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# ON THE CHARACTERIZATION OF PARTIAL METRICS AND QUASIMETRICS

MIRA-CRISTIANA ANISIU* AND VALERIU ANISIU**

* *T. Popoviciu* Institute of Numerical Analysis, Romanian Academy,
P.O. Box 68, 400110 Cluj-Napoca, Romania
E-mail: mira@ubbcluj.ro

** Department of Mathematics, *Babeş-Bolyai* University,
1 Kogălniceanu St., 400084 Cluj-Napoca, Romania
E-mail: anisiu@ubbcluj.ro

**Abstract.** We present the relationship between the notion of partial metric, which has applications in Computer Science, that of quasimetric (which lacks symmetry) and that of standard metric. In this process the nonexpansive functions play an important role. We give some simple formulations of the sequence convergence and of the 0-completeness in partial metric spaces. We apply the results to the characterization of completeness in terms of Caristi's theorem in quasimetric spaces.

**Key Words and Phrases**: metric spaces, partial metric, quasimetric, fixed points, nonexpansive mappings, Caristi's theorem.

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## 1. Introduction

The notion of partial metric belongs to Matthews [7], [8], being aimed as a generalization of the usual metric spaces $X$ in which the distance $d(x,x)$ is not necessarily zero for $x \in X$. Matthews provided also a partial metric version of the Banach fixed point theorem. His study was related to denotational semantics of dataflow networks, and is still applied to models in the theory of computation.

There is a strong relationship between partial metrics and quasimetrics, partially noted already by Matthews. A quasimetric is not symmetric, but it satisfies the triangle inequality (Wilson [13]).

In section 2 we present the notions of partial metric and (weighted) quasimetric and we provide the characterization of partial metrics in terms of usual metrics and nonexpansive functions, and then in terms of weighted quasimetrics with nonexpansive weights. A characterization of weighted quasimetrics in terms of metrics is also given. The new element here is the nonexpansiveness of the functions which appear in the characterization of the partial metrics and quasimetrics.

Section 3 begins with some simple formulations of the sequence convergence and of 0-completeness in partial metric spaces. It contains applications to the characterization of completeness in terms of Caristi's theorem in quasimetric spaces.

## 2. Characterizations for partial metrics and weighted quasimetrics

We begin by giving the definitions of partial metrics and quasimetrics.

**Definition 2.1.** *Let $X$ be a nonempty set. A* partial metric *is a function $p : X \times X \to [0, \infty)$ satisfying, for all $x, y, z \in X$, the following conditions:*
*($p_1$) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;*
*($p_2$) $p(x, x) \le p(x, y)$;*
*($p_3$) $p(x, y) = p(y, x)$;*
*($p_4$) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.*

**Remark 2.1.** *The condition ($p_4$) generalizes the triangle inequality of a metric.*

The notion of quasimetric is much older, having been used by Hausdorff at the beginning of 20th century, but its name belongs to Wilson [13]. It is a generalization of a metric by dropping the symmetry property (see [6]).

**Definition 2.2.** *Let $X$ be a nonvoid set. A* quasimetric *is a function $q : X \times X \to [0, \infty)$ satisfying, for all $x, y, z \in X$, the following conditions*
*($q_1$) $x = y$ if and only if $q(x, y) = q(y, x) = 0$;*
*($q_2$) $q(x, y) \le q(x, z) + q(z, y)$.*
*If $q$ is a quasimetric, then $d : X \times X \to [0, \infty)$ given by*

$$d(x, y) = (q(x, y) + q(y, x))/2 \tag{2.1}$$

*is a metric named the* associated metric *to $q$.*

Matthews ([7], [8]) defined also the notion of weighted quasimetric.

**Definition 2.3.** *Let $X$ be a nonvoid set. A* weighted quasimetric *is a quasimetric $q : X \times X \to [0, \infty)$ for which there exists a function $\alpha : X \to [0, \infty)$, called* weight*, so that*

$$q(x, y) + \alpha(x) = q(y, x) + \alpha(y) \text{ for all } x, y \in X.$$

To each partial metric, one can associate a usual metric in a natural way.

