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SOME FIXED POINT THEOREMS OF A KRASNOSEL'SKII TYPE AND APPLICATION TO DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we establish some fixed point results of Krasnosel'skii type fixed point theorem for the sum of $B + G$, where B is expansive linear operator and G is a weakly continuous map or weakly-weakly upper semi-continuous multivalued operator. Finally, our results are used to prove the existence of solution for multivalued Dirihlet problem in reflexive Banach spaces.
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1. Introduction

Many problems arising from diverse areas of natural science, when modeled from a mathematical point of view, involve the study of solutions of nonlinear differential equations or inclusions of the respective forms,

$$Bu + Au = u, \qquad u \in M, \tag{1}$$

or

$$u \in Bu + G(u), \qquad u \in M, \tag{2}$$

where G is a multivalued map and M is a closed convex subset of a Banach space X. In particular, many integral equations and inclusions can be formulated in terms of (1) or (2); see, for example [21]. In 1958, Krasnosel'skii [19] established that the equation(1) has a solution in M where A and B satisfy:

(i) $Ax + By \in M$ for all $x, y \in M$.
(ii) A is continuous on M and $\overline{A(M)}$ is a compact set in X.
(iii) B is a $k-$contraction on X.

That result combined the Banach contraction principle and Schauder's fixed point theorem. The existence of fixed points for the sum of two operators has attracted tremendous interest, and their applications are frequent in nonlinear analysis. Many improvements of Krasnosel'skii's theorem have been established in the literature in the course of time by modifying the above assumptions; see, for example [2, 3, 4, 7, 8, 9, 10, 11, 12, 20].

Fixed point theory for multivalued mappings is an important topic of set-valued analysis. Several well-known fixed point theorems of single-valued mappings such as Banach's and Schauder's have been extended to multivalued mappings in Banach spaces; see [1, 14].

Recently, a multivalued analogue of Krasnosel'skii's fixed point theorem was obtained by Boriceanu [5], Graef *et al* [15] and [16].

Very recently, several authors extended the classical Krasnoselskii fixed point theorem [3, 9, 11, 12]. Our goal in this work is to give some single and multivalued version of a Krasnoselskii type fixed point theorem and application to partial differential inclusion.

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. Let $(E, |\cdot|)$ be a locally convex space.

Denote by

$\mathcal{P}(E) = \{Y \subset E\colon\ Y \neq \emptyset\}$, $\mathcal{P}_{cl}(E) = \{Y \in \mathcal{P}(E)\colon\ Y \text{ closed}\}$,
$\mathcal{P}_b(E) = \{Y \in \mathcal{P}(E)\colon\ Y \text{ bounded}\}$, $\mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E)\colon\ Y \text{ convex }\}$,
$\mathcal{P}_{cp}(E) = \{Y \in \mathcal{P}(E)\colon\ Y \text{ compact}\}$, and
$\mathcal{P}_{wk,cp}(E) = \{Y \in \mathcal{P}(E)\colon\ Y \text{weakly compact}\}$.

Let X and Y be two locally convex spaces and $G : X \to \mathcal{P}_{cl}(Y)$ be a multi-valued map. A single-valued map $g : X \to Y$ is said to be a selection of G and we write $g \subset G$ whenever $g(x) \in G(x)$ for every $x \in X$.

G is called *upper semi-continuous (u.s.c. for short)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq Y$. That is, if the set $G^{-1}(V) = \{x \in X,\ G(x) \cap V \neq \emptyset\}$ is closed for any closed set V in Y. Equivalently, G is *u.s.c.* if the set $G^{+1}(V) = \{x \in X,\ G(x) \subset V\}$ is open for any open set V in Y.

Assume now X and Y are normed linear spaces. We consider as locally convex topological Hausdorff spaces with their weak topologies $\sigma(X, X^*)$ and $\sigma(Y, Y^*)$, respectively.

The multifunction $F : X \to \mathcal{P}(Y)$ is weakly-weakly upper semicontinuous ($w.-w.u.s.c$) on X if for every weakly closed set $M \subseteq Y$ the set $F^-(M) = \{x \in X : F(x) \cap M \neq \emptyset\}$ is sequentially weakly closed in X. We say that $F : X \to \mathcal{P}(Y)$ is weakly-strongly upper semicontinuous ($w.-s.u.s.c$) on X if for every weakly closed set $M \subseteq Y$ the set $F^-(M)$ is closed (with the norm topology) in X.

