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# BIFURCATION INDEX FOR ACYCLIC MAPPINGS OF ANR-S

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**Abstract.** The notion of bifurcation index and some of its consequences were studied in [5]. There was considered the case of compact mappings on arbitrary ANR-s. In this paper we generalize the above mentioned results for compact acyclic mappings.

**Key Words and Phrases**: ANR-s, fixed point index, bifurcation index, acyclic mappings.
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## 1. Introduction

The bifurcation theory is very useful in the qualitive theory of differential equations (compare [1]). The key to do it is the topological bifurcation index and its properties (see [5]).

In the case of multivalued mappings above problems were considered only for maps of open subsets of normed spaces (see [4], [6]).

We shall show that is also possible for multivalued mappings of arbitrary absolute neigbourhood retracts (ANR-spaces). We are doing this by recalling preliminary notions from geometrical topology. Then in section 3 we consider acyclic mappings and the fixed point index for acyclic mappings. Finally in section 4 the notion of the bifurcation index and its properties for acyclic mappings of ANR-s are presented.

## 2. Preliminaries

In this paper, we assumed that all topological spaces are metric and all single-valued mappings are continous. Let $TOP$ be the category of topological spaces and continous mappings. Moreover we denote by $\mathcal{V}$ the category of graded vector spaces over the field of rational numbers $\mathbb{Q}$ and linear maps.

By $H : TOP \to \mathcal{V}$ we understand the $\tilde{C}$ech homology functor with compact carries and coefficients in $\mathbb{Q}$. Then for every metric spaces $\mathbb{X}, \mathbb{Y}$ we have $H(\mathbb{X}) = \{H_q(\mathbb{X})\}_{q\geq 0}$ a graded vector space in $\mathcal{V}$, and for any $f : \mathbb{X} \to \mathbb{Y}$ we have induced map $f_* = \{f_{*q}\} : H(\mathbb{X}) \to H(\mathbb{Y})$, where $f_{*q} : H_q(\mathbb{X}) \to H_q(\mathbb{Y})$ is a linear map for every $q \geq 0$. For the properties of, $H$ we recomended, [3, p. 27-35].

A compact nonempty set $\mathbb{X}$ is caled acyclic if:

(i) $H_q(\mathbb{X}) = 0$, for every $q > 0$,
(ii) $H_0(\mathbb{X}) = \mathbb{Q}$.

**Definition 2.1.** A map $p : \Gamma \to \mathbb{X}$ is called a Vietoris map if:

(i) p is proper, i.e, $f^{-1}(K)$ is compact for every compact set $K \subset \mathbb{X}$,
(ii) $p^{-1}(x)$ is acyclic for every $x \in \mathbb{X}$.

**Theorem 2.2.** (Vietoris) (see e.g. [3]) *If $p : \Gamma \to \mathbb{X}$ is a Vietoris map, then the induced linear map $p_* : H(\Gamma) \to H(\mathbb{X})$ is an isomorphism, i.e., for every $q \geq 0$ the linear map $p_{*q} : H_q(\Gamma) \to H_q(\mathbb{X})$ is a linear isomorphism. For more proporties of Vietoris mappings, see e.g.*[3].

**Definition 2.3.** Recall that a space $\mathbb{X}$ is an absolute neigbourhood retract ($\mathbb{X} \in ANR$) if for every continous map $f : A \to \mathbb{X}$ where $A$ is closed subset of metric space $\mathbb{Y}$ there exists $\tilde{f} : U \to \mathbb{X}$, to be a continous extension of $f$ onto an open $U \subset Y$ such that $A \subset U$.

**Remark 2.4.** It is well known that $\mathbb{X} \in ANR$ if and only if $\mathbb{X}$ is homeomorphic to a retract of an open set $U$ in some normed space compare [3, p. 5-9].

**Definition 2.5.** A multivalued mapping $\varphi : \mathbb{X} \multimap \mathbb{Y}$ is called acyclic if:

(i) $\varphi(x)$ is an acyclic set, for every $x \in \mathbb{X}$,
(ii) for every open subset $V \subset Y$, the set $\varphi^{-1}(V) = \{x \in \mathbb{X}, \varphi(x) \subset V\}$ is open subset of $\mathbb{X}$, i.e., $\varphi$ is u.s.c.
(iii) $\varphi$ is a compact map, i.e., the closure $\overline{\varphi(\mathbb{X})}$ of $\varphi(\mathbb{X})$ is compact.

If $\varphi : \mathbb{X} \multimap \mathbb{X}$, then by $Fix(\varphi)$ we denote the set $\{x \in X,\ x \in \varphi(x)\}$. The set $Fix(\varphi)$ is called the set of all fixed points of $\varphi$.