**Theorem 2.1.** ([7]) *If $p$ is a partial metric, the function $d : X \times X \to [0, \infty)$ given by*

$$d(x, y) = p(x, y) - \frac{p(x, x) + p(y, y)}{2} \text{ for all } x, y \in X \tag{2.2}$$

*is a metric on $X$, called the* associated metric *to $p$.*

**Remark 2.2.** *Turinici* ([12]) *showed that, in addition,*

$$|p(x,x) - p(y,y)| \le 2d(x,y) \text{ for all } x, y \in X.$$

*In fact, the metric associated to the partial metric $p$ was originally $d_p = 2d$, but using $d$ simplifies the form of the results.*

First, we give a characterization of partial metrics in terms of usual metrics and nonexpansive functions.

**Theorem 2.2.** *A function $p : X \times X \to [0, \infty)$ is a partial metric on $X$ if and only if there exist a metric $d$ and a nonexpansive function $\varphi : X \to [0, \infty)$ with respect to $d$, so that*

$$p(x,y) = d(x,y) + \varphi(x) + \varphi(y) \text{ for all } x, y \in X. \tag{2.3}$$

*Furthermore, $d$ and $\varphi$ are uniquely determined by $p$.*

*Proof.* Let a partial metric $p$ be given. We define $d : X \times X \to [0, \infty)$ by (2.2) and $\varphi : X \to [0, \infty)$ by $\varphi(x) = p(x,x)/2$.

Using $(p_2)$ and the definitions of $d$ and $\varphi$, we get

$$2\varphi(x) = p(x,x) \le p(x,y) = d(x,y) + \varphi(x) + \varphi(y),$$

hence

$$\varphi(x) - \varphi(y) \le d(x,y).$$

By interchanging $x \leftrightarrow y$ we obtain $\varphi(y) - \varphi(x) \le d(x,y)$, and finally

$$|\varphi(x) - \varphi(y)| \le d(x,y),$$

so the function $\varphi$ is nonexpansive. The uniqueness of $d$ and $\varphi$ follows from the fact that $x = y$ in (2.3) implies $\varphi(x) = p(x,x)/2$.

Conversely, given a metric $d$ and a nonexpansive function $\varphi$, we have to prove that $p$ given by (2.3) satisfies the conditions $(p_1)$-$(p_4)$.

$(p_1)\Leftrightarrow(x = y$ if and only if $2\varphi(x) = d(x,y) + \varphi(x) + \varphi(y) = 2\varphi(y))$;

"$\Rightarrow$" is obvious; "$\Leftarrow$" follows from the fact that $\varphi(x) = \varphi(y)$ and then $d(x,y) = 0$, hence $x = y$.

$(p_2)\Leftrightarrow 2\varphi(x) \le d(x,y) + \varphi(x) + \varphi(y) \Leftrightarrow \varphi(x) - \varphi(y) \le d(x,y)$, and this is true since $\varphi$ is nonexpansive;

$(p_3)$ is obvious;

$(p_4)\Leftrightarrow d(x,y)+\varphi(x)+\varphi(y) \le d(x,z)+\varphi(x)+\varphi(z)+d(z,y)+\varphi(z)+\varphi(y)-2\varphi(z) \Leftrightarrow \varphi(x) - \varphi(y) \le d(x,y) \Leftrightarrow d(x,y) \le d(x,z) + d(z,y)$ and this is true because $d$ is a metric. □

**Remark 2.3.** *Beside the interest in itself of $\varphi$ being a nonexpansive function in the above representation of a partial metric, this fact allows to make shorter some proofs where the continuity of $\varphi$ is needed, as in Corollaries 4.1 and 4.2 of* [10].

**Remark 2.4.** *By applying theorem 2.2, we obtain the nonexpansive functions $\varphi$ and the metrics $d$ corresponding to some partial metrics $p$, which originate in Computer Science.*

(1) *The set $X = \{[a,b] : a \leq b\}$ represents the* vague real numbers.
*Then for the partial metric (Matthews [8]) $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$, $\varphi([a,b]) = (b-a)/2$ and $d([a,b],[c,d]) = 1/2\ (|a-c| + |b-d|)$.*