Theorem 2.1. [18] A multivalued map $F : X \to \mathcal{P}(Y)$ is $w.-w.u.s.c$ (or $w.-s.u.s.c$) on X if for every sequence $\{x_n\}$ of X weakly converging (converging in the norm topology of X resp.) to x and every sequence $\{y_n\}$ of Y with $y_n \in F(x)$ for $n \in \mathbb{N}$, there exists a subsequence of $\{y_n\}$ weakly converging to any $y \in F(x)$. If furthermore, there is a weakly compact set $C \subset Y$ such that $F(x) \subset C$ for $x \in X$, then the above conditions are also necessary for F to be $w.-w.u.s.c.$ on X.

Theorem 2.2. [18] Let X be a Banach space and M a weakly compact convex subset of X. Suppose that $F : M \to \mathcal{P}_{wk,cl,cv}(M)$ is a $w. - w.u.s.c.$ multivalued operator, then there exists $x \in M$ such that $x \in F(x)$.
Theorem 2.3. [18] Let X be a Banach space and M a weakly compact convex subset of X. Suppose that $F : M \to M$ is a weakly continuous, then there exists $x \in M$ such that $x \in F(x)$.

3. Expansive Krasnosel'skii-type fixed point theorem

In this section, we will provide some expansive Krasnosel'skii-type fixed point theorem. Our result extends some results presented in [22].
Definition 3.1. Let (X, d) be a metric space and M be a subset of X. The mapping $B : M \to X$ is said to be expansive, if there exists a constant $k > 1$ such that

$$d(B(x), B(y)) \geq kd(x, y) \quad \text{for all } x, y \in M.$$

Now, we present some auxiliary results of this subsection.
Theorem 3.1. [22] Let X be a complete metric space and M be a closed subset of X. Assume $B : M \to X$ is expansive and $M \subseteq B(M)$. Then there exists a unique point $x \in M$ such that $x = B(x)$.
Lemma 3.1. Let $B : X \to X$ be a map such that B^n($n-$power)is an expansive for some $n \in \mathbb{N}$. Assume further that there exist a closed subset M of X such that M is contained in $B(M)$. Then there exists a unique fixed point of B.
Proof. Since B^n is expansive map and $M \subseteq B^n(M)$. From theorem 3 there exists unique fixed point of B^n. Let $x \in M$ be a fixed point of B^n. Using the fact that B^n is expansive map, then there exist $k > 1$ such that

$$d(B^n(x), B^n(y)) \geq kd(x, y) \quad \text{for all } x, y \in M.$$

Hence

$$d(x, B(x)) = d(B^n(x), B^{n+1}(x)) \geq kd(x, B(x)) \Rightarrow d(x, B(x)) = 0.$$

Then B has a unique fixed point in M.
Lemma 3.2. [22]Let X be a linear normed space and $M \subseteq X$. Assume the mapping $B : M \to X$ is expansive with constant $k > 1$. Then the inverse of $I - B : M \to (I - B)(M)$ exists and

$$|(I - B)^{-1}(x) - (I - B)^{-1}(y)| \leq \frac{1}{k-1}|x - y|, \ x, y \in (I - B)(M).$$

Theorem 3.2. Let $(X, |\cdot|)$ be a Banach space and M be a nonempty weakly compact convex subset of X. Assume that $A : M \to X$ is weakly continuous and $B \in L(X)$ satisfy

($\mathcal{H}_1$) $\|B^p\| > 1$ for some $p \in \mathbb{N}$,
($\mathcal{H}_2$) for each $x, y \in M$ such that

$$x = B(x) + A(y) \Rightarrow x \in M.$$

Then there exists $y \in M$ such that $y = By + A(y)$.
Proof. Let $y \in M$. Let $F_y : M \to X$ be a operator defined by

