, \lambda_0)$ in $X \times J$ there exists $(x, t) \in U$ such that $x \in \text{Fix}(\varphi_t) \setminus \{p_0\}$. The set of all bifurcation points is denoted by $B_{\varphi}$.

It is easy to observe the following two corollaries:
Corollary 4.4. If \((p_0, \lambda_0) \in \mathbb{B}_\varphi\), then \(\lambda_0 \in \Lambda_\varphi\).

Corollary 4.5. Let \(N_\varphi = \{(x,t) \in X \times J \mid x \in \text{Fix}(\varphi_t) \setminus \{p_0\}\}\) be the set of nontrivial solutions. With the above designations we have \(\mathbb{B}_\varphi = T_\varphi \cap N_\varphi\); it means the only trivial solutions in \(N_\varphi\) are bifurcation points.

Now we are able to prove proposition:

Proposition 4.6. If \(\lambda_0 \in \Lambda_\varphi\) and \((p_0, \lambda_0) \notin \mathbb{B}_\varphi\), then the index \(\text{Ind}(\varphi_t, B_{r(t)})\) is constant in some neighbourhood of \(\lambda_0\).

Proof. Indeed, if \((p_0, \lambda_0) \notin \mathbb{B}_\varphi\), then there exists neighbourhood \(V\) of \((p_0, \lambda_0)\) in \(X \times J\) we have \(V \cap N_\varphi = \emptyset\). By (4.2) and the homotopy property of the fixed point index, our assertion follows.

Presently we are ready to formulate main definition of this paper.

Definition 4.7. Assume that \(\lambda_0 \in \Lambda_\varphi\), \(J_0 = (a,b) \subset J\) and \(r(t) > 0\) are as in (4.2). Choose any \(t_1 \in (a, \lambda_0)\), \(t_2 \in (\lambda_0, b)\). We define the bifurcation index \(\Xi(\lambda_0)\) of \(\varphi\) at \(\lambda_0\) by
\[
\Xi(\lambda_0) = \text{Ind}(\varphi_{t_1}, B_{r(t_1)}) - \text{Ind}(\varphi_{t_2}, B_{r(t_2)}).
\]

In view of [4.2], it is obvious that \(\Xi(\lambda_0)\) does not depends on the choice of \(t_1, t_2\). Moreover, the radius \(r(t_i)\), \(i = 1,2\), in the definition (4.7) can be changed by any \(\theta > 0\) such that \(\text{Fix}(\varphi_{t_i}|B_{\theta}) = \{p_0\}\), \(i = 1,2\).

The following theorem is a consequence of above reasoning:

Theorem 4.7. (Local bifurcation). If \(\lambda_0 \in \Lambda_\varphi\) and \(\Theta(\lambda_0) \neq 0\), then \((p_0, \lambda_0)\) is a bifurcation point.

It is an immediate consequence of (4.6) and (4.7).

Finally, let us add that further properties and applications of the bifurcation index, defined in this paper, will be treated in our next work.

References


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