(2) *The* Baire metric *on sequences over a set $S$ is $d : S^{\mathbb{N}} \times S^{\mathbb{N}} \to [0,\infty)$ ($\mathbb{N} = \{0, 1, ..., \}$), given by $d(x,y) = 2^{-\sup\{i: i\in\mathbb{N}, \forall j<i, x_j = y_j\}}$.*
*Matthews ([8]) considered also the set of finite sequences $S^*$ (called* partially defined streams of information*) and extended the Baire metric to $S^* \cup S^{\mathbb{N}}$. This can be done in a simpler way in the setting of* words *(for definitions and notations, see [2]).*
*Let us denote by $|x|$ the length of a finite or infinite word $x \in S^* \cup S^{\mathbb{N}}$ ($|x| = \infty$ if $x$ is infinite, and $|x| = 0$ for the empty word $\varepsilon$), and by $x \cap y$ the longest common prefix ($x \cap y = \varepsilon$ if there is no common prefix for $x$ and $y$). Then, the Baire metric on $S^{\mathbb{N}}$ is given by $d(x,y) = 2^{-|x\cap y|}$ ($2^{-\infty} := 0$).*
*It follows that $p : \left(S^* \cup S^{\mathbb{N}}\right) \times \left(S^* \cup S^{\mathbb{N}}\right) \to [0,\infty)$, $p(x,y) = 2^{-|x\cap y|}$ is a partial metric, $\varphi(x) = 2^{-|x|-1}$ and $d(x,y) = 2^{-|x\cap y|} - 2^{-|x|-1} - 2^{-|y|-1}$.*

(3) *([11]) On the complexity space*

$$\mathcal{C} = \left\{ f : \mathbb{N} \to (0,\infty] : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\}, \tag{2.4}$$

*the function $p : \mathcal{C} \times \mathcal{C} \to [0,\infty)$ defined by*

$$p(f,g) = \sum_{n=0}^{\infty} 2^{-n} \max\left(\frac{1}{f(n)}, \frac{1}{g(n)}\right) \tag{2.5}$$

*is a partial metric,*

$$\varphi(f) = 1/2 \sum_{n=0}^{\infty} 2^{-n}/f(n)$$

*and*

$$d(f,g) = 1/2 \sum_{n=0}^{\infty} 2^{-n} \left|1/f(n) - 1/g(n)\right|.$$

**Corollary 2.1.** *For each partial metric $p$ given by (2.3), we have*

$$|p(x,y) - p(x',y')| \leq 2\left(d(x,x') + d(y,y')\right) \text{ for all } x,y,x',y' \in X. \tag{2.6}$$

*Proof.* $|p(x,y) - p(x',y')| =$
$|d(x,y) - d(x',y) + d(x',y) - d(x',y') + \varphi(x) - \varphi(x') + \varphi(y) - \varphi(y')|$
$\leq d(x,x') + d(y,y') + d(x,x') + d(y,y') = 2\left(d(x,x') + d(y,y')\right)$. □

**Remark 2.5.** *The constant 2 is the best for partial metrics, as the following example shows (for a metric, the inequality (2.6) holds with the constant 1).*

**Example 2.1.** *Let* $X = \mathbb{R}$*,* $d(x,y) = |x-y|$ *and* $\varphi(x) = |x|$*. For* $x = -1$*,* $y = 1$*,* $x' = y' = 0$*,* $|p(x,y) - p(x',y')| = 4 = 2\left(d(x,x') + d(y,y')\right)$*.*

We give a characterization of weighted quasimetrics in terms of metrics and of partial metrics.

**Theorem 2.3.** *Let* $q$ *be a quasimetric. The following assertions are equivalent:*
***(a)*** $q$ *is weighted;*
***(b)*** *there exist a metric* $d$ *and a function* $\varphi : X \to [0,\infty)$ *so that the relation*

$$q(x,y) = d(x,y) - \varphi(x) + \varphi(y) \text{ for all } x,y \in X \tag{2.7}$$

*holds;*
***(c)*** *there exists a function* $\psi : X \to [0,\infty)$ *so that* $(x,y) \mapsto q(x,y) + 2\psi(x)$ *is a partial metric.*

*Furthermore,*
***1.*** $d$ *is unique (and equals the metric associated to* $q$*);*
***2.*** $\varphi$ *is unique up to an additive constant and nonexpansive with respect to* $d$*;*
***3.*** $\psi - \varphi$ *is constant.*