$$F_y(x) = B(x) + A(y), \quad x \in M.$$

From theorem 3.1 there exist unique $x(y) \in M$ such that

$$x(y) = B(x(y)) + A(y).$$

By $(\mathcal{H}_1)$, we can prove that $I-B$ is invertible and $(I-B)^{-1} = (I-B^p)^{-1}\sum_{k=0}^{k=p-1} B^k$. This operator is well defined and $(I-B)^{-1} \in L(X)$. Moreover, $(I-B)^{-1}$ is weakly continuous (see, Theorem 3.10 [6]). Let us define $N : M \to M$ by

$$y \to N(y) = (I-B)^{-1}A(y).$$

Let $x \in M$ and $h = (I-B)^{-1}(A(x))$. Then

$$h = (I-B)^{-1}(A(x)) \Rightarrow h = B(h) + A(x),$$

and thus $(\mathcal{H}_2)$ implies that $h \in M$. Let $\{y_n : n \in \mathbb{N}\} \subset M$ be a sequence converging weakly to y in M we show that $N(y_n)$ converge weakly to $N(y)$.
Set $x_n = (I-B)^{-1}A(y_n)$, then $(I-B)(x_n) = A(y_n),\ n \in \mathbb{N}$.

Since M is weakly compact, there exists a subsequence of $\{x_n\}$ converging weakly for some $x \in M$. Then $(I-B)(x_n)$ converges weakly to $(I-B)(x)$. Hence there exists a subsequence of y_n converging weakly to $(I-B)(x)$. Then $N(y_n)$ converge weakly to $N(y)$. Hence from Theorem 2.3, there exists $y \in M$ such that $y = (I-B)^{-1}A(y)$, and we deduce that $B+G$ has a fixed point in M.

Now we are ready to state our results of this part.

Theorem 3.3. Let X be a Banach space, M be a weakly compact convex subset of X, $A : M \to X$ be an weakly continuous map and $B \in L(X)$ be a linear continuous operator single-valued mapping. Assume that G and B satisfy the following hypotheses:

- $(\mathcal{C}_1)$ $\|B\| \geq k > 1$ is an expansive mapping.
- $(\mathcal{C}_2)$ for each $x, y \in coA(M)$ such that

$$x = B(x) + A(y) \Rightarrow x \in coA(M).$$

Then the abstract equation $x = B(x) + A(x)$ has a solution.

Proof. Let $\widetilde{M} = \overline{co}A(M)$ be weakly compact convex. Now we prove only that $N(\widetilde{M}) \subseteq \widetilde{M}$, where N is defined in the proof of Theorem 3.2. Indeed, let $x \in N(\widetilde{M})$. Then there exists $y \in \widetilde{M}$ such that

$$x = N(y).$$

Hence

$$x = (I-B)^{-1}A(y) \Rightarrow x \in \overline{co}A(M).$$

Then

$$N(\widetilde{M}) \subseteq \widetilde{M}.$$

So, by Theorem 3.2, there exists $x \in X$ which is fixed point of N.

4. Krasnosel'skii-type fixed point theorem for $w.-w.u.s.c.$

In this section, we use Theorem 2 and Lemma 3 to obtain a multivalued version of the Krasnosel'skii theorem presented by Xiang and Yuan [22].

Theorem 4.1. Let $(X, |\cdot|)$ be a Banach space and M be a nonempty weakly compact convex subset of X. Assume that $G : M \to \mathcal{P}_{wcl,cv}(X)$ is $w.-w.u.s.c.$ and $B \in L(X)$ satisfy

($\mathcal{H}_1$) $\|B\| > 1$

($\mathcal{H}_2$) for each $x, y \in M$ such that

$$x \in B(x) + G(y) \Rightarrow x \in M.$$

Then there exists $y \in M$ such that $y \in By + G(y)$.

Proof. Let $y \in M$. Let $F_y : M \to \mathcal{P}(X)$ be the multivalued operator defined by

$$F_y(x) = B(x) + G(y), \quad x \in M.$$

Since $G(y) \in \mathcal{P}_{wk,cp,cv}(X)$ and G is $w-w-u.s.c.$. Then $F_y(x) \in \mathcal{P}_{wcp,cv}(X)$. By ($\mathcal{H}_2$) we have $F_y(M) \subseteq M$. Now we show that F_y is $w-w-u.s.c.$

Let $(x_n, y_n) \in Gr(F_y) = \{(x, z) \in M \times M : z \in F_y(x)\}$ be a sequence such that

$$y_n \in F_y(x_n),\ x_n \rightharpoonup x \text{and } y_n \rightharpoonup y_*.$$