For given $\varphi : \mathbb{X} \multimap \mathbb{X}$ by $\Gamma_\varphi = \{(x, \varphi(x)),\ x \in \mathbb{X}\}$ we shall denote a graph of $\varphi$. Then we have two natural projections:

$$p_\varphi(x, y) = x,\ q_\varphi(x, y) = y.$$

Observe that, if $\varphi$ is acyclic, then $p_\varphi$ is a Vietoris map.

**Definition 2.6.** Let $\varphi : \mathbb{X} \multimap \mathbb{X}$ be an acyclic map. We define the induced mapping $\varphi_* : H(\mathbb{X}) \to H(\mathbb{Y})$, by putting: $\varphi_* = q_{\varphi *} \circ p_{\varphi *}^{-1}$. In what follows by $L(\varphi) = L(\varphi_*)$ we shall denote the *Lefschetz number* of $\varphi$, compare [5, p. 418-423].

## 3. Fixed point index

The fixed point index theory for multivalued mappings was studied by several authors (see [3] and references there in). Below we recall the notion of fixed point index for acyclic mappings following [2]. We shall formulate only these properties which are needed in bifurcation problems. For more details see ([2], [3]).

Let $\mathbb{X} \in$ ANR, $U \subset \mathbb{X}$ open and $\varphi : \mathbb{X} \multimap \mathbb{X}$ an acyclic map such that $Fix(\varphi) \cap \partial U = \emptyset$ ($\partial U$ is the boundary of $U$ in $\mathbb{X}$). Then the fixed point index $Ind(\varphi, U)$ of $\varphi$ with respect to $U$ is well defined (see [2] and [3]).

Below, we shall list some properties of the index needed in what follows:

(3.1) (Existence). If $Ind(\varphi, U) \neq 0$ then $Fix(\varphi) \cap U \neq \emptyset$.
(3.2) (Additivity). If $V_1,\ V_2$ are open subsets of $\mathbb{X}$ such that $V_1 \cap V_2 = \emptyset$ and $Fix(\varphi) \cap V_1,\ Fix(\varphi) \cap V_2$ are compact sets, then

$$Ind(\varphi, V_1 \cup V_2) = Ind(\varphi, V_1) + Ind(\varphi, V_2).$$

(3.3) (Homotopy). If $\chi : [0,1] \times \mathbb{X} \multimap \mathbb{X}$, $\chi(t,\cdot) : \mathbb{X} \multimap \mathbb{X}$ is an acyclic mapping such that $Fix(\chi(t,\cdot)) \cap \partial U = \emptyset$ for every $t \in [0,1]$. Then

$$Ind(\chi(0,\cdot), U) = Ind(\chi(1,\cdot), U).$$

(3.4) (Excision) If $Fix(\varphi) \cap W \subset V \subset W$ is compact, then $Ind(\varphi, V) = Ind(\varphi, W)$.

(3.5) (Normalization) If $U = \mathbb{X}$, then $Ind(\varphi, U) = L(\varphi)$.

**Remark 3.6.** If $\mathbb{X}$ is a contractible space then for every acyclic mapping $\varphi : \mathbb{X} \multimap \mathbb{X}$ we have $L(\varphi) = 1$.

## 4. Bifurcation index

Numerous questions, ranging from nonlinear Sturm-Liouville problems in ordinary differential equations to eigenvalue problems for eliptical partial differential equations, reduce in a natural way to the study of solutions of equations of the form

$$x = F(x, \lambda)$$

where $x$ is an element of a Banach space $E$, $\lambda$ is a real parameter, and $F : E \times \mathbb{R} \to E$ is a completely continuous mapping with $F(0,\lambda) = 0$ for each $\lambda \in \mathbb{R}$.

The above problem was considered in [5, p. 338-350].

In this paper we are considering the following inclusion

$$x \in \varphi(x, \lambda),$$

where $\varphi : \mathbb{X} \times J \multimap \mathbb{X}$ is an acyclic mapping, $J = (\alpha, \beta)$ is an open interval, $\mathbb{X}$ is an ANR with some based point $x_0$ such that $x_0 \in \varphi(x_0, \lambda)$ for every $\lambda \in J$. The set $\mathcal{T} = \{p_0\} \times J$ is called *the set of trivial solutions.* If $t \in J$ then we let $\varphi_t = \varphi(\cdot, t) : \mathbb{X} \multimap \mathbb{X}$. Moreover we denote $B_r = B(p_0, r)$ and $K_r = K(p_0, r)$ to be open and closed balls in $\mathbb{X}$ respectively. If $U \subset \mathbb{X} \times J$, then $U_t = \{x \in \mathbb{X}, (x,t) \in U\}$.