*Proof.*
**(a)⇒(b).** Let $d$ be the the associated metric to $q$ given by (2.1). If $\alpha$ is a weight for $q$,
$q(x,y) - d(x,y) = \left(q(x,y) - q(y,x)\right)/2 = \left(\alpha(y) - \alpha(x)\right)/2$. Taking $\varphi(x) = \alpha(x)/2$ one obtains the relation (2.7).
**(b)⇒(c).** Take $\psi = \varphi$. Notice first that the function $\varphi$ in (2.7) is $d$-nonexpansive. In fact, $q \geq 0$ implies $\varphi(x) - \varphi(y) \leq d(x,y)$, and permuting $x$ and $y$ gives $|\varphi(x) - \varphi(y)| \leq d(x,y)$.
We have $q(x,y) + 2\varphi(x) = d(x,y) + \varphi(x) + \varphi(y)$ and this is a partial metric by theorem 2.2.
**(c)⇒(a).** Using theorem 2.2, there exist metric $d$ and a $d$-nonexpansive function $\varphi_1 : X \to [0,\infty)$ such that $q(x,y) + 2\psi(x) = d(x,y) + \varphi_1(x) + \varphi_1(y)$.
Taking here $x = y$ one obtains $\varphi_1(x) = \psi(x)$, hence

$$q(x,y) = d(x,y) - \psi(x) + \psi(y). \tag{2.8}$$

It results $q(x,y) - q(y,x) = 2\left(-\psi(x) + \psi(y)\right)$, and $q$ is weighted with $\alpha = 2\psi$. We have $q(x,y) + q(y,x) = 2d(x,y)$, so $d$ is the associated metric to $q$.

The uniqueness of $d$ in (2.7) follows from the fact that $q(x,y) + q(y,x) = 2d(x,y)$, so $d$ is actually the metric associated to $q$. The nonexpansiveness of $\varphi$ was already noticed.

For a fixed $x_0$ in $X$, we obtain from (2.7) that $\varphi(x) = d(x,x_0) - q(x,x_0) + \varphi(x_0)$, hence the uniqueness of $\varphi$ up to an additive constant follows. Using this fact and (2.8) one obtains that $\psi - \varphi$ is constant. □

**Corollary 2.2.** *The function* $q : X \times X \to [0, \infty)$ *is a weighted quasimetric if and only if there exist a metric* $d$ *and a* $d$*-nonexpansive function* $\varphi : X \to [0, \infty)$ *so that*

$$q(x, y) = d(x, y) - \varphi(x) + \varphi(y). \tag{2.9}$$

*In addition, the metric* $d$ *is unique (and equals the metric associated to* $q$*), and the function* $\varphi$ *is unique up to an additive constant.*

*Proof.* The direct implication and the uniqueness follow from the previous theorem.

Conversely, given a metric $d$ and a nonexpansive function $\varphi : X \to [0, \infty)$, the function $q$ defined by (2.9) is obviously a weighted quasimetric with the weight $\alpha = 2\varphi$. □

**Remark 2.6.** *The formula (2.9) defines a quasimetric even if the nonexpansive function* $\varphi : X \to \mathbb{R}$ *is not nonnegative, but in this case* $q$ *may fail to be weighted. For example, for* $X = \mathbb{R}$, $q(x, y) = |x - y| - x + y$ *is not weighted; in fact, a weight* $\alpha$ *would satisfy* $\alpha(y) - 2y = \alpha(x) - 2x$, *implying* $\alpha(x) = 2x + const$, *contradicting* $\alpha \geq 0$.

We give an example of a quasimetric $q$ for which, whatever the function $\varphi$ is, $q(x, y) + 2\varphi(x)$ is not a partial metric. This is equivalent with the fact that the quasimetric $q$ is not weighted; notice that if $q(x, y) + 2\varphi(x)$ is a partial metric then $\varphi \geq 0$ (by taking $x = y$ in $q(x, y) + 2\varphi(x) \geq 0$).

**Example 2.2.** *Let* $X = [0, 1]$ *(or* $[0, \infty)$*) and*

$$q(x, y) = \begin{cases} 0, & \text{if } x \leq y \\ 1, & \text{if } x > y. \end{cases}$$

*Let us suppose that there is a metric* $d$ *so that relation (2.7) holds.*

*For* $x < y$ *we get*

$$0 = d(x, y) - \varphi(x) + \varphi(y); \tag{2.10}$$

*for* $x > y$ *we get*

$$1 = d(x, y) - \varphi(x) + \varphi(y),$$

*which implies for* $x < y$

$$1 = d(y, x) - \varphi(y) + \varphi(x). \tag{2.11}$$

*By adding (2.10) and (2.11) we get* $d(x, y) = 1/2 = \varphi(x) - \varphi(y)$ *for each* $x < y$. *It follows that* $\varphi$ *is strictly decreasing and nowhere continuous, which is a contradiction.*

The next theorem gives a connection between partial metrics and quasimetrics.