Thus there exists $z_n \in G(y)$ such that

$$y_n = B(x_n) + z_n$$

Since $B \in L(X)$, then $B(x_n)$ converge weakly to $B(x)$(see, Theorem 3.10 [6]). this implies that $z_n \rightharpoonup y_* - B(x) \in G(y)$. Hence F_y has a weakly closed graph. By ($\mathcal{H}_2$) and theorem 2.1, we deduced that F_y is $w-w-u.s.c.$ From theorem 2.2, there exists $x(y) \in M$ such that

$$x(y) \in B(x(y)) + G(y).$$

By ($\mathcal{H}_1$), we can prove that $I - B$ is invertible and $(I - B)^{-1} \in L(X)$. Moreover, $(I - B)^{-1}$ is weakly continuous (see, Theorem 3.10 [6]).

Let us define $N : M \to \mathcal{P}_{wcl,cv}(M)$ by

$$y \to N(y) = (I - B)^{-1}G(y).$$

Since $G(\cdot) \in \mathcal{P}_{wcl,cv}(X)$ and $(I - B)^{-1} \in L(X)$, then $N(\cdot) \in \mathcal{P}_{wcl,cv}(X)$. Now we show that $N(\cdot)$ is $w.-w.u.s.c.$ Let $x \in M$ and $h \in (I - B)^{-1}(G(x))$. Then there exists $y \in G(x)$ such that

$$x = (I - B)^{-1}(y) \Rightarrow x = Bx + y \subseteq B(x) + G(y),$$

and thus ($\mathcal{H}_2$) implies that $x \in M$. Let $\{y_n : n \in \mathbb{N}\} \subset M$ be a sequence converging weakly to x in M and $y_n \in N(y),\ n \in \mathbb{N}$. Then there exists $x_n \in G(y)$ such that

$$(I - B)(y_n) = x_n,\ n \in \mathbb{N}.$$

Since G is $w.-w.u.s.c.$, there exists a subsequence of $\{x_n\}$ converging weakly for some $x \in G(y)$. Then $(I-B)^{-1}(x_n)$ converges weakly to $(I-B)^{-1}(x) \in N(y)$. Hence there exists a subsequence of y_n converging weakly to $(I - B)^{-1}(x)$. By Theorem 2.1, N is $w.-w.u.s.c.$ Hence from Theorem 2.2, there exists $y \in M$ such that $y \in (I-B)^{-1}G(y)$, and we deduce that $B + G$ has a fixed point in M.

Now, we can easily prove the next result.

Theorem 4.2. Let $(X, |\cdot|)$ be a Banach space and M be a nonempty weakly closed bounded convex subset of X. Assume that $G : M \to \mathcal{P}_{wcl,cv}(X)$ is $w.-w.u.s.c.$, that B satisfies $(\mathcal{H}_1)$- $(\mathcal{H}_2)$, and the condition

$(\mathcal{H}_3)$ $G(M)$ is weakly relatively compact and for each $x, y \in coG(M)$ such that
$$x \in B(x) + G(y) \Rightarrow x \in coG(M).$$

Then the operator $B + G$ has at least one fixed point.

Proof. Let $\widetilde{M} = \overline{co}G(M)$ be weakly compact convex. Now we prove only that $N(\widetilde{M}) \subseteq \widetilde{M}$, where N is defined in the proof of Theorem 4.1. Indeed, let $x \in N(\widetilde{M})$. Then there exists $y \in \widetilde{M}$ such that
$$x \in N(y).$$
Hence
$$x = (I - B)^{-1}z,\ z \in G(y) \Rightarrow x \in \overline{co}G(M).$$
Then
$$N(\widetilde{M}) \subseteq \widetilde{M}.$$
So, by Theorem 4.1, there exists $x \in X$ which is fixed point of N.

5. Application

In this section, we present some applications of above results. The first one is devoted to a Dirichlet problem with multivalued nonlinearities.