With above assumptions we can formulate the definition of a *singular set.*

**Definition 4.1.** A set $\Lambda \subset J$ is called a *singular set associated with* $\varphi$ if:

(i) $\Lambda$ is finite or countable,

(ii) for any closed $J_0 = [a,b] \subset J$, the intersection $\Lambda \cap J_0$ is finite,

(iii) for any closed $J_0 = [a,b] \subset J \setminus \Lambda$, there is a ball $B_\varepsilon = B(p_0, \varepsilon)$, with a sufficiently small radius $\varepsilon = \varepsilon(J_0) > 0$, for which

$$Fix(\varphi_t) \cap B_\varepsilon = Fix(\varphi_t | B_\varepsilon) = \{p_0\} \text{ for all } t \in J_0.$$

A map $\varphi$ is said to be *allowable* if it is equipped with a singular set $\Lambda = \Lambda_\varphi$.

**Proposition 4.2.** *Let $\lambda_0 \in \Lambda_\varphi$, and $J_0 = (a,b) \subset J$ be such that $\Lambda_\varphi \cap J_0 = \{\lambda_0\}$. Then:*

(4.2.1) *for each $t \in J_0 \setminus \{\lambda_0\}$, there exists an $r(t) > 0$ such that $Fix(\varphi_t | B_{r(t)}) = \{p_0\}$,*

(4.2.2) *$Ind(F_t, B_{r(t)})$ is constant in two open intervals $(a, \lambda_0)$, $(\lambda_0, b)$.*

See (3.3).

**Definition 4.3.** A point $(p_0, \lambda_0)$ is called a bifurcation point of following inclusion:

$$x \in \varphi_t(x) = \varphi(x, t),$$

if for every open neigbourhood $U$ of $(p_0, \lambda_0)$ in $\mathbb{X} \times J$ there exists $(x,t) \in U$ such that $x \in Fix(\varphi_t) \setminus \{p_0\}$. The set of all bifurcation points is denoted by $\mathbb{B}_\varphi$.

It is easy to observe the following two corollaries:

**Corollary 4.4.** *If* $(p_0, \lambda_0) \in \mathbb{B}_\varphi$, *then* $\lambda_0 \in \Lambda_\varphi$.

**Corollary 4.5.** *Let* $\mathcal{N}_\varphi = \{(x,t) \in \mathbb{X} \times J \mid x \in Fix(\varphi_t) \setminus \{p_0\}\}$ *be the set of nontrivial solutions. With the above designations we have* $\mathbb{B}_\varphi = \mathcal{T}_\varphi \cap \overline{\mathcal{N}_\varphi}$*; it means the only trivial solutions in* $\overline{\mathcal{N}_\varphi}$ *are bifurcation points.*

Now we are able to prove proposition:

**Proposition 4.6.** *If* $\lambda_0 \in \Lambda_\varphi$ *and* $(p_0, \lambda_0) \notin \mathbb{B}_\varphi$, *then the index* $Ind(\varphi_t, B_{r(t)})$ *is constant in some neigbourhood of* $\lambda_0$.

*Proof.* Indeed, if $(p_0, \lambda_0) \notin \mathbb{B}_\varphi$, then there exists neigbourhood $V$ of $(p_0, \lambda_0)$ in $\mathbb{X} \times J$ we have $V \cap \mathcal{N}_\varphi = \emptyset$. By (4.2) and the homotopy property of the fixed point index, our assertion follows.

Presently we are ready to formulate main definition of this paper.

**Definition 4.7.** Assume that $\lambda_0 \in \Lambda_\varphi$, $J_0 = (a, b) \subset J$ and $r(t) > 0$ are as in (4.2). Choose any $t_1 \in (a, \lambda_0)$, $t_2 \in (\lambda_0, b)$. We define the *bifurcation index* $\Xi(\lambda_0)$ of $\varphi$ at $\lambda_0$ by

$$\Xi(\lambda_0) = Ind(\varphi_{t_1}, B_{r(t_1)}) - Ind(\varphi_{t_2}, B_{r(t_2)}).$$

In view of [4.2], it is obvious that $\Xi(\lambda_0)$ does not depends on the choice of $t_1, t_2$. Moreover, the radius $r(t_i)$, $i = 1, 2$, in the definition (4.7) can be changed by any $\theta > 0$ such that $Fix(\varphi_{t_i} | B_\theta) = \{p_0\}$, $i = 1, 2$.

The following theorem is a consequence of above reasoning:

**Theorem 4.7.** (Local bifurcation). *If* $\lambda_0 \in \Lambda_\varphi$ *and* $\Theta(\lambda_0) \neq 0$, *then* $(p_0, \lambda_0)$ *is a bifurcation point.*

It is an immediate consequence of (4.6) and (4.7).

Finally, let us add that further properties and applications of the bifurcation index, defined in this paper, will be treated in our next work.

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