**Theorem 2.4.** *For every partial metric* $p$ *on* $X$ *there exist a quasimetric* $q$ *and a function* $\varphi : X \to [0, \infty)$ *so that*

$$p(x, y) = q(x, y) + 2\varphi(x) \text{ for all } x, y \in X. \tag{2.12}$$

*In addition, the functions* $q$ *and* $\varphi$ *are unique,* $q$ *is weighted and* $\varphi$ *is nonexpansive with respect to* $d$, *where* $d$ *is the metric associated to the partial metric* $p$, *given by (2.2). Furthermore, the associated metric to the quasimetric* $q$ *is precisely* $d$.

*Conversely, given a weighted quasimetric $q$, the function $p$ given by (2.12) is a partial metric.*

*Proof.* In [7] it was shown that the axioms $(q_1)$ and $(q_2)$ are fulfilled for

$$q(x,y) = p(x,y) - p(x,x),$$

and this fact is also a consequence of theorem 2.2. Therefore we may put $\varphi(x) = p(x,x)/2$, and it follows from the proof of theorem 2.2 that $\varphi$ is nonexpansive with respect to $d$.

For the uniqueness, we consider $x = y$ in (2.12) and get $\varphi(x) = p(x,x)/2$ and then $q(x,y) = p(x,y) - p(x,x)$.

The associated metric to the quasimetric $q$ is

$$\left(q(x,y) + q(y,x)\right)/2 = p(x,y) - p(x,x)/2 - p(y,y)/2 = d(x,y).$$

The converse is contained in the implication (a)⇒(c) of theorem 2.3. □

From corollary 2.2 we obtain, as examples, the quasimetrics corresponding to the partial metrics from Remark 2.4.

**Example 2.3.** *For each $t \in \mathbb{R}$, denote $t^+ = \max(t,0)$.*

(1) *On the set $X = \{[a,b] : a \le b\}$ of the vague real numbers $q([a,b],[c,d]) = (d-b)^+ + (c-a)^+$ is a weighted quasimetric, with the weight $\alpha([a,b]) = b-a$.*
(2) *On the set $S^* \cup S^{\mathbb{N}}$, $q(x,y) = 2^{-|x\cap y|} - 2^{-|x|-1} + 2^{-|y|-1}$ is a weighted quasimetric, with the weight $\alpha(x) = 2^{-|x|}$.*
(3) ([11]) *On the complexity space $\mathcal{C}$ given by (2.4), the function $q : \mathcal{C}\times\mathcal{C} \to [0,\infty)$ defined by*

$$q(f,g) = \sum_{n=0}^{\infty} 2^{-n}\left(\frac{1}{g(n)} - \frac{1}{f(n)}\right)^+$$

*is a weighted quasimetric with the weight*

$$\alpha(f) = \sum_{n=0}^{\infty} 2^{-n}\frac{1}{f(n)}.$$

## 3. Applications

A partial metric $p$ can be viewed as a pair $(d,\varphi)$, where $d$ is a metric and $\varphi$ is $d$-nonexpansive, and many properties for $p$ have simple formulations with respect to $d$ and $\varphi$.

- In a partial metric space $(X,p)$,

$$x_n \xrightarrow{p} x \Longleftrightarrow \limsup_{n\to\infty}\left(d(x_n,x) + \varphi(x_n) - \varphi(x)\right) \le 0.$$

- The 0-Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ (see [9]) are characterized by

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 \text{ and } \lim_{n\to\infty} \varphi(x_n) = 0.$$

- $(X, p)$ is 0-complete iff for each 0-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ there exists $x \in X$ such that $\varphi(x) = 0$ and $d(x_n, x) \to 0$.

In [9], the 0-completeness was showed to be equivalent with Caristi's theorem in partial metric spaces:

**Theorem 3.1.** *Let $(X, p)$ be a partial metric space. The following are equivalent:*
*(a) $(X, p)$ is 0-complete;*
*(b) Each $f : X \to X$ for which there exists a function $\Phi : X \to [0, \infty)$ lower semicontinuous with respect to $d$ such that*

$$p(x, f(x)) \le \Phi(x) - \Phi(f(x)) \text{ for all } x \in X \tag{3.1}$$

*has a fixed point in $X$.*

Haghi et al. [5] remarked that the *only if* part of the above theorem follows from Caristi's theorem applied to the complete metric space $X$ endowed with the metric

$$d(x, y) = \begin{cases} 0, & \text{for } x = y \\ p(x, y), & \text{for } x \ne y. \end{cases}$$

Note that there exist in the literature other definitions for Cauchy sequences and for completeness, but they cannot be used in the above theorem.

Generally, in partial metric spaces the completeness with respect to the associated metric $d$ does not follow from the validity of Caristi's theorem. For example, for $p = d + 1$ $(> 0)$, there is no function with fixed points which verifies the inequality (3.1).

A less trivial example is given by Romaguera in [9]. Let us consider $X = [0, \infty) \cap \mathbb{Q}$ endowed with the partial metric $p(x, y) = \max(x, y)$, hence $d(x, y) = |x - y|/2$. In the partial metric space $(X, p)$, which is 0-complete, Caristi's theorem holds, but the metric space $(X, d)$ is not complete. We remark that in this case the set of Caristi's functions is nonvoid (for example, $f(x) = 0$ and $\Phi(x) = x$ for $x \in X$).

We shall give a characterization of completeness in terms of Caristi's theorem [3] for spaces $X$ endowed with a weighted quasimetric $q$ with weight $\alpha$. We remind that $X$ is a metric space with respect to the metric $d$ defined by (2.9), where $\varphi = \alpha/2$, and we refer to $d$-completeness. Results of this type for other kinds of completeness can be found for example in [4] or [1].

**Theorem 3.2.** *Let $(X, q)$ be a weighted quasimetric space and denote by $d$ the associated metric. The following assertions are equivalent:*
*(a) $X$ is complete with respect to $d$;*

*(b) Each $f : X \to X$ for which there exists a function $\Phi : X \to [0, \infty)$ lower semicontinuous with respect to $d$ such that*

$$q(x, f(x)) \le \Phi(x) - \Phi(f(x)) \text{ for all } x \in X \tag{3.2}$$

*has a fixed point in $X$.*

*Proof.* From Corollary 2.2 we get $q(x, y) = d(x, y) - \varphi(x) + \varphi(y)$, $\varphi$ being $d$-nonexpansive, and the condition (3.2) becomes

$$d(x, f(x)) \le \varphi(x) + \Phi(x) - \varphi(f(x)) - \Phi(f(x)) \text{ for all } x \in X. \tag{3.3}$$

By applying the standard Caristi theorem for the lower semicontinuous function $\varphi+\Phi$ we obtain the conclusion.

For the converse, suppose that $(X, d)$ is not complete. Denoting by $(\tilde{X}, \tilde{d})$ its completion, there exists a sequence $(a_n)$ in $X$ such that $a_n \to a_\infty \in \tilde{X} \setminus X$.

For $n \in \mathbb{N}$, $x \in X$, $q(x, a_n) = d(x, a_n) - \varphi(x) + \varphi(a_n) \le 2d(x, a_n) \to 2\tilde{d}(x, a_\infty) > 0$ (here again $\varphi = \alpha/2$). Then $3\tilde{d}(x, a_\infty) - 3\tilde{d}(a_n, a_\infty) \to 3\tilde{d}(x, a_\infty) > 2\tilde{d}(x, a_\infty)$.

It follows that there exists $n \in \mathbb{N}$ such that

$$q(x, a_n) < 3\tilde{d}(x, a_\infty) - 3\tilde{d}(a_n, a_\infty).$$

Denote by $n(x)$ the smallest $n$ for which this inequality holds.

Defining $f : X \to X$, $f(x) = a_{n(x)}$ and $\Phi : X \to [0, \infty)$, $\Phi(x) = 3\tilde{d}(x, a_\infty)$, one obtains

$$q(x, f(x)) < \Phi(x) - \Phi(f(x)).$$

The function $\Phi$ is obviously continuous (actually Lipschitz) and $f$ has no fixed point (the previous inequality being strict), contradiction. □

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