Let $\Omega \subset \mathbb{R}^m$, $m \geq 3$, be a bounded domain with $C^{1,1}$ boundary. Consider the following nonlinear multivalued Dirichlet problem:
$$\begin{cases} -(1+\lambda)\Delta u - \lambda u & \in & F(x, u(x)), & x \in \Omega, \\ u(x) & = & 0, & x \in \partial\Omega, \end{cases} \tag{3}$$
where $\lambda > 1$ and $F : \Omega \times \mathbb{R}^m \to \mathcal{P}_{wk,cpcv}(\mathbb{R})$ is a $w{-}w{-}u.s.c.$ Carathédory multivalued map.

Definition 5.1. A function $u : \Omega \to \mathbb{R}$ is said to be solution of (3) if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and there exists $f \in S_{F,u}$ such that
$$-(1+\lambda)\Delta u - \lambda u = f(x),\ \text{a.e. } x \in \Omega,$$
and u satisfies the boundary condition.

Lemma 5.1. [13] We define $\phi : E = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega)$ by $\phi(u) = -\Delta u$ then ϕ is one to one from E to $L^2(\Omega)$.

Theorem 5.2. Assume that there exists $R > 0$ such that
$$S_F(B(0,R)) \subseteq B(0,R),\ B(0,R) \subset L^2(\Omega),$$
where
$$S_F(u) = \{f \in L^2(\Omega) : f(x) \in F(x, u(x)) \text{ a.e. } x \in \Omega\} \neq \emptyset.$$
Then problem (3) has at least one solutions.

Proof. Let $M = \{u \in L^2(\Omega) : \|u\|_2 \leq R\}$ and define $G : M \to \mathcal{P}(L^2(\Omega))$ by
$$G(u) = \phi^{-1}(S_F(u))$$

and $B : L^2(\Omega) \to L^2(\Omega)$ by

$$B(u) = \lambda(\phi^{-1}(u) - u).$$

For the proof it is enough to show that $B + G$ has fixed point.

Step 1. Let $u \in E$ then $\langle u, -\Delta u\rangle = -\int_\Omega u(x)\Delta u(x)dx = \int_\Omega |\nabla u(x)|^2 dx \geq 0$, then ϕ^{-1} is dissipative. Hence

$$\|u\|_2 \leq \|u - \phi^{-1}(u)\|_2 \Rightarrow \lambda\|u\|_2 \leq \|B(u)\|_2.$$

Step 2. S_F weakly closed graph. Let $\{u_n\}_{n\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ are two sequence, $u_n \rightharpoonup u$ and $f_n \rightharpoonup f$ such that $f_n \in S_F(u_n),\ n \in \mathbb{N}$. From proposition 3.9 [17], we deduce that

$$f(x) \in \overline{co} - w - \overline{\lim}\{f_n(x) : n \in \mathbb{N}\}.$$

Since $F(.,.) \in \mathcal{P}_{wcp,cv}(\mathbb{R})$ and $w - w - u.s..c.$, then

$$f(x) \in \overline{co} - w - \overline{\lim}\{f_n(x) : n \in \mathbb{N}\} \subseteq F(x, u(x)) \text{ a.e.} x \in \Omega.$$

Mazur's Lemma and implies the existence of $\alpha_i^n \geq 0,\ i = n, \ldots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n f_i(\cdot),\ n \in \mathbb{N}$ such that

$$\|f_n - f\|_2 \to 0, \quad \text{as} \quad n \to \infty.$$

It is clear that M is weakly compact. Hence the set $S_F(M)$ weakly compact. Therefore S_F is $w - w - u.s.c.$

Step 3. Let $u, v \in M$ then there exists $f \in S_F(v)$ such that

$$-(\lambda + 1)\Delta u + \lambda u = f(x), \text{a.e. } x \in \Omega.$$

Then

$$\begin{aligned}(\lambda + 1)\langle u, -\Delta u\rangle + \lambda^2\langle u, u\rangle &= \langle v, f\rangle \\ (\lambda + 1)\int_\Omega |\nabla u(x)|^2 dx + \lambda^2\|u\|_2^2 &= \langle v, f\rangle.\end{aligned}$$

Thus

$$\lambda^2\|u\|_2^2 \leq \|v\|_2\|f\|_2 \Rightarrow \|u\|_2 \leq R \Rightarrow u \in M.$$

Finally by using Theorem 4.1, we conclude that $B + G$ has a fixed point in M